Efficient Global and Local Force Calculations Based on Continuum Sensitivity Analysis

Dong-Hun Kim[1], David A. Lowther[2], and Jan K. Sykulski[3]

[1]School of Electrical Engineering and Computer Science, Kyungpook National University, Daegu 702-701, Korea
[2]Department of Electrical and Computer Engineering, McGill University, Montreal, QC H3A 2A7, Canada
[3]School of Electronics and Computer Science, University of Southampton, Southampton SO17 1BJ, U.K.

Equations derived from the continuum design sensitivity analysis (CDSA), in conjunction with the material derivatives for a continuous medium and using the energy-based approach, have been successfully applied to the calculation of both total force and force distributions. The resultant expressions are similar to the Maxwell Stress Tensor, Magnetic Charge Method, and Virtual Work Method but have several advantages over the traditional approaches. Numerical implementation of the scheme leads to efficient calculations and improved accuracy.

***Index Terms*—Electromagnetic force calculation, finite elements, sensitivity analysis.**

I. Introduction

The knowledge of local force distributions, as well as global force, is essential in the design of many electromagnetic (EM) systems. Thus, the importance of reliable and efficient techniques for extraction of such information from numerical solutions cannot be overestimated. Existing popular methods include the Virtual Work Principle (VWP), Maxwell Stress Tensor (MST), and Magnetic Charge Method (MCM), which are well documented in the literature [1]–[9]. Each method has advantages and some implementation related drawbacks and there is no consensus of opinions regarding the best calculation methods for the local force distribution.

Recently, a new approach to the calculation of EM fields has been proposed using differential geometry [7], [8]. Under the assumption of continuous media, it utilizes the material derivatives and differential forms to derive energy-based descriptions of materials and forces. However, the resultant forces are expressed in terms of EM fields only, like the MST, and so it is very difficult to establish direct relationships between the magnetic material properties (permeability, permanent magnetization, and current density) and their corresponding force distributions.

In this paper, the force expressions derived in our previous paper [9], based on the continuum design sensitivity analysis (CDSA), are extended to allow for calculation of the local force distributions as well as the global forces. Although the usage of material derivatives and the VWP approach are similar to those presented in [7] and [8], the proposed scheme gives better understanding and direct insight into the mechanism of electromechanical forces acting on the materials due to the virtual work displacement. Moreover, the scheme offers easy implementation and improved accuracy over traditional force calculation methods. Its validity has been tested using numerical models for both linear and nonlinear materials.

Digital Object Identifier 10.1109/TMAG.2006.890949

II. Derivation

In the previous paper [9], analytical expressions for global forces in magnetic systems were developed using CDSA, which itself stems from the VWP. Although these expressions could be used to calculate the force acting on a nonlinear material, they did not actually include a volume force term generated inside such a material. This is because the derivation was performed from the viewpoint of the energy sensitivity with the respect to the variation of the interface shape between two different materials only. To accomplish a more general formulation, expressions considering the contribution of changes of nonlinear material properties to the energy sensitivity are proposed here.

A. Energy Sensitivity Formula

When dealing with the variation of the system energy in response to changes of the shape and material properties, it is convenient to think of the analysis domain as a continuous medium and utilize the material derivative idea of continuum mechanics [10]. Thus, the material derivative concept with the augmented Lagrangian method and the adjoint variable method are used as a vehicle to develop the sensitivity formula.

In order to derive the energy sensitivity formula, first, an objective function W is mathematically expressed as

$$W = \int_\Omega g\left(\mathbf{A}(\mathbf{p}), \nabla \times \mathbf{A}(\mathbf{p})\right) d\Omega \tag{1}$$

where g means an energy function of the magnetostatic system, differentiable with respect to the magnetic vector potential, $\mathbf{A}$, and $\nabla \times \mathbf{A}$, that are themselves implicit functions of the design variable vector $\mathbf{p}$. To deduce the sensitivity formula and the adjoint system equation systematically, the variational form of Maxwell's equation, referred to as the primary system, is added to (1) based on the augmented Lagrangian method

$$\overline{W} = \int_\Omega g(\mathbf{A}, \nabla \times \mathbf{A}) d\Omega + \int_\Omega \boldsymbol{\lambda} \left\{ -\nabla \times (\upsilon \nabla \times \mathbf{A} - \mathbf{M}) + \mathbf{J} \right\} d\Omega \tag{2}$$

where λ is the Lagrange multiplier vector interpreted as the adjoint variable. For simplicity in dealing with a nonlinear system, the solution could be considered as equivalent to a linear problem with the reluctivities in the problem "frozen" at an incremental value determined by the nonlinear solution from the point-of-view of the virtual work perturbation. Under this scenario, each point inside the nonlinear material has a reluctivity equal to its incremental value and a magnetization equal to the effective coercive force for the linear $B-H$ relationship. Thus, the space distribution of the reluctivity υ in (2) can be assumed to be already known for the given source, permanent magnetization $\mathbf{M}$, and current density $\mathbf{J}$.

To obtain an explicit expression for the deformation of the interface boundary between different materials, Ω_1 and Ω_2, and accordingly for the variation of the reluctivity distribution inside the materials, the second integral on the right-hand side of (2) is split into two regions. Then, we take the material derivative on both sides of (2) as

$$\begin{aligned}\frac{d\overline{W}}{dt} &= \dot{\overline{W}} \\ &= \int_\Omega \{\mathbf{g_A}\cdot\overline{\boldsymbol{\lambda}}+\mathbf{g}_{\nabla\times\mathbf{A}}\cdot\nabla\times\overline{\mathbf{A}}\}d\Omega \quad (\Omega=\Omega_1+\Omega_2)\\ &\quad-\int_{\Omega_1}\upsilon_1\{\nabla\times\mathbf{A}_1\cdot\nabla\times\overline{\boldsymbol{\lambda}}_1+\nabla\times\boldsymbol{\lambda}_1\cdot\nabla\times\overline{\mathbf{A}}_1\}d\Omega\\ &\quad-\int_{\Omega_2}\upsilon_2\{\nabla\times\mathbf{A}_2\cdot\nabla\times\overline{\boldsymbol{\lambda}}_2+\nabla\times\boldsymbol{\lambda}_2\cdot\nabla\times\overline{\mathbf{A}}_2\}d\Omega\\ &\quad-\int_{\Omega_1}\{\overline{\upsilon}_1\nabla\times\mathbf{A}_1\cdot\nabla\times\boldsymbol{\lambda}_1\}d\Omega\\ &\quad-\int_{\Omega_2}\{\overline{\upsilon}_2\nabla\times\mathbf{A}_2\cdot\nabla\times\boldsymbol{\lambda}_2\}d\Omega\\ &\quad+\int_{\Omega_1}\{\mathbf{J}_1\cdot\overline{\boldsymbol{\lambda}}_1+\mathbf{M}_1\cdot\nabla\times\overline{\boldsymbol{\lambda}}_1]d\Omega\\ &\quad+\int_{\Omega_2}\{\mathbf{J}_2\cdot\overline{\boldsymbol{\lambda}}_2+\mathbf{M}_2\cdot\overline{\boldsymbol{\lambda}}_2\}d\Omega\\ &\quad+\int_\gamma\{\upsilon_1\nabla\times\mathbf{A}_1\cdot\nabla\times\boldsymbol{\lambda}_1-\upsilon_2\nabla\times\mathbf{A}_2\cdot\nabla\times\boldsymbol{\lambda}_2\}V_n d\gamma\\ &\quad-\int_\gamma\{\mathbf{M}_1\cdot\nabla\times\boldsymbol{\lambda}_1-\mathbf{M}_2\cdot\nabla\times\boldsymbol{\lambda}_2\\ &\qquad+\mathbf{J}_1\cdot\boldsymbol{\lambda}_1-\mathbf{J}_2\cdot\boldsymbol{\lambda}_2\}V_n d\gamma \end{aligned}\tag{3}$$

where $\mathbf{g_A}\equiv\partial g/\partial\mathrm{A}$, $\mathbf{g}_{\nabla\times\mathbf{A}}\equiv\partial g/\partial(\nabla\times\mathrm{A})$, $\overline{A}\equiv(\dot{\boldsymbol{\lambda}}-\nabla\lambda\cdot\mathbf{V})$, $\overline{\lambda}\equiv(\dot{\mathbf{A}}-\nabla\mathbf{A}\cdot\mathbf{V})$, $\overline{\upsilon}\equiv(\dot{\upsilon}-\nabla\upsilon\cdot\mathbf{V})$, and $\mathbf{V}$ denotes a design velocity vector. In order to make the previous equation simpler, the space distributions of the sources of $\mathbf{M}$ and $\mathbf{J}$ inside the materials were assumed constant at a certain time. γ denotes the part of the interface boundary that is allowed to move. The integrands related to $\overline{\lambda}$ and $\overline{A}$ in (3) vanish because they have the same variational forms as the primary system and the adjoint system, respectively [10]. Moreover, the material derivative of υ can be assumed to be zero only if its distribution is expressed by using pointwise continuous functions inside Ω_1 and Ω_2.

Finally, the energy sensitivity formula applicable to linear and nonlinear magnetostatic problems is given by

$$\begin{aligned}\frac{d\overline{W}}{d\mathbf{p}} &= \int_\gamma\{(\upsilon_1-\upsilon_2)\nabla\times\mathbf{A}_1\cdot\nabla\times\boldsymbol{\lambda}_2-(\mathbf{M}_1-\mathbf{M}_2)\\ &\qquad\cdot\nabla\times\boldsymbol{\lambda}_2-(\mathbf{J}_1-\mathbf{J}_2)\cdot\boldsymbol{\lambda}_2\}V_n d\gamma\\ &\quad+\int_{\Omega_1}\{(\nabla\upsilon_1\cdot\mathbf{V})(\nabla\times\mathbf{A}_1\cdot\nabla\times\boldsymbol{\lambda}_1)\}d\Omega\\ &\quad+\int_{\Omega_2}\{(\nabla\upsilon_2\cdot\mathbf{V})(\nabla\times\mathbf{A}_2\cdot\nabla\times\boldsymbol{\lambda}_2)\}d\Omega\end{aligned}\tag{4}$$

where the surface and the volume integrals related to υ represent the variation of the stored total energy (magnetic energy W_m and co-energy W'_m) experienced over the interface and inside the material region, respectively. On the other hand, the surface integrals concerned with $\mathbf{M}$ and $\mathbf{J}$ express the variation of the total input energy over γ. When dealing with the objective functions related to the system energy, the dual system consisting of the primary and the adjoint systems is self-adjoint. In other words, the variational of the adjoint system is the same as that of the primary system [9]. Thus, $\mathbf{A}=\boldsymbol{\lambda}$ and there is no need to solve the adjoint problem. The variation of W (interpreted as the total energy stored and the input energy supplied to the system) can now be expressed as

$$\begin{aligned}\frac{d\overline{W}}{d\mathbf{p}} &= \Bigg[\int_\gamma\{(\upsilon_1-\upsilon_2)\mathbf{B}_1\cdot\mathbf{B}_2\}V_n d\gamma\\ &\qquad+\int_{\Omega_1}\{(\nabla\upsilon_1\cdot\mathbf{V})(\mathbf{B}_1\cdot\mathbf{B}_1)\}d\Omega\\ &\qquad+\int_{\Omega_2}\{(\nabla\upsilon_2\cdot\mathbf{V})(\mathbf{B}_2\cdot\mathbf{B}_2)\}d\Omega\Bigg]\\ &\quad+\int_\gamma\{(\mathbf{M}_2-\mathbf{M}_1)\cdot\mathbf{B}_2\}V_n d\gamma\\ &\quad+\int_\gamma\{(\mathbf{J}_2-\mathbf{J}_1)\cdot\boldsymbol{\lambda}_2\}V_n d\gamma\end{aligned}\tag{5}$$

where the integrals inside the square brackets express variation of the total magnetic energy over γ and inside Ω_1 and Ω_2.

B. Expression of Force

To associate the energy sensitivity (5) with the mechanical force $\mathbf{F}_{\mathrm{iron}}$ acting on the interface γ between two different magnetic materials of μ_1 and μ_2, we assume a constant current condition and a virtual displacement $\mathbf{p}$, which yields

$$\mathbf{F}_{\mathrm{iron}}=\frac{\partial W'_m(\mathbf{p},\mathbf{I})}{\partial\mathbf{p}}\Bigg|_{\mathbf{I}=\mathrm{constant}}=\frac{\partial W}{\partial\mathbf{p}}-\frac{\partial W_m}{\partial\mathbf{p}}\cong\frac{dW}{d\mathbf{p}}-\frac{dW_m}{d\mathbf{p}}.\tag{6}$$

Moreover, the total magnetic force due to the magnetization in the material, expressed in terms of the reluctivity difference across the interface and the gradient of the reluctivity distribution in the domain Ω_2, can be written as

$$\mathbf{F}_{\text{iron}} = \int_{\gamma} \mathbf{f}_{s,\text{iron}} d\gamma + \int_{\Omega_2} \mathbf{f}_{v,\text{iron}} d\Omega = \int_{\gamma} \left\{ \frac{1}{2}(\upsilon_1 - \upsilon_2)\mathbf{B}_1 \cdot \mathbf{B}_2 \right\} \mathbf{n} d\gamma + \int_{\Omega_2} \left\{ \frac{1}{2} \nabla \upsilon_2 (\mathbf{B}_2 \cdot \mathbf{B}_2) \right\} d\Omega \quad (7)$$

where $\mathbf{f}_{s,\text{iron}}$ and $\mathbf{f}_{v,\text{iron}}$ are surface or volume force distributions, respectively, and the direction of each surface force is decided by an arbitrary design velocity $\mathbf{V}$. In the case of $\mathbf{f}_{s,\text{iron}}$, $\mathbf{V}$ is set to the same direction as a unit normal vector, $\mathbf{n}$, outward to γ.

Meanwhile, the electromechanical energy conversion due to the variation of the magnetic and electric input energy follows a different argument from that of the magnetic material. The instantaneous change of the input energy supplied by a virtual displacement is always equal to the mechanical force under the assumption that the flux linkage of the system is invariant (i.e., there is no change of the stored energy). A loudspeaker voice coil (an *energy-transfer* transducer) [11] serves as a good example. Accordingly, the force due to permanent magnet magnetizations on the side of Ω_2 may be written as

$$\mathbf{F}_{\text{magnet}} = \int_{\gamma} \mathbf{f}_{s,\text{magnet}} d\gamma = \int_{\gamma} \{(\mathbf{M}_2 - \mathbf{M}_1) \cdot \mathbf{B}_2\} \mathbf{n} d\gamma \quad (8)$$

and the force due to the currents on the side of Ω_2 equals

$$\mathbf{F}_{\text{conductor}} = \int_{\gamma} \mathbf{f}_{s,\text{conductor}} d\gamma = \int_{\gamma} \{(\mathbf{J}_2 - \mathbf{J}_1) \cdot \mathbf{A}_2\} \mathbf{n} d\gamma. \quad (9)$$

III. IMPLICATIONS OF THE FORCE EXPRESSIONS

Expressions (7)–(9) have several interesting implications. The first integral on the right-hand side of (7) gives the force on an interface between materials of two different reluctivities. The expression is consistent with the MST calculation in the air surrounding Ω_2 if Ω_1 is assumed to be air. On the other hand, the second integral of (7) corresponds to the volume force generated by the gradient of the reluctivity distribution in the material, i.e., $(\partial W/\partial \alpha_i)\nabla \alpha_i$ as appearing in the Korteweg–Helmholtz force expression in [3]. As the volume force densities are normally much smaller than the surface forces in a nonlinear magnetic material, this term is not considered in the examples that follow.

Equation (8) provides the force due to the presence of permanent magnets only and is similar to a MCM calculation of $\sigma_s \mathbf{H}_s = \mu_o (\mathbf{M} \cdot \mathbf{n}) \mathbf{H}_s$ [3]. But there are significant differences in the directions and distributions of the forces given by the two expressions; these are compared in the examples that follow. Equation (9) represents the force due to current carrying conductors only, i.e., it is a surface integral equivalent to the volume integral of $\mathbf{J} \times \mathbf{B}$. Unlike the MST approach, these equations clearly illustrate the contributions to the global force, as well as force distributions, on a body in terms of each source of the magnetic field.

IV. EXAMPLES

The formulations given by (7)–(9) have been compared with other calculation methods based on variants of the MCM and

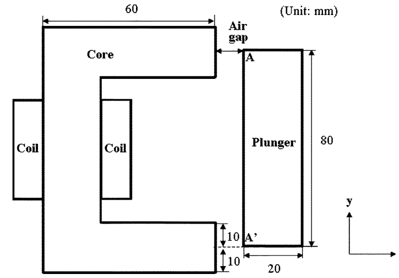

Fig. 1. C-core actuator.

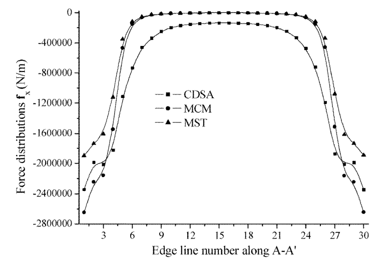

Fig. 2. Force distributions in the air gap for a linear material.

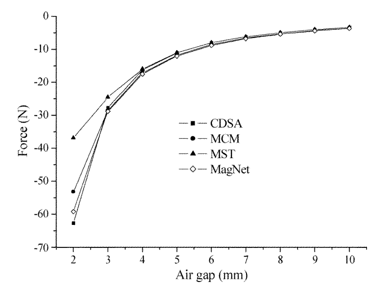

Fig. 3. Global forces versus distance for a linear material.

the MST. In order to ensure fairness of comparison, the different force calculations were carried out by using the same numerical field solution provided by a commercial electromagnetic software package (MagNet 6).

A. Force Comparisons for a Simple C-Core Actuator

A simple C-core actuator shown in Fig. 1 is considered. The core is 60-mm wide × 100-mm high and the poles are 20-mm wide. The "plunger" has dimensions of 20 mm × 80 mm. In the first example, the core and plunger are assumed to be constructed from a linear material with a relative permeability of 1000. The force distributions with the air gap length of 2 mm are shown in Fig. 2 and the global forces calculated on the basis of the force distributions are compared in Fig. 3.

As can be easily seen, the global force results of CDSA show a good agreement with those of MagNet 6 as the air gap length decreases from 10 to 2 mm. This means that the proposed method yields accurate force results even when the fields abruptly change on the test line $A - A'$ in Fig. 1. Figs. 4 and 5 show similar results but with the core and the plunger assumed to be made of nonlinear electric steel. Compared to the linear case, much better agreement is observed between the new method and the conventional approaches. This results from the fact that the changes of the field on the interface of the plunger are not so abrupt as the air gap decreases. Some of the errors might be attributed to the inaccuracy of the interpolation used on the magnetization curve for the material to compute the incremental reluctivities.

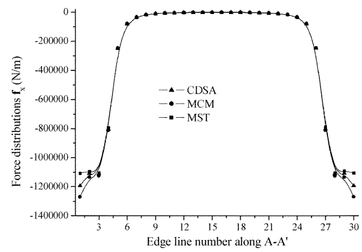

Fig. 4. Force distributions in the air gap for a nonlinear material (M19).

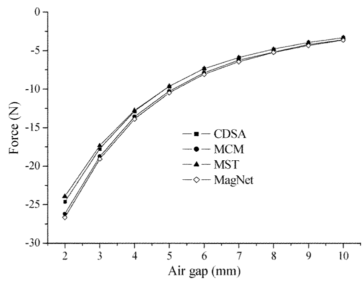

Fig. 5. Global forces versus distance for a nonlinear material (M19).

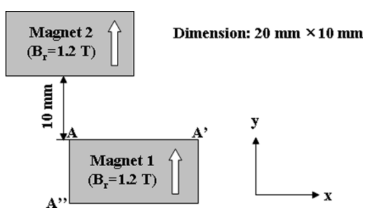

Fig. 6. Permanent magnet model.

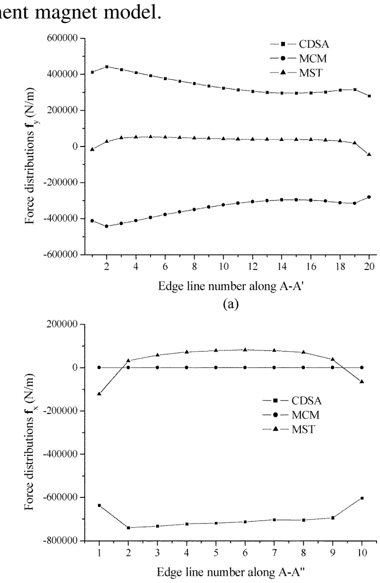

Fig. 7. Force distributions generated between two permanent magnets. (a) Along $A - A'$. (b) Along $A - A''$.

B. Comparison of Forces Acting on Permanent Magnets

It is well known that the traditional methods such as VWP, MCM, and MST often lead to different force distributions even if they agree in terms of the global force. One might concur with the statements in [5] and [6] that only the methods based on the VWP provide results which have true physical meaning. It, therefore, appeared reasonable to verify our proposed formulation using a simple model of two permanent magnets with residual magnetic flux density of 1.2 T and constant magnetization as shown in Fig. 6. Force distributions produced by three methods (CDSA, MST, and MCM) are depicted in Figs. 7 and 8. Clearly, as seen especially in Fig. 8, the distributions obtained

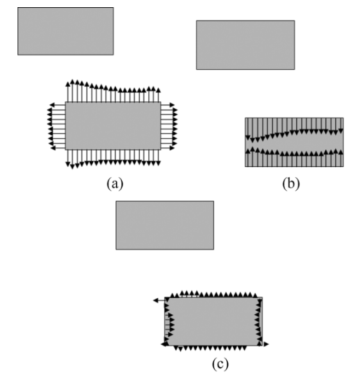

Fig. 8. Illustration of different force distributions by three methods. (a) CDSA. (b) MCM. (c) MST.

by MST and MCM are wrong, whereas those obtained using our new CDSA approach are in good agreement with the results reported in [5] and [6], as all these formulations are based on the virtual work principle. However, it should be emphasised that our method—unlike the other reported formulations based on the VWP—does not require additional cumbersome manipulations, such as differentiation of the jacobian matrix in the finite element (FE) formulation, and is, in fact, independent of the FE formulation used.

V. Conclusion

This paper has proposed a new derivation of the global and distributed force formulations for a computational algorithm based on the CDSA. The force expressions clearly indicate the contributions to the global forces and force distributions from different sources of the magnetic field. The implementation is simple and is independent of the numerical analysis approach taken. The method has been shown to agree well with the tested implementations of traditional techniques.

Acknowledgment

This work was supported by KESRI under Grant R-2005-7-068, which is funded by the Ministry of Commerce, Industry, and Energy.

References

[1] J. L. Coulomb and G. Meunier, *IEEE Trans. Magn.*, vol. MAG-20, no. 5, pp. 1894–1896, Sep. 1984.
[2] S. McFee, J. P. Webb, and D. A. Lowther, *IEEE Trans. Magn.*, vol. 24, no. 1, pp. 439–442, Jan. 1988.
[3] S.-H. Lee *et al., IEEE Trans. Appl. Supercond.*, vol. 14, no. 2, pp. 682–685, Jun. 2004.
[4] Z. Ren, *IEEE Trans. Magn.*, vol. 30, no. 5, pp. 3471–3474, Sep. 1994.
[5] L. H. de Medeiros, G. Reyne, G. Meunier, and J. P. Yonnet, *IEEE Trans. Magn.*, vol. 34, no. 5, pp. 3012–3015, Sep. 1998.
[6] L. H. de Medeiros, G. Reyne, and G. Meunier, *IEEE Trans. Magn.*, vol. 35, no. 3, pp. 1215–1218, May 1999.
[7] F. Henrotte, H. V. Sande, G. Deliege, and K. Hameyer, *IEEE Trans. Magn.*, vol. 40, no. 2, pp. 553–556, Mar. 2004.
[8] F. Henrotte, *ICS Newslett.*, vol. 11, pp. 3–9, 2004.
[9] D.-H. Kim, D. A. Lowther, and J. K. Sykulski, *IEEE Trans. Magn.*, vol. 41, no. 2, pp. 1404–1407, May 2005.
[10] E. J. Haug, K. K. Choi, and V. Komkov, *Design Sensitivity Analysis of Structural Systems*. Orlando, FL: Academic, 1986.
[11] S. H. Crandall *et al., Dynamics of Mechanical and Electromechanical Systems*. New York: McGraw-Hill, 1968.

Manuscript received April 24, 2006 (e-mail: dh29kim@ee.knu.ac.kr).