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Controllable and Uncontrollable Poles and Zeros of nD Systems*

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Abstract. We use the behavioural approach to define and characterize controllable and uncontrollable poles and zeros of multidimensional (nD) linear systems. We show a strong relationship between controllable poles and zeros and properties of the transfer function matrix, and we give characterizations of uncontrollable poles and zeros, in particular demonstrating that these have an input decoupling property.

Key words. Multidimensional systems, Behavioural approach, Poles, Zeros, Input decoupling zeros.

1. Introduction

The behavioural approach due to Willems [W1]–[W3] aims to reinterpret systems theory by emphasizing and formalizing the role of the system trajectories. This approach has had particular impact in the field of multidimensional (nD) systems, which are systems defined by sets of partial differential or multidimensional difference equations. The use of behaviours has allowed previously obscure relationships between nD systems concepts to emerge.

Any complete theory of systems should contain an analysis of pole/zero structure. In the case of a one-dimensional (1D) linear system, the pole/zero structure provides fundamental information on system structure and control. For example, the presence of a right half-plane zero will cause difficulties with feedback. In the nD context also we expect the location of zeros to prove significant for control purposes, and thus the development of the underlying theory of zeros becomes an important fundamental task. Furthermore, it is known that the poles of an nD system correspond to certain oscillating/exponential trajectories. Such trajectories provide important general structural information. In a system defined by linear PDEs with constant coefficients they in fact characterize the system as a whole [O2].

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Poles of nD systems were introduced via the behavioural approach in [WORO]. In that paper definitions of controllable, uncontrollable, observable and unobservable poles of such systems were provided, and these were characterized dynamically and in terms of system representations. The definitions in [WORO] agree with the classical definitions in the 1D case. They also have module-theoretic interpretations, which agree for 1D systems with the definitions proposed by Bourlès and Fliess [BF] and in the nD case with those suggested by Pommaret and Quadrat [PQ].

In the current work we extend the study in [WORO] to include a new theory of zeros. The theory of system zeros is more complex than the theory of poles, since the system inputs may still contain free variables ("completely unobservable inputs") when the outputs vanish. A zero is therefore defined as a frequency which may arise when not only the outputs but also any combination of completely unobservable inputs vanish.

The layout of the paper is as follows. In Section 3 we re-introduce the characteristic variety, which is fundamental to any multidimensional poles/zeros work. In this section we derive new properties of the characteristic points of a factor $\mathscr{B}/\mathscr{B}'$ of behaviours, which will prove useful in what follows. Then in Section 4 of this paper we discuss controllable and uncontrollable poles. In particular, the new Theorem 4.2 shows that an uncontrollable pole is precisely a frequency which can appear in some observed function of the system (a linear combination of the system variables and their derivatives), independently of the value of the input. This is a natural generalization of the input decoupling property to any system given by linear PDEs with constant coefficients.

In Section 5 we introduce the theory of zeros, in particular showing that a zero is precisely a rank-loss point of the operator describing the zero output behaviour. We also define controllable zeros and uncontrollable zeros. Controllable zeros are the zeros of the transfer matrix, and uncontrollable zeros are a special class of uncontrollable poles.

Our work takes a quite different approach to zeros to that in the classical theory of distributed parameter systems (e.g. [CCD] and [CZ]). In particular, here we study systems described in a completely general form by linear PDEs with constant coefficients. The inputs and outputs to such systems are n-dimensional signals, and their exponential characteristics are therefore described by points in $\mathbb{C}^n$. In a physical setting we might say that both temporal and spatial exponential behaviour is captured in our approach; however we make no such distinction of independent variables in this work.

2. Background

In this section we cover the necessary background on 1D/nD behavioural theory. Recall that the behaviour of a system is the set of all possible system trajectories [W1], [W2]. Formally, we define a **system** to be a triple $(\mathscr{A}, q, \mathscr{B})$, where $\mathscr{A}$ is a set, $q \in \mathbb{Z}^+$ and $\mathscr{B} \subseteq \mathscr{A}^q$ is a subset of **trajectories**, called the system **behaviour**.

In practice, we assume $\mathscr{A}$ to be a vector space over a field k which is either $\mathbb{R}$ or $\mathbb{C}$; $\mathscr{A}$ is therefore called the **signal space**. Throughout this paper, in the continuous case we take $\mathscr{A}$ to be either $\mathscr{C}^{\infty}(\mathbb{R}^n, k)$, the set of all k-valued smooth functions on $\mathbb{R}^n$, or else $\mathscr{D}'(\mathbb{R}^n, k)$, the set of all k-valued distributions (continuous k-linear maps from k-valued compactly supported smooth functions to k) on $\mathbb{R}^n$. In the discrete case, we always take $\mathscr{A}$ to be either $k^{\mathbb{N}^n}$ or $k^{\mathbb{Z}^n}$. These particular continuous and discrete signal spaces have certain algebraic properties [O1] which are crucial for our purposes.

Throughout the paper we consider behaviours specified by sets of linear differential equations (or difference equations) with constant coefficients. Thus the notation $\mathscr{B} \subseteq \mathscr{A}^q$ implicitly assumes that the behaviour $\mathscr{B}$ is of this type. Accordingly, let $\mathscr{D}$ denote the polynomial ring $k[z_1, \ldots, z_n]$, and let $R(z_1, \ldots, z_n) \in \mathscr{D}^{g,q}$ be a $g \times q$ polynomial matrix.[1] Then the **differential behaviour** defined by R is given by

$$\mathscr{B} = \ker_{\mathscr{A}} R := \left\{ w \in \mathscr{A}^q \,\middle|\, R\left(\frac{\partial}{\partial t_1}, \ldots, \frac{\partial}{\partial t_n}\right) w = 0 \right\}, \tag{1}$$

and R is said to be a **kernel representation** of this behaviour $\mathscr{B}$. **Difference behaviours** are the discrete equivalent and are defined analogously, using the backward shift operator σ_i, defined by

$$(\sigma_i w)(t_1, \ldots, t_n) := w(t_1, \ldots, t_i + 1, \ldots, t_n), \tag{2}$$

instead of the partial derivative $\partial/\partial t_i$. In either case, we write $\mathscr{B} = \ker_{\mathscr{A}} R$, the meaning being implicitly given by the choice of signal space $\mathscr{A}$. We also drop the operator notation and write $Rw = R(z_1, \ldots, z_n)w$ for a given polynomial matrix R applied to a given trajectory w, where the meaning is given according to $\mathscr{A}$.

For example, the 2D differential behaviour over $\mathscr{A} = \mathscr{C}^{\infty}(\mathbb{R}^2, \mathbb{R})$ with three dependent variables described by the single PDE:

$$\frac{\partial w_1}{\partial t_1}(t_1, t_2) + \frac{\partial w_1}{\partial t_2}(t_1, t_2) - \frac{\partial^2 w_2}{\partial t_1 t_2}(t_1, t_2) - 2\frac{\partial^3 w_3}{\partial t_1^3}(t_1, t_2) + w_3(t_1, t_2) = 0,$$

can be written as

$$\mathscr{B} = \ker_{\mathscr{A}} R, \qquad R = ((z_1 + z_2) \quad (-z_1 z_2) \quad (1 - 2z_1^3)).$$

Given a behaviour $\mathscr{B}$ and sub-behaviour $\mathscr{B}'$, treating them as k-vector spaces it is possible to form the factor space $\mathscr{B}/\mathscr{B}' = \{w + \mathscr{B}'; w \in \mathscr{B}\}$. As shown in Theorem 2.56(iii) of [O1], this factor itself admits the structure of a behaviour. This can be seen by choosing a kernel representation $R' \in \mathscr{D}^{g',q}$ for $\mathscr{B}'$. The restriction of this operator R' to $\mathscr{B}$ has kernel $\mathscr{B}'$, and so its image $R'\mathscr{B}$ is isomorphic to $\mathscr{B}/\mathscr{B}'$. The image of any differential operator has a kernel representation (e.g. [O1, Corollary 2.38]), so $\mathscr{B}/\mathscr{B}'$ is in this way a behaviour, and properties such as controllability, autonomy, etc. of $\mathscr{B}/\mathscr{B}'$ are well defined, i.e. independent of R'.

We are particularly interested in the case where R' is a single polynomial row vector x. In this case $R'\mathscr{B} = x\mathscr{B}$ is a single component behaviour, which we can think of as a behaviour describing the possible values of an "observable function"

[1] In the case $\mathscr{A} = k^{\mathbb{Z}^n}$ only, it is necessary to take $\mathscr{D} = k[z_1, \ldots, z_n, z_1^{-1}, \ldots, z_n^{-1}]$ instead.

x measuring a quantity derived from the system variables. Note that $\mathcal{B}/\mathcal{B}'$ is not to be confused with the set-theoretic difference $\mathcal{B}\backslash\mathcal{B}'$. See [W4] for a further discussion of factor behaviours.

We next turn to autonomous behaviours. Recall that a subset of variables $\{w_i \mid i \in \Phi\}$ is said to be a **set of free variables** if the projection $\mathcal{B} \mapsto \mathcal{A}^{\Phi}$, which projects onto the components indexed by Φ, is surjective [R, Definition III.11]. In other words, free variables can take on any combination of values in $\mathcal{A}$. The maximum size of a set of free variables is called the **number of free variables** of $\mathcal{B}$, and is denoted by $m = m(\mathcal{B})$. Given $\mathcal{B} = \ker_{\mathcal{A}} R \subseteq \mathcal{A}^q$, we have [O1, Theorem 2.69]

$$m(\mathcal{B}) = q - \text{rank } R, \tag{3}$$

where the rank is defined over the field $k(z_1, \dots, z_n)$.

When $m(\mathcal{B}) = 0$, $\mathcal{B}$ is called **autonomous**. Equivalently, the **annihilator** of $\mathcal{B}$,

$$\text{ann } \mathcal{B} = \{r \in \mathcal{D} \mid rw = 0, \forall w \in \mathcal{B}\} \tag{4}$$

is non-zero. Given a kernel representation R of $\mathcal{B}$, another equivalent condition is that R has full column rank [FRZ], [WRO2]. An example of an autonomous behaviour is given by any behaviour of the form

$$\mathcal{B}_{0,y} = \{(u, y) \in \mathcal{B} \mid u = 0\}, \tag{5}$$

where u is a maximal set of free variables. Such a partitioning of variables is called a **(free) input/output structure** on $\mathcal{B}$, and we write $\mathcal{B} = \mathcal{B}_{u,y}$. The sub-behaviour (5) is called the **zero-input behaviour**. Equivalently, we can consider a partitioning $R = (-Q \;\; P)$ of any kernel representation R of $\mathcal{B}$, where the columns of Q correspond to the input variables u, and the columns of P to the output variables y; since $\mathcal{B}_{0,y}$ is autonomous, P has full column rank. Note that the number of inputs is necessarily equal to $m(\mathcal{B})$, and this number is in particular independent of the input/output structure.

For a given free input/output structure, any behaviour $\mathcal{B}$ has a unique **transfer (function) matrix** $G \in k(z_1, \dots, z_n)^{p,m}$ characterized by the equation $PG = Q$; see Theorem 2.69 of [O1] and also p. 75 of [R] and Section VIII of [W2] for the 2D/1D cases.

Finally, we recall the definition of controllability for continuous nD behaviours [PS]. For brevity we refer for the discrete definition to [R], [RW], [WRO2], and [WZ]. A differential behaviour $\mathcal{B}$ is **controllable** if, for any two open sets $T_1, T_2 \subseteq \mathbb{R}^n$ with disjoint closures, and any pair of trajectories $w^{(1)}, w^{(2)} \in \mathcal{B}$,

$$\exists w \in \mathcal{B} \qquad \text{with} \quad w|_{T_1} = w^{(1)}|_{T_1} \quad \text{and} \quad w|_{T_2} = w^{(2)}|_{T_2}, \tag{6}$$

where $w'|_T$ denotes the restriction of w' to a set T. Controllability has many interesting characterizations due to many authors; see Theorem 3.8 of [WORO] for a partial list.

The **controllable part** of a behaviour $\mathcal{B}$ is uniquely defined as the controllable sub-behaviour $\mathcal{B}^{\text{c}}$ of $\mathcal{B}$ satisfying $\mathcal{B} = \mathcal{B}^{\text{c}} + \mathcal{B}^{\text{a}}$ for some autonomous $\mathcal{B}^{\text{a}}$ [FRZ], [WRO2], [Z]. The controllable part can be shown to be the (unique) maximal controllable sub-behaviour of $\mathcal{B}$. Also, it is the (unique) minimal sub-behaviour possessing the same transfer matrix as $\mathscr{B}$, and in particular it has the same input/output structures as $\mathscr{B}$ [O1, Theorem 7.21], [R, Lemma IV.14], [WRO2, Corollary 6]. The controllable part can be algorithmically constructed by computing syzygy modules [O1, Theorem 7.24], [PQ, Section 4.1], [WRO2, Corollary 6], [Z, Lemma 4].

Since $\mathscr{B}$ and $\mathscr{B}^c$ have the same input/output structure, they also have the same number of free variables, from which it follows by additivity of $m(\cdot)$ [WORO, Section 3.1] that the factor $\mathscr{B}/\mathscr{B}^c$ is an autonomous behaviour. As we will see, this behaviour describes input decoupling properties of $\mathscr{B}$.

3. Characteristic Points

The interpretation of poles and zeros is in terms of trajectories of a certain structure, which we now recall. Such trajectories can be defined for all signal spaces which we consider in this paper (and the following theory holds in all cases), but for brevity we provide the definitions here only for $\mathscr{A} = \mathbb{C}^{\mathbb{Z}^n}$, $\mathscr{A} = \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{C})$ and $\mathscr{A} = \mathscr{D}'(\mathbb{R}^n, \mathbb{C})$. The other cases are covered in [O2] and also summarized in Theorems 4.1 and 4.2 of [WORO].

Definition 3.1. Let $w(t_1, \dots, t_n) \in \mathscr{A}^q$. Then w is said to be an **exponential trajectory of frequency** $(a_1, \dots, a_n) \in \mathbb{C}^n$ if it is of the form

$$w(t_1, \dots, t_n) = \begin{cases} v_0 a_1^{t_1} \cdots a_n^{t_n}, & \mathscr{A} = \mathbb{C}^{\mathbb{Z}^n}, \\ v_0 \exp(a_1 t_1 + \cdots + a_n t_n), & \mathscr{A} = \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{C}) \\ & \text{or} \quad \mathscr{A} = \mathscr{D}'(\mathbb{R}^n, \mathbb{C}), \end{cases} \tag{7}$$

where $v_0 \in \mathbb{C}^q$. Also, w is said to be a **polynomial exponential trajectory of pure frequency** $(a_1, \dots, a_n)$ if it is of the form

$$w(t_1, \dots, t_n) = \begin{cases} p(t_1, \dots, t_n) a_1^{t_1} \cdots a_n^{t_n}, & \mathscr{A} = \mathbb{C}^{\mathbb{Z}^n}, \\ p(t_1, \dots, t_n) \exp(a_1 t_1 + \cdots + a_n t_n), & \mathscr{A} = \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{C}) \\ & \text{or} \quad \mathscr{A} = \mathscr{D}'(\mathbb{R}^n, \mathbb{C}), \end{cases} \tag{8}$$

where $p(t_1, \dots, t_n) \in \mathbb{C}[t_1, \dots, t_n]^q$.

The sub-behaviour of a given behaviour $\mathscr{B}$ generated linearly by the polynomial exponential trajectories of all pure frequencies determines $\mathscr{B}$ uniquely (see Section 4 of [WORO], referring to results in [O2]). The set of all frequencies of exponentials or polynomial exponentials in a given behaviour is not in general a finite set, as in the 1D case, but has the structure of an algebraic variety.

Given an ideal $I \subseteq \mathscr{D}$, we can define the variety $V(I)$ of all points $a \in \mathbb{C}^n$ such that $p(a) = 0$ for all $p \in I$. Note that in the case $\mathscr{A} = k^{\mathbb{Z}^n}$, $\mathscr{D}$ is different and it is necessary to consider points in $(\mathbb{C}\backslash 0)^n$ only, throughout the paper. Note also that the variety is always defined in complex space even when the field k is real.

This enables the theory to cover real sinusoidal trajectories [WORO, Section 4.1]. We now recall the definition and characterization of the characteristic variety [WORO], which is from the theory of PDEs [B], [P]. The equivalences in the following theorem hold for all $\mathscr{A}$ in question.

Theorem 3.2. *The* **characteristic variety** *of a behaviour* $\mathscr{B} = \ker_{\mathscr{A}} R$ *is the set* $\mathscr{V}(\mathscr{B})$ *of all points* $(a_1, \dots, a_n) \in \mathbb{C}^n$ *such that the following equivalent conditions hold*:

1. $(a_1, \dots, a_n) \in V(\operatorname{ann} \mathscr{B})$.
2. $R(a_1, \dots, a_n)$ *has less than full column rank*.
3. $\mathscr{B}$ *contains a non-zero exponential trajectory of frequency* $(a_1, \dots, a_n)$.

The points in $\mathscr{V}(\mathscr{B})$ *are called the* **characteristic points** *of* $\mathscr{B}$.

If there is a non-zero polynomial exponential trajectory of pure frequency $(a_1, \dots, a_n)$ in $\mathscr{B}$, then by application of an appropriate scalar differential or shift operator there is also a non-zero exponential trajectory of the same frequency in $\mathscr{B}$. Thus Theorem 3.2 also characterizes the frequencies of polynomial exponentials in a given behaviour.

From condition 2 of Theorem 3.2 it is clear that a non-autonomous behaviour is precisely a behaviour which contains a non-zero exponential trajectory of every frequency. At the other extreme, only the zero behaviour has an empty characteristic variety.

The ideal ann $\mathscr{B}$ which describes the characteristic variety can be constructed by means of Gröbner bases as described for example in [WRO1].

For any behaviour $\mathscr{B}$ with sub-behaviour $\mathscr{B}'$ we find [WORO, Lemma 4.7]

$$\mathscr{V}(\mathscr{B}) = \mathscr{V}(\mathscr{B}') \cup \mathscr{V}(\mathscr{B}/\mathscr{B}'). \tag{9}$$

The next result, which is new, deals with the characterization of the characteristic variety of such a factor behaviour $\mathscr{B}/\mathscr{B}'$:

Lemma 3.3. *Let* $\mathscr{B}' \subseteq \mathscr{B}$ *be behaviours and* $(a_1, \dots, a_n) \in \mathbb{C}^n$. *Then the following are equivalent*:

1. $(a_1, \dots, a_n)$ *is a characteristic point of* $\mathscr{B}/\mathscr{B}'$.
2. *There exists a polynomial vector* x *such that* $xw' = 0$ *for all* $w' \in \mathscr{B}'$ *but* xw *equals a non-zero exponential trajectory of frequency* $(a_1, \dots, a_n)$ *for some* $w \in \mathscr{B}$.
3. *There exists a polynomial exponential trajectory of pure frequency* $(a_1, \dots, a_n)$ *in* $\mathscr{B} \backslash \mathscr{B}'$.

Proof. The equivalence of conditions 1 and 2 is a direct generalization of Corollary 8 of [W4]. Now suppose that condition 2 holds, so that for some x with $xw = 0$ for all $w \in \mathscr{B}'$, there exists a non-zero exponential $r \in x\mathscr{B}$ of frequency $(a_1, \dots, a_n)$. Write I for the ideal of all polynomials vanishing at $(a_1, \dots, a_n)$, and consider the signal space $\mathscr{A}_1$ of all elements of $\mathscr{A}$ annihilated by some power of

I. We have that $\mathscr{A}_1$ is an injective module [O2, Theorem 1.14], and therefore the exact sequence

$$\mathscr{B} \xrightarrow{x} x\mathscr{B} \to 0$$

restricts to an exact sequence

$$\mathscr{B} \cap \mathscr{A}_1^q \xrightarrow{x} x\mathscr{B} \cap \mathscr{A}_1 \to 0.$$

Now the trajectory r is in $x\mathscr{B} \cap \mathscr{A}_1$, and must therefore be the image under x of some element $w^{(1)}$ of $\mathscr{B} \cap \mathscr{A}_1^q$. Furthermore, $w^{(1)} \neq 0$ as $r \neq 0$, and $w^{(1)}$ is annihilated by some power of I. It is shown in [O2] (see also Theorem 4.2 of [WORO]) that such trajectories are precisely the polynomial exponential trajectories of pure frequency $(a_1, \ldots, a_n)$. However, $w^{(1)}$ cannot be in $\mathscr{B}'$ because $x\mathscr{B}' = 0$, so $\mathscr{B} \backslash \mathscr{B}'$ contains a polynomial exponential trajectory of pure frequency $(a_1, \ldots, a_n)$.

Finally, if such a $w \in \mathscr{B} \backslash \mathscr{B}'$ exists, then there is some system equation x of $\mathscr{B}'$ such that $xw \neq 0$. By the nature of the shift or derivative operators, xw must also be a (non-zero) polynomial exponential trajectory of pure frequency $(a_1, \ldots, a_n)$, so $(a_1, \ldots, a_n)$ must be a characteristic point of $x\mathscr{B}$. Condition 2 now follows on applying Theorem 3.2. ■

We can construct the ideal $\mathrm{ann}(\mathscr{B}/\mathscr{B}')$ by first constructing a representation of the factor $\mathscr{B}/\mathscr{B}'$, as described in [W4], and then computing the annihilator.

Example 3.4. Consider $\mathscr{B} = \mathscr{A} = \mathscr{C}^\infty(\mathbb{R}, \mathbb{C})$ and $\mathscr{B}' = \ker_{\mathscr{A}}(z_1)$, the sub-behaviour of all constant trajectories. Now $\mathscr{B}/\mathscr{B}'$ is naturally isomorphic to $z_1\mathscr{B}$, which is the behaviour of all $\mathscr{C}^\infty$ trajectories with $\mathscr{C}^\infty$ integrals. This includes some (all) non-zero constant functions, so $\mathscr{B}/\mathscr{B}'$ has zero as a characteristic point. Taking $x = z_1$ confirms condition 2 in Lemma 3.3; x vanishes on $\mathscr{B}'$ but $x\mathscr{B}$ contains non-zero constant functions. Condition 3 of Lemma 3.3 is also easily confirmed; $\mathscr{B} \backslash \mathscr{B}'$ contains polynomial functions, and these are the polynomial exponential trajectories of pure frequency zero. However $\mathscr{B} \backslash \mathscr{B}'$ does not contain any exponential trajectories of frequency zero, so condition 3 of Lemma 3.3 cannot be strengthened in this way.

Lemma 3.3 can be applied to various classes of poles and zeros; in the next section we use it to demonstrate that uncontrollable poles are a generalization of input decoupling zeros.

4. Controllable and Uncontrollable Poles

Poles are frequencies which can occur in the output when the input is zero, and are therefore given by the characteristic variety of the zero-input behaviour.

Definition 4.1 [WORO].

1. The **pole variety** and **pole points** of a behaviour $\mathscr{B}$ with a given free input/output structure are the characteristic variety and characteristic points of $\mathscr{B}_{0,y}$.

2. The **controllable pole variety** and **controllable pole points** are the characteristic variety and characteristic points of $(\mathcal{B}^c)_{0,y} = \mathcal{B}_{0,y} \cap \mathcal{B}^c$.
3. The **uncontrollable pole variety** and **uncontrollable pole points** are the characteristic variety and characteristic points of $\mathcal{B}/\mathcal{B}^c$.

It follows from Theorem 3.2 that, for a system with free input/output structure given by the equation $Qu = Py$, the pole points are the points where P has less than full column rank. We also recall from Corollary 5.4 and Theorem 5.8 of [WORO] that the union of the controllable pole variety and the uncontrollable pole variety is the pole variety, and also that the controllable pole points are precisely the poles of the transfer matrix.

Uncontrollable pole points are also interesting; to begin with, a behaviour is controllable precisely when it has no uncontrollable pole points. Furthermore, for a behaviour given by a 1D state space model, the uncontrollable pole points are precisely the input decoupling zeros [WORO, Lemma 6.5]. The following new result shows that this interpretation can be generalized to an arbitrary nD behaviour.

Theorem 4.2. *The following are equivalent for any behaviour* $\mathcal{B}$ *and point* $(a_1, \ldots, a_n) \in \mathbb{C}^n$:

1. $(a_1, \ldots, a_n)$ *is an uncontrollable pole point.*
2. *There exists a polynomial exponential trajectory of pure frequency* $(a_1, \ldots, a_n)$ *in* $\mathcal{B} \backslash \mathcal{B}^c$.
3. $(a_1, \ldots, a_n)$ *is a characteristic point of some behaviour of the form* $x\mathcal{B}$, *where also* $x\mathcal{B} \neq \mathcal{A}$.
4. *There exists a polynomial vector* x *and a non-zero exponential trajectory* $r \in \mathcal{A}$ *of frequency* $(a_1, \ldots, a_n)$ *such that, for any input* u, *there exists an output* y *with*
$$\begin{pmatrix} u \\ y \end{pmatrix} \in \mathcal{B} \quad \text{and} \quad x\begin{pmatrix} u \\ y \end{pmatrix} = r. \tag{10}$$

Proof. Equivalence of conditions 1 and 2 is immediate from Lemma 3.3. We now prove equivalence of conditions 1 and 3. For this purpose we need to recall the module $\mathcal{B}^\perp$ of system equations defined by

$$\mathcal{B}^\perp = \{v \in \mathcal{D}^{1,q} \mid vw = 0 \text{ for all } w \in \mathcal{A}^q\},$$

and the corresponding factor $M = \mathcal{D}^{1,q}/\mathcal{B}^\perp$ (e.g. [O] and [W4]). Suppose now that $(a_1, \ldots, a_n) \in \mathcal{V}(x\mathcal{B})$, where $x\mathcal{B} \neq \mathcal{A}$. Then $x + \mathcal{B}^\perp$ is in the torsion submodule $t(M)$ of M [W4, Lemma 3], and also the module element $x + \mathcal{B}^\perp$ and the behaviour $x\mathcal{B}$ have the same annihilator [W4, Lemma 4]. Hence

$$(a_1, \ldots, a_n) \in V(\operatorname{ann}(x + \mathcal{B}^\perp)) \subseteq V(\operatorname{ann} t(M)). \tag{11}$$

However, $\mathcal{B}/\mathcal{B}^c$ is the dual of $t(M)$ under the correspondence given by Oberst [O1, Theorem 7.21], and therefore $\operatorname{ann} t(M) = \operatorname{ann} \mathcal{B}/\mathcal{B}^c$ [W4, Lemma 4]. So (11) tells us that $(a_1, \ldots, a_n)$ is an uncontrollable pole point.

Conversely, suppose that $\underline{a} = (a_1, \ldots, a_n) \in V(\operatorname{ann} \mathscr{B}/\mathscr{B}^c) = V(\operatorname{ann} t(M))$. Thus $\operatorname{ann} t(M) \subseteq I(\underline{a})$, the ideal of all polynomials in $\mathscr{D}$ vanishing at $\underline{a}$. Let P be a minimal prime divisor of $\operatorname{ann} t(M)$ contained in $I(\underline{a})$. Then P is an associated prime of $t(M)$, and so annihilates an element $x + \mathscr{B}^\perp$ of $t(M) \subseteq M$. Now $\operatorname{ann}(x + \mathscr{B}^\perp) \subseteq I(\underline{a})$, so $\underline{a} \in V(\operatorname{ann}(x + \mathscr{B}^\perp))$. However, $V(\operatorname{ann}(x + \mathscr{B}^\perp)) = V(\operatorname{ann} x\mathscr{B})$, so $\underline{a}$ is a characteristic point of $x\mathscr{B}$. Since $x + \mathscr{B}^\perp$ is a torsion element of M, $x\mathscr{B} \neq \mathscr{A}$ [W4, Lemma 3]. This establishes equivalence of conditions 1 and 3.

We now show equivalence of conditions 3 and 4. So suppose that $\underline{a} \in \mathscr{V}(x\mathscr{B})$ and $x\mathscr{B} \neq A$. Then $x + \mathscr{B}^\perp$ is a torsion element of M, and so vanishes on $\mathscr{B}^c$ [W4, Corollary 2]. Hence $\mathscr{B}^c \subseteq \ker_{\mathscr{A}}(x) \cap \mathscr{B}$.

Now let $(u_r, y_r) \in \mathscr{B}$ be such that $r = x(u_r, y_r)$, and choose an arbitrary input u. Since $\mathscr{B}^c$ has the same transfer matrix as $\mathscr{B}$, it in particular shares the same free input/output structures. Hence there must exist a y^* with $(u - u_r, y^*) \in \mathscr{B}^c$, and so in particular $x(u - u_r, y^*) = 0$. Now

$$\begin{pmatrix} u \\ y^* + y_r \end{pmatrix} = \begin{pmatrix} u - u_r \\ y^* \end{pmatrix} + \begin{pmatrix} u_r \\ y_r \end{pmatrix} \in \mathscr{B},$$

and applying the operator x to this trajectory we get r as required.

Conversely, suppose that condition 4 holds, so that x and r have the required properties. Define $\mathscr{B}' = \mathscr{B} \cap \ker_{\mathscr{A}}(x)$, and let $(0, y^*) \in \mathscr{B}$ be such that $x(0, y^*) = r$; it exists by the supposition. Now let u be an arbitrary input. Then there exists y with $(u, y) \in \mathscr{B}$ and $x(u, y) = r$, and so

$$x\begin{pmatrix} u \\ y - y^* \end{pmatrix} = x\begin{pmatrix} u \\ y \end{pmatrix} - x\begin{pmatrix} 0 \\ y^* \end{pmatrix} = 0$$

and also $(u, y - y^*)$ is in $\mathscr{B}$ by the same decomposition. Thus $(u, y - y^*) \in \mathscr{B}'$. Since this holds for any u, $\mathscr{B}'$ has the same number of free variables as $\mathscr{B}$. However, by Theorem 7 of [W4] $\mathscr{B}^c$ is the unique minimal sub-behaviour of $\mathscr{B}$ with the same number of free variables as $\mathscr{B}$, which proves that $\mathscr{B}^c \subseteq \mathscr{B}'$. Therefore x vanishes on $\mathscr{B}^c$, so $x + \mathscr{B}^\perp \in t(M)$ [W4, Corollary 2], and so $x\mathscr{B} \neq \mathscr{A}$. We already have that $\underline{a} \in \mathscr{V}(x\mathscr{B})$, so this completes the proof. ■

Note condition 2, which states the existence of a polynomial exponential trajectory (of the given pure frequency) not in $\mathscr{B}^c$. It can be shown, at least in the continuous case, that only trajectories $w^{(1)}$ in $\mathscr{B}^c$ are concatenable with $w^{(2)} = 0$ in the sense of (6) (for arbitrary open sets T_1 and T_2 with disjoint closures); see the proof of Theorem 3.9 in [PS] for this argument. So uncontrollable poles are frequencies which correspond to polynomial exponentials that cannot be controlled to zero.

From condition 3 of the theorem, if the behaviour $x\mathscr{B}$ of some observable function x includes a non-zero exponential trajectory with frequency not an uncontrollable pole point, then this behaviour must be equal to $\mathscr{A}$.

The last condition of Theorem 4.2 is perhaps the most interesting, particularly since the observed trajectory r can be fixed independently of the system input. In the continuous case, we can paraphrase this condition as follows: there is some

k-linear combination of the inputs, outputs and their derivatives which can take on an exponential value (of the given frequency), and which furthermore can take this value independently of the values of the inputs. In particular, for a state-space model, the given k-linear combination can be expressed entirely as a combination of the inputs, their derivatives, and the states. This is a clear generalization of input decoupling.

Example 4.3. Take the behaviour $\mathscr{B}$ over the signal space $\mathscr{A} = \mathscr{C}^\infty(\mathbb{R}^3, \mathbb{R})$ defined by

$$\mathscr{B} = \{w = (u, y) \mid Qu = Py\},$$

where

$$Q = \begin{pmatrix} -z_1 z_3 & z_1 z_2 & z_2^2 + z_3^2 \\ -z_1 & 0 & z_3 \\ -z_2 & z_1 - z_3 & z_2 \end{pmatrix}, \qquad P = \begin{pmatrix} z_2 & z_3 & z_3 \\ 0 & 1 & 1 \\ 1 & z_1 & 1 \end{pmatrix}.$$

The controllable part turns out to be

$$\mathscr{B}^c = \{w = (u, y) \mid Q^c u = P^c y\},$$

$$Q^c = \begin{pmatrix} 0 & z_1 & z_2 \\ -z_1 & 0 & z_3 \\ -z_2 & -z_3 & 0 \end{pmatrix}, \qquad P^c = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & z_1 & 1 \end{pmatrix},$$

and we have a matrix L with $Q = LQ^c$, $P = LP^c$ given by

$$L = \begin{pmatrix} z_2 & z_3 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Since $(-Q^c \;\; P^c)$ has full row rank, it is not hard to show that L is a kernel representation of $(-Q^c \;\; P^c)\mathscr{B} \cong \mathscr{B}/\mathscr{B}^c$. Hence by Theorem 3.2 the uncontrollable pole points are the places where the determinant of L vanishes. Hence

$$\text{uncontrollable pole points} = \{(\alpha, 0, \gamma) \mid \alpha, \gamma \in \mathbb{C}\}.$$

Now consider the polynomial vector and non-zero trajectory

$$x = (0 \;\; -z_1 \;\; -z_2 \mid 1 \;\; 0 \;\; 0), \qquad r = \begin{cases} (\alpha - 1)\exp(\alpha t_1 + \gamma t_3), & \alpha \neq 1, \\ \exp(\alpha t_1 + \gamma t_3), & \alpha = 1. \end{cases}$$

We claim that r can occur as $r = x(u, y)$ for any u. For example, consider the input

$$u = \begin{pmatrix} t_1 e^{2t_3} \\ 0 \\ -e^{t_1 - t_2} \end{pmatrix}.$$

Extending it to the trajectory

$$(u, y) = \begin{pmatrix} t_1 e^{2t_3} \\ 0 \\ -e^{t_1 - t_2} \\ \hline e^{t_1 - t_2} + (\alpha - 1) e^{\alpha t_1 + \gamma t_3} \\ -e^{2t_3} - e^{\alpha t_1 + \gamma t_3} \\ e^{\alpha t_1 + \gamma t_3} \end{pmatrix}, \quad \alpha \neq 1,$$

$$\text{or} \qquad \begin{pmatrix} t_1 e^{2t_3} \\ 0 \\ -e^{t_1 - t_2} \\ \hline e^{t_1 - t_2} + e^{\alpha t_1 + \gamma t_3} \\ -e^{2t_3} - t_1 e^{\alpha t_1 + \gamma t_3} \\ t_1 e^{\alpha t_1 + \gamma t_3} \end{pmatrix}, \quad \alpha = 1,$$

we find that $(u, y) \in \mathscr{B}$ and $x(u, y) = r$, verifying condition 4 of the theorem.

An example of a polynomial exponential trajectory of pure frequency $(\alpha, 0, \gamma)$ in $\mathscr{B} \backslash \mathscr{B}^{\mathrm{c}}$ is given by

$$\begin{pmatrix} u \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \hline (\alpha - 1) e^{\alpha t_1 + \gamma t_3} \\ -e^{\alpha t_1 + \gamma t_3} \\ e^{\alpha t_1 + \gamma t_3} \end{pmatrix}, \quad \alpha \neq 1, \qquad \text{or} \qquad \begin{pmatrix} 0 \\ 0 \\ 0 \\ \hline e^{\alpha t_1 + \gamma t_3} \\ -t_1 e^{\alpha t_1 + \gamma t_3} \\ t_1 e^{\alpha t_1 + \gamma t_3} \end{pmatrix}, \quad \alpha = 1.$$

For $\alpha = 1$ it is not hard to see that there is no exponential trajectory of frequency $(\alpha, 0, \gamma)$ in $\mathscr{B} \backslash \mathscr{B}^{\mathrm{c}}$, although this is an uncontrollable pole point.

5. Controllable and Uncontrollable Zeros

Roughly speaking, zeros are frequencies which occur in the input when the output vanishes (see, e.g. [DS]). However, in contrast to $\mathscr{B}_{0,y}$, the behaviour

$$\mathscr{B}_{u,0} := \{(u, y) \in \mathscr{B} \mid y = 0\} \tag{12}$$

may not be autonomous, in which case defining the zero variety to be the characteristic variety of $\mathscr{B}_{u,0}$ would be inappropriate, as it would lead to the conclusion that for such a behaviour all frequencies are zeros. Instead, zeros can be defined as rank-loss points of the corresponding representation matrices, and can be characterized by the property that not only are the outputs zero, but also a certain number of the inputs, the rest being exponential of the given frequency. The definitions and results in this section are new.

Consider the free variables of $\mathscr{B}_{u,0}$. A set of such free variables could be called **completely unobservable inputs**, since we can deduce absolutely no information about their values by looking at the outputs. Let $m'(\mathscr{B})$ denote the number of free variables of $\mathscr{B}_{u,0}$, i.e. the number of completely unobservable inputs. Note that this is dependent on the given free input/output structure on $\mathscr{B}$. For a given kernel representation $Qu = Py$ of $\mathscr{B}$, we clearly have that $m'(\mathscr{B}) = m(\mathscr{B}) - \operatorname{rank} Q$. Also, for any subset Γ of $\{1, \ldots, m(\mathscr{B})\}$, let $\mathscr{B}_{u(\Gamma),0}$ denote the behaviour obtained by setting the outputs and those inputs u_i with $i \in \Gamma$ to 0. When Γ specifies a complete set of completely unobservable inputs of $\mathscr{B}_{u,0}$, $\mathscr{B}_{u(\Gamma),0}$ is autonomous.

We now provide a definition of zeros in the behavioural approach.

Definition 5.1. Let $\mathscr{B}$ be a behaviour with a given free input/output structure. The **zero variety** of $\mathscr{B}$, denoted $\mathscr{Z}(\mathscr{B}_{u,y})$, is defined by

$$\mathscr{Z}(\mathscr{B}_{u,y}) = \bigcap_{\Gamma \subseteq \{1,\ldots,m\},\, |\Gamma| = m'(\mathscr{B})} \mathscr{V}(\mathscr{B}_{u(\Gamma),0}). \tag{13}$$

The **controllable zero variety** is the zero variety of $\mathscr{B}^{\mathrm{c}}$, and the **uncontrollable zero variety** is the characteristic variety of $\mathscr{B}_{u,0}/(\mathscr{B}^{\mathrm{c}})_{u,0}$, where $(\mathscr{B}^{\mathrm{c}})_{u,0} = \mathscr{B}_{u,0} \cap \mathscr{B}^{\mathrm{c}}$. **(Controllable/uncontrollable) zero points** are defined as the points of the corresponding varieties.

Note that although (13) is an intersection over all sets of inputs of size $m'(\mathscr{B})$, it is only those Γ corresponding to complete sets of completely unobservable inputs which actually contribute (for other Γ, the variety will be all of $\mathbb{C}^n$). In the case where $\mathscr{B}_{u,0}$ is autonomous, $m'(\mathscr{B}) = 0$ and the zero variety is simply the characteristic variety of $\mathscr{B}_{u,0}$. In this special case the zeros theory will mirror the poles theory.

Proposition 5.2. *The following are equivalent for a behaviour* $\mathscr{B} = \ker_{\mathscr{A}}(-Q \;\; P)$ *and* $(a_1, \ldots, a_n) \in \mathbb{C}^n$:

1. $(a_1, \ldots, a_n)$ *is a zero point of* $\mathscr{B}$.
2. *The rank of* $Q(a_1, \ldots, a_n)$ *is less than the rank of* $Q(z_1, \ldots, z_n)$.
3. *For any set of up to* $m'(\mathscr{B})$ *inputs* u_i, $\mathscr{B}_{u,0}$ *contains a non-zero exponential trajectory of frequency* $(a_1, \ldots, a_n)$ *which is zero in the specified components.*

Proof. Write $m = m(\mathscr{B})$, $m' = m'(\mathscr{B})$ throughout, and let $p' = m - m'$ denote the rank of Q. Now the rank loss points of Q are the points where the ideal $I_{p'}(Q)$ generated by all p'th order minors of Q vanishes. For any selection $\{j_1, \ldots, j_{p'}\} \subseteq \{1, \ldots, m\}$ of p' columns of Q, let $I_{(j_1,\ldots,j_{p'})}$ denote the ideal of p'th order minors of the corresponding submatrix of Q. We have

$$I_{p'}(Q) = \sum_{\{j_1,\ldots,j_{p'}\} \subseteq \{1,\ldots,m\}} I_{(j_1,\ldots,j_{p'})}$$

and therefore

$$V(I_{p'}(Q)) = \bigcap_{\{j_1,\dots,j_{p'}\}\subseteq\{1,\dots,m\}} V(I_{(j_1,\dots,j_{p'})}).$$

However, $V(I_{(j_1,\dots,j_{p'})})$ is precisely the characteristic variety of $\mathscr{B}_{u(\Gamma),0}$, where $\Gamma = \{1,\dots,m\}\backslash\{j_1,\dots,j_{p'}\}$. This gives us the equivalence of conditions 1 and 2. Equivalence of conditions 1 and 3 is immediate by application of Theorem 3.2 to (13). ■

So a zero is a frequency which can occur in the input when not only the output but up to $m'(\mathscr{B})$ inputs are set to zero. Note that, in the special case where $\mathscr{B}_{u,0}$ is autonomous, Proposition 5.2 reduces to a direct application of Theorem 3.2. We also see that zeros are a type of rank-loss point; such points are considered from a formal algebraic point of view on pp. 155–158 of [O1]. This characterization of zeros also enables us to compute an ideal representing the zero variety (i.e. an ideal I with $V(I) = \mathscr{Z}(\mathscr{B}_{u,y})$); we just compute the minors of Q of order m'. Controllable zero points can be computed by first computing the controllable part and then computing the zeros; uncontrollable zero points can be computed as the characteristic variety of a factor behaviour.

Example 5.3. Consider again the behaviour given in Example 4.3:

$$\mathscr{B} = \{w = (u, y) \mid Qu = Py\},$$

where

$$Q = \begin{pmatrix} -z_1z_3 & z_1z_2 & z_2^2+z_3^2 \\ -z_1 & 0 & z_3 \\ -z_2 & z_1-z_3 & z_2 \end{pmatrix}, \qquad P = \begin{pmatrix} z_2 & z_3 & z_3 \\ 0 & 1 & 1 \\ 1 & z_1 & 1 \end{pmatrix}.$$

In this case, Q has rank 2, i.e. the number of completely unobservable inputs is 1 (any one of u_1, u_2 and u_3 will do). The ideal of second-order minors of Q is given by

$$I_2(Q) = (z_1^2z_2, -z_1z_2^2, z_1z_2z_3, -z_1z_2^2 + z_1^2z_3 - z_1z_3^2, z_2^3 - z_1z_2z_3 + z_2z_3^2,$$
$$-z_2^2z_3 + z_1z_3^2 - z_3^3, -z_1^2 + z_1z_3, z_1z_2 - z_2z_3, -z_1z_3 + z_3^2).$$

We observe that for all these polynomials to vanish, we need $z_2 = 0$, and in this case we also deduce $z_1 = z_3$. Thus the set of zero points is

$$\mathscr{Z}(\mathscr{B}_{u,y}) = \{(\alpha, 0, \alpha) \mid \alpha \in \mathbb{C}\}.$$

Note that for any α, the inputs

$$u^{(1)} = \begin{pmatrix} 0 \\ e^{\alpha t_1+\alpha t_3} \\ 0 \end{pmatrix}, \qquad u^{(2)} = \begin{pmatrix} e^{\alpha t_1+\alpha t_3} \\ 0 \\ e^{\alpha t_1+\alpha t_3} \end{pmatrix}$$

(together with corresponding zero outputs) are in $\mathscr{B}_{u,0}$, so any single input variable can be set to zero, while the input trajectory as a whole remains non-zero of frequency $(\alpha, 0, \alpha)$. This confirms condition 3 of Proposition 5.2.

The next result shows that every zero point is either controllable, uncontrollable or both. It also relates the uncontrollable zero points to the uncontrollable pole points of both $\mathscr{B}$ and the output behaviour

$$\mathscr{B}_y = \left\{ y \in \mathscr{A}^p \mid \exists u \text{ with } \begin{pmatrix} u \\ y \end{pmatrix} \in \mathscr{B} \right\}. \tag{14}$$

The set $(\mathscr{B}^{\mathrm{c}})_y$ is defined analogously with respect to $\mathscr{B}^{\mathrm{c}}$.

Theorem 5.4. *The union of the controllable zero variety and the uncontrollable zero variety is the zero variety. The union of the uncontrollable zero variety of $\mathscr{B}$ and the uncontrollable pole variety of $\mathscr{B}_y$ is the uncontrollable pole variety of $\mathscr{B}$. A point $(a_1, \dots, a_n)$ is an uncontrollable zero point if and only if there exists a polynomial exponential trajectory of pure frequency $(a_1, \dots, a_n)$ which is in $\mathscr{B}_{u,0}$ but not in $\mathscr{B}^{\mathrm{c}}$.*

Proof. To show the first claim, pick any kernel representation $\mathscr{B} = \ker_{\mathscr{A}}(-Q \;\; P)$. Then the controllable part $\mathscr{B}^{\mathrm{c}}$ is given as the image of any minimal right annihilator C of $(-Q \;\; P)$ [WRO2, Corollary 6], and therefore as the kernel of any minimal left annihilator R' of C. Thus the relations on the columns of $(-Q \;\; P)$ are the same as the relations on the columns of R', so in particular any maximal set of linearly independent columns of Q corresponds to a maximal set of linearly independent columns of the first $m(\mathscr{B})$ columns of R'. In other words, $\mathscr{B}_{u,0}$ and $(\mathscr{B}^{\mathrm{c}})_{u,0}$ have the same free input/output structures. Hence for any $(u_1, u_2, 0) \in \mathscr{B}_{u,0}$, where the variables u_1 correspond to a set of inputs of $\mathscr{B}_{u,0}$, there exists a u_2' with $(u_1, u_2', 0) \in (\mathscr{B}^{\mathrm{c}})_{u,0}$, and we have

$$\mathscr{B}_{u,0} = (\mathscr{B}^{\mathrm{c}})_{u,0} + \mathscr{B}_{u(\Gamma),0}$$

for any subset Γ of $\{1, \dots, m\}$ specifying a set of inputs of $\mathscr{B}_{u,0}$. Applying a standard isomorphism theorem, we now obtain

$$\frac{\mathscr{B}_{u(\Gamma),0}}{(\mathscr{B}^{\mathrm{c}})_{u(\Gamma),0}} = \frac{\mathscr{B}_{u(\Gamma),0}}{\mathscr{B}_{u(\Gamma),0} \cap (\mathscr{B}^{\mathrm{c}})_{u,0}} \cong \frac{(\mathscr{B}^{\mathrm{c}})_{u,0} + \mathscr{B}_{u(\Gamma),0}}{(\mathscr{B}^{\mathrm{c}})_{u,0}} = \frac{\mathscr{B}_{u,0}}{(\mathscr{B}^{\mathrm{c}})_{u,0}} \tag{15}$$

and it now follows from (9) that for any such Γ we have

$$\mathscr{V}(\mathscr{B}_{u(\Gamma),0}) = \mathscr{V}((\mathscr{B}^{\mathrm{c}})_{u(\Gamma),0}) \cup \mathscr{V}(\mathscr{B}_{u,0}/(\mathscr{B}^{\mathrm{c}})_{u,0}). \tag{16}$$

Intersecting over all Γ which correspond to sets of inputs of $\mathscr{B}_{u,0}$ gives us the first desired result.

To show the second claim, observe that $\mathscr{B}_{u,0}/(\mathscr{B}^{\mathrm{c}})_{u,0}$ can be considered as a sub-behaviour of $\mathscr{B}/\mathscr{B}^{\mathrm{c}}$ according to the isomorphism

$$\frac{\mathscr{B}_{u,0}}{(\mathscr{B}^{\mathrm{c}})_{u,0}} = \frac{\mathscr{B}_{u,0}}{\mathscr{B}^{\mathrm{c}} \cap \mathscr{B}_{u,0}} \cong \frac{\mathscr{B}^{\mathrm{c}} + \mathscr{B}_{u,0}}{\mathscr{B}^{\mathrm{c}}} \subseteq \frac{\mathscr{B}}{\mathscr{B}^{\mathrm{c}}}.$$

The corresponding factor is $\mathscr{B}/(\mathscr{B}^{\mathrm{c}} + \mathscr{B}_{u,0})$, which maps to $\mathscr{B}_y/(\mathscr{B}^{\mathrm{c}})_y$ under

$$\varphi\colon \begin{pmatrix} u \\ y \end{pmatrix} + (\mathscr{B}^{\mathrm{c}} + \mathscr{B}_{u,0}) \mapsto y + (\mathscr{B}^{\mathrm{c}})_y.$$

It is easy to show that φ is well defined, surjective and injective, therefore an isomorphism. Now the characteristic variety of $\mathscr{B}_{u,0}/(\mathscr{B}^{\mathrm{c}})_{u,0}$ is the uncontrollable zero variety, and that of $\mathscr{B}/\mathscr{B}^{\mathrm{c}}$ is the uncontrollable pole variety. The behaviour $(\mathscr{B}^{\mathrm{c}})_y$ is equal to the controllable part of $\mathscr{B}^{\mathrm{c}}$ (see Theorem 6.4 of [WORO], which shows that elimination of variables commutes with taking the controllable part), and so $\mathscr{V}(\mathscr{B}_y/(\mathscr{B}^{\mathrm{c}})_y)$ is the uncontrollable pole variety of $\mathscr{B}_y$. The second claim now follows from (9).

The final claim is immediate from Lemma 3.3. ■

The last statement of Theorem 5.4 tells us that an uncontrollable zero point is a frequency which can appear in the input when the output is zero, and which furthermore corresponds to a trajectory outside $\mathscr{B}^{\mathrm{c}}$. As in the case of uncontrollable pole points, in the continuous case this implies that it cannot be controlled to zero.

Example 5.5. We return to the earlier example:

$$\mathscr{B} = \{w = (u, y) \mid Qu = Py\},$$

where

$$Q = \begin{pmatrix} -z_1 z_3 & z_1 z_2 & z_2^2 + z_3^2 \\ -z_1 & 0 & z_3 \\ -z_2 & z_1 - z_3 & z_2 \end{pmatrix}, \qquad P = \begin{pmatrix} z_2 & z_3 & z_3 \\ 0 & 1 & 1 \\ 1 & z_1 & 1 \end{pmatrix}$$

with controllable part

$$\mathscr{B}^{\mathrm{c}} = \{w = (u, y) \mid Q^{\mathrm{c}} u = P^{\mathrm{c}} y\},$$

$$Q^{\mathrm{c}} = \begin{pmatrix} 0 & z_1 & z_2 \\ -z_1 & 0 & z_3 \\ -z_2 & -z_3 & 0 \end{pmatrix}, \qquad P^{\mathrm{c}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & z_1 & 1 \end{pmatrix}.$$

We can construct a kernel representation of $\mathscr{B}_{u,0}/(\mathscr{B}^{\mathrm{c}})_{u,0} \cong Q^{\mathrm{c}}\mathscr{B}_{u,0}$ as follows:

$$Q^{\mathrm{c}}\mathscr{B}_{u,0} = \ker_{\mathscr{A}} \begin{pmatrix} z_2 & z_3 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \\ z_3 & -z_2 & z_1 \end{pmatrix}.$$

This matrix fails to have full column rank at precisely the points

$$\mathscr{V}(\mathscr{B}_{u,0}/(\mathscr{B}^{\mathrm{c}})_{u,0}) = \{(\alpha, 0, \alpha) \mid \alpha \in \mathbb{C}\}.$$

These are the uncontrollable zero points of $\mathscr{B}$; in this case the uncontrollable zero points coincide with the zero points. Note also from Example 4.3 that every uncontrollable zero point is an uncontrollable pole point, as we expect from Theorem 5.4.

Next, for any given $\alpha \in \mathbb{C}$ look at the two trajectories identified in Example 5.3 as being input trajectories of $\mathscr{B}$ which may result in output zero:

$$u^{(1)} = \begin{pmatrix} 0 \\ e^{\alpha t_1 + \alpha t_3} \\ 0 \end{pmatrix}, \qquad u^{(2)} = \begin{pmatrix} e^{\alpha t_1 + \alpha t_3} \\ 0 \\ e^{\alpha t_1 + \alpha t_3} \end{pmatrix}.$$

We can see that $(u^{(2)}, 0)$ is in $\mathscr{B}^c$, but $(u^{(1)}, 0)$ is not. Thus $\mathscr{B}_{u,0} \backslash (\mathscr{B}^c)_{u,0}$ contains a non-zero exponential trajectory of frequency $(\alpha, 0, \alpha)$, as predicted by Theorem 5.4.

Finally, the controllable zero points are the points where Q^c has rank less than its usual rank, i.e. 2. Since z_1^2, z_2^2 and z_3^2 are all order 2 minors of Q^c, the set of rank-loss points is just $\{(0,0,0)\}$. This point happens to be both a controllable and an uncontrollable zero point.

Controllable zero points correspond to trajectories in $(\mathscr{B}^c)_{u,0}$, i.e. to inputs which can be controlled to zero while keeping the output at zero. The final result shows that the controllable zero points are also the zeros of the transfer matrix in a suitable sense.

Lemma 5.6. *Let $\mathscr{B}$ be a behaviour with transfer matrix G, and take $(a_1, \ldots, a_n) \in \mathbb{C}^n$. Then $G(a_1, \ldots, a_n)$ is well defined and has rank less than that of $G(z_1, \ldots, z_n)$ if and only if $(a_1, \ldots, a_n)$ is a controllable zero point but not a controllable pole point.*

Proof. As remarked following Definition 4.1, the controllable pole points are precisely the points $(a_1, \ldots, a_n)$ where $G(a_1, \ldots, a_n)$ is not well defined. Therefore let $(a_1, \ldots, a_n)$ be a point which is not a controllable pole point. It suffices to prove that $(a_1, \ldots, a_n)$ is a controllable zero point if and only if G loses rank at $(a_1, \ldots, a_n)$. Let $(-Q^c \;\; P^c)$ be a kernel representation of $\mathscr{B}^c$; then we know that P^c has full column rank at $(a_1, \ldots, a_n)$. Also, $P^c G = Q^c$ by definition of the transfer matrix. Therefore rank $G =$ rank Q^c and rank $G(a_1, \ldots, a_n) =$ rank $Q^c(a_1, \ldots, a_n)$. Thus G loses rank at the same points as Q^c, which by Proposition 5.2 are precisely the controllable zero points. ■

The controllable zero points are clearly related to the classical transmission zeros; the difference is essentially that there are no states involved in the current analysis. Future work incorporating the states should provide a closer analogy to the transmission zeros.

Example 5.7. Continuing with the same example, the transfer matrix of the behaviour $\mathscr{B}$ is computed as

$$G = P^{-1}Q = \begin{pmatrix} 0 & z_1 & z_2 \\ \dfrac{z_2 - z_1}{1 - z_1} & \dfrac{z_3}{1 - z_1} & \dfrac{z_3}{1 - z_1} \\ \dfrac{z_1^2 - z_2}{1 - z_1} & \dfrac{-z_3}{1 - z_1} & \dfrac{-z_1 z_3}{1 - z_1} \end{pmatrix}.$$

This transfer matrix has rank 2; the column vector $(z_3, -z_2, z_1)$ is in the right-hand kernel. The determinant of the bottom-right 2×2 submatrix of G is $z_3^2/(1 - z_1)$, so z_3 must vanish for the matrix to lose rank. Substituting $z_3 = 0$ we soon deduce that the set of points where G is well defined but loses rank is $\{(0, 0, 0)\}$, which is the set of controllable zero points constructed in Example 5.5, as predicted by the lemma.

6. Summary

We have provided characterizations of the characteristic points of a factor behaviour. We have applied this to the closer examination of the uncontrollable pole points of behaviours given by linear partial differential equations with constant coefficients, and shown that they have an input decoupling property. We have also extended the pole structure theory by including controllable and uncontrollable zero points in a manner which does not depend upon the relative numbers of inputs and outputs. We showed that the controllable zero points are the zeros of the transfer matrix, and the uncontrollable zero points are a special case of uncontrollable pole points. Further work will involve the description of observable and unobservable zeros.

All the definitions we have given of different types of poles and zeros are equivalent to module-theoretic properties, and these properties are similar (and in some cases equivalent) in the 1D case to the definitions of poles and zeros given by Bourlès and Fliess in [BF]. Many of our results can equivalently be obtained in this dual framework by applying techniques from commutative algebra.

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