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On invariant zeros of linear systems of PDEs

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Abstract

In this paper we provide a behavioral framework in which to describe and extend the concept of linear dynamics introduced by Fliess, from the one dimensional (1D) to the multidimensional (nD) framework. We provide an alternative description of the invariant zeros of a system, equivalent to the Smith zero description in the 1D case and use this to generalize the concept and characterization of invariant zeros to the nD case. In particular we show that the definitions are equivalent in the 1D case to those in the classical literature. We provide new results on the structural relations of nD invariant and transmission zeros.

Keywords: Multidimensional linear system; Behavioral approach; Invariant zeros; Oberst duality; Characteristic variety

1. Introduction

Consider the 1D system defined over some field $\mathbb{K}$, given by the polynomial matrix description

$$\begin{aligned} A(s)x &= B(s)u, \\ y &= C(s)x + D(s)u, \end{aligned} \tag{1}$$

where $s = \mathrm{d}/\mathrm{d}t$ and A, B, C, D are polynomial matrices over $\mathbb{K}[s]$ and x, u, y are state variables, input and output variables respectively. This standard formulation of a linear system can be extended to the linear multidimensional system, Σ, given by equations of the form:

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$$\begin{aligned} A(z)x &= B(z)u, \\ y &= C(z)x + D(z)u, \end{aligned} \tag{2}$$

where $x = x(t)$ is a vector of latent variables, $u = u(t)$ is a vector of system inputs, and $y = y(t)$ is a vector of system outputs. The entries in the polynomial matrices A, B, C and D are elements of the ring $\mathbb{K}[z_1, z_2, \ldots, z_n] = \mathbb{K}[z]$ and are taken to be partial differential operators with $z_i = \partial/\partial t_i$. In the nD case it is important that the variables x are treated as latent variables rather than as state variables. We then see in the 1D framework this corresponds to the state space model where the variables $x(t)$ are state variables [6,5]. In this paper we shall consider the system using the behavioral framework introduced by Willems [10,11] and show that the system corresponds to a unique finitely generated module. Using this module theoretic framework we show that the poles and zeros can be well defined in terms of the exterior algebra of the module.

2. A module framework

We view the system as a triple $(\mathscr{A}, q, \mathscr{B})$, where $\mathscr{A}$ the *signal space* is a vector space over the field $k = \mathbb{C}$ (or $\mathbb{R}$) or more generally a $\mathbb{K}[z]$-module of n-dimensional mappings. The signal space $\mathscr{A}$ is one of the discrete spaces $k^{\mathbb{N}}, k^{\mathbb{Z}}$ or one of the continuous spaces $C^{\infty}(\mathbb{R}^n, k)$ or $\mathscr{D}'(\mathbb{R}^n, k)$, the space of all k-valued distributions on $\mathbb{R}^n$. Then q is the number of system variables and the *behavior* $\mathscr{B} \subseteq \mathscr{A}^q$ is the solution space of the finite set of n-dimensional differential or difference equations describing the system. For the system Σ:

$$\mathscr{B}_{x,u,y} = \left\{ \begin{pmatrix} x \\ u \\ y \end{pmatrix} \in \mathscr{A}^{\bullet} \,\middle|\, \begin{pmatrix} x \\ u \\ y \end{pmatrix} \text{satisfy}(2) \right\}.$$

Note that $\mathscr{A}^{\bullet}$ stands for the appropriate number of copies of the signal space $\mathscr{A}$.

For the polynomial ring $\mathbb{K}[z]$, the ring action $\mathbb{K}[z] \times \mathscr{A} \longrightarrow \mathscr{A}$ for discrete systems is defined as the shift operator σ_i and for continuous systems defined by the differential operators $\partial/\partial t_i$. Using this notation we can write any linear system in the form of a behavior, and similarly for any sub-system, we can write in terms of sub-behaviors. For example, we can write the system Σ in a behavioral kernel representation [13,5]:

$$\mathscr{B}_{x,u,y} = \mathrm{Ker}_{\mathscr{A}} \begin{pmatrix} A & -B & 0 \\ C & D & -I \end{pmatrix} \subseteq \mathscr{A}^q, \tag{3}$$

where we use the suffix notation $\mathrm{Ker}_{\mathscr{A}}$ to denote the kernel of the ring action of the matrix. An important sub-behavior is the one containing all outputs that are zero, that is the sub-behavior

$$\mathscr{B}_{x,u,0} := \left\{ \begin{pmatrix} x \\ u \\ y \end{pmatrix} \in \mathscr{B}_{x,u,y} | y = 0 \right\}, \tag{4}$$

which we see is given by the kernel representation

$$\mathscr{B}_{x,y,0} \cong \mathrm{Ker}_{\mathscr{A}} \begin{pmatrix} A & -B \\ C & D \end{pmatrix}, \tag{5}$$

where the sub-matrix $P(z) = \begin{pmatrix} A & -B \\ C & D \end{pmatrix}$ is the Rosenbrock system matrix [5].

This sub-behavior is very important when considering invariant zeros, for example the invariant zeros in the 1D case are given by the set of points in $\mathbb{C}$ such that the matrix $P(s)$ loses rank [6,5]. We will show that the invariant zeros in the nD case are the varieties in $\mathbb{C}^n$ such that $P(z)$ loses rank. In Section 2.3, we therefore consider the rank loss points of a polynomial matrix. We first outline some preliminary results concerning behaviors.

2.1. Preliminary results

For any matrix $E \in \mathscr{R}^{g,q}$, where $\mathscr{R}$ is some ring, define the modules:

$$\begin{aligned}
\mathrm{Ker}_{\mathscr{R}}\, E &:= \{v \in \mathscr{R}^{1,g} \,|\, vE = 0\},\\
\mathrm{Im}_{\mathscr{R}}\, E &:= \{v \in \mathscr{R}^{1,q} \,|\, v = xE \text{ for some } x \in \mathscr{R}^{1,q}\},\\
\mathrm{Coker}_{\mathscr{R}}\, E &:= \mathscr{R}^{1,q}/\mathrm{Im}_{\mathscr{R}}\, E,\\
\mathrm{Im}_{\mathscr{A}}\, E &:= \{w \in \mathscr{A}^{g} \,|\, w = El \text{ for some } l \in \mathscr{A}^{q}\},\\
\mathrm{Ker}_{\mathscr{A}}\, E &:= \{w \in \mathscr{A}^{q} \,|\, Ew = 0\}.
\end{aligned}$$

Note the different subscripts used to denote different ring actions. Also, the modules $\mathrm{Ker}_{\mathscr{R}}\, E$, $\mathrm{Im}_{\mathscr{R}}\, E$, and $\mathrm{Coker}_{\mathscr{R}}\, E$ are defined with respect to a left action on E, whereas $\mathrm{Ker}_{\mathscr{A}}\, E$ and $\mathrm{Im}_{\mathscr{A}} E$ are defined with respect to a right action.

Let M be a finitely generated $\mathscr{R}$-module. The *dual* of M with respect to $\mathscr{A}$, denoted $D(M)$, is defined by

$$D(M) := \mathrm{Hom}_{\mathscr{R}}(M, \mathscr{A}). \tag{6}$$

If $\psi : M \to N$ is a morphism of finitely generated $\mathscr{R}$-modules, then the dual map $D(\psi) : D(N) \to D(M)$ is given by $\forall v \in D(N)$, $(D(\psi))(v) := v \circ \psi$. The next result tells us precisely what the module M is.

Theorem 1 [7, 2.5.4, 2.56]. *Differential/discrete behaviors are precisely the dual modules of finitely generated $\mathscr{R}$-modules. Specifically, if $\mathscr{B} = \mathrm{Ker}_{\mathscr{A}}\, E$ then $\mathscr{B} = D(M)$, where M is the finitely generated module $\mathrm{Coker}_{\mathscr{R}} E$. For each signal space $\mathscr{A}$ (an injective cogenerator) we have the important property that given a complex of modules*

$$\cdots \longrightarrow M_{i+2} \xrightarrow{\phi_{i+1}} M_{i+1} \xrightarrow{\phi_i} M_i \longrightarrow \cdots \tag{7}$$

and its dual complex

$$\cdots \longrightarrow D(M_i) \xrightarrow{D(\phi_i)} D(M_{i+1}) \xrightarrow{D(\phi_{i+1})} D(M_{i+2}) \longrightarrow \cdots \tag{8}$$

then (7) is exact if and only if (8) is exact. We define the submodule $\mathscr{B}^{\perp} \subset \mathscr{R}^{1,q}$ called the orthogonal module as

$$\mathscr{B}^{\perp} := \{v \in \mathscr{R}^{1,q} \,|\, vw = 0 \textit{ for all } w \in \mathscr{B}\}. \tag{9}$$

In consequence if $\mathscr{B} = \mathrm{Ker}_{\mathscr{A}}\, E$ then $\mathscr{B}^{\perp} = \mathrm{Im}_{\mathscr{R}}\, E$ and therefore M is the finitely generated module $\mathrm{Coker}_{\mathscr{R}}\, E = \mathscr{R}^{1,q}/\mathscr{B}^{\perp}$.

The set of variables $\{w_i | i \in \Phi\}$ for some subset Φ of $\{1, \ldots, q\}$ is said to be a set of *free variables* if the mapping $\rho : \mathscr{A}^q \longrightarrow \mathscr{A}^\Phi$, which projects a trajectory onto the components of Φ, is epic when restricted to $\mathscr{B}$. The maximum cardinality of such a set Φ is an invariant of the behavior and is denoted by $m(\mathscr{B})$. It is given by

$$m(\mathscr{B}) = q - \operatorname{rank}(E),$$

where $\mathscr{B} = \operatorname{Ker}_{\mathscr{A}} E$. The number of free variables is an additive property, that is, given the sub-behavior $\mathscr{B}' \subset \mathscr{B}$ and the short exact sequence

$$0 \longrightarrow \mathscr{B}' \longrightarrow \mathscr{B} \longrightarrow \mathscr{B}/\mathscr{B}' \longrightarrow 0, \tag{10}$$

where $\mathscr{B}/\mathscr{B}'$ has the structural properties of a behavior [12], then $m(\mathscr{B}) = m(\mathscr{B}') + m(\mathscr{B}/\mathscr{B}')$.

Consider the generators $\{e_1, \ldots, e_q\}$ of the free module $\mathscr{R}^{1,q}$. The module M is generated by the set of generators $e_1 + \mathscr{B}^\perp, \ldots, e_q + \mathscr{B}^\perp$, and for a maximal set of free variables indexed by $\Phi \subset \{1, \ldots, q\}$ the corresponding set of elements $\{e_i + \mathscr{B}^\perp | i \in \Phi\}$ form a set of $m(\mathscr{B})$ linearly independent elements of M, that is they generate a free submodule, Λ of M. The system variables are assumed to be partitioned into inputs u, which are free and outputs x, y, which contain no free elements (once the inputs are fixed). Such a partitioning is called an *input/output structure* on the behavior. We have the following construction on M. Suppose $\mathscr{B}$ has l latent variables x (which are to be treated as outputs—in the 1D case, these form the state variables), m input variables u and p output variables y. Let $\Phi = \{l + 1, \ldots, l + m\}$, then

$$\Lambda = \langle e_{l+1} + \mathscr{B}^\perp, \ldots, e_{l+m} + \mathscr{B}^\perp \rangle. \tag{11}$$

Similarly, let

$$\Omega_1 = \langle e_1 + \mathscr{B}^\perp, \ldots, e_l + \mathscr{B}^\perp \rangle, \tag{12}$$

$$\Omega_2 = \langle e_{l+m+1} + \mathscr{B}^\perp, \ldots, e_q + \mathscr{B}^\perp \rangle, \tag{13}$$

where Ω_1 and Ω_2 correspond to x and y respectively. Then $M = \Omega_1 + \Lambda + \Omega_2$ and

$$\mathscr{B}_{x,u,y} = D(M) = D(\Omega_1 + \Lambda + \Omega_2). \tag{14}$$

We define the *annihilator of a behavior* $\mathscr{B}$ as

$$\operatorname{ann} \mathscr{B} = \{s \in \mathscr{R} | sw = 0 \; \forall w \in \mathscr{B}\}. \tag{15}$$

From [13] we have $\operatorname{ann} \mathscr{B} = \operatorname{ann} M$. A behavior containing no free variables is an *autonomous* behavior and is precisely one which has a non-zero annihilator. In the continuous case, we define a *controllable* behavior as follows [9]; given any two trajectories $\omega_1(t), \omega_2(t)$ in the behavior and any two open domains T_1, T_2 with disjoint closures, there exists a trajectory, $\omega_3(t)$ in the behavior, such that $\omega_3(t)|_{T_1} = \omega_1(t)|_{T_1}$ and $\omega_3(t)|_{T_2} = \omega_2(t)|_{T_2}$. For a given behavior, we define the *controllable part* as the unique maximal controllable sub-behavior, and we denote this, the controllable part of $\mathscr{B}$ by $\mathscr{B}^c$. (See also [12] for a similar definition for discrete systems.) It is well known that for $\mathscr{B} = D(M)$:

$$\mathscr{B}^c = D(M/tM) \quad \text{and} \quad \mathscr{B}/\mathscr{B}^c = D(tM),$$

where tM is the torsion submodule of M. For any behavior we can write its corresponding dual module representation using the fact that $M = \Omega_1 + \Lambda + \Omega_2$. As one possible example we see that

$$\mathscr{B}_{x,u,0} = D(M/\Omega_2)$$

and

$$\mathscr{B}^c_{x,u,0} := \mathscr{B}^c_{x,u,y} \cap \mathscr{B}_{x,u,0} = D(M/tM) \cap D(M/\Omega_2) = D(M/(tM + \Omega_2)).$$

2.2. Characteristic varieties

Linear systems with constant coefficients are entirely characterized by the exponential trajectories contained in their behavior. We now give the definition of such trajectories.

Definition 1. Let $w(t) = w(t_1, \ldots, t_n) \in \mathscr{A}^q$. Then w is said to be an *exponential trajectory of frequency* $(a_1, \ldots, a_n) \in \mathbb{C}^n$ if it is of the form:

$$w(t) = \begin{cases} v_0 a_1^{t_1} \ldots a_n^{t_n} \, \mathscr{A} = \mathbb{C}^{\mathbb{Z}^n} \\ v_0 \mathrm{e}^{a_1 t_1 + \cdots + a_n t_n} \begin{cases} \mathscr{A} = \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{C}), \\ \text{or } \mathscr{D}'(\mathbb{R}^n, \mathbb{C}), \end{cases} \end{cases} \tag{16}$$

where $v_0 \in \mathscr{C}^q$. Also w is said to be a *polynomial exponential trajectory of pure frequency* $(a_1, \ldots, a_n)$ if it is of the form:

$$w(t) = \begin{cases} p(t) a_1^{t_1} \ldots a_n^{t_n} \, \mathscr{A} = \mathbb{C}^{\mathbb{Z}^n}, \\ p(t) \mathrm{e}^{a_1 t_1 + \cdots + a_n t_n} \begin{cases} \mathscr{A} = \mathscr{C}^\infty(\mathbb{R}^n, \mathbb{C}), \\ \text{or } \mathscr{D}'(\mathbb{R}^n, \mathbb{C}), \end{cases} \end{cases} \tag{17}$$

where $p(t) = p(t_1, \ldots, t_n) \in \mathbb{C}[t_1, \ldots, t_n]^q$. *A polynomial exponential trajectory* is any trajectory which is a finite sum of polynomial trajectories of pure frequencies.

Let $J \subseteq \mathscr{R}$, be an ideal where $k = \mathbb{R}$ or $\mathbb{C}$. Define the variety $V(J)$ as

$$V(J) := \{a \in \mathbb{C}^n | p(a) = 0 \; \forall p \in J\}. \tag{18}$$

Note that $V(J)$ is defined as a subset of $\mathbb{C}^n$ even when $k = \mathbb{R}$. If $\mathscr{A} = k^{\mathbb{Z}^n}$, we consider all points $a \in (\mathbb{C}\backslash 0)^n$.

Definition 2 [13]. The *characteristic variety* of a behavior $\mathscr{B} = \mathrm{Ker}_{\mathscr{A}} R$ is the set $\mathscr{V}(\mathscr{B})$ of all points $(a_1, \ldots, a_n) \in \mathbb{C}^n$ such that the following equivalent conditions hold:

1. $(a_1, \ldots, a_n) \in \mathscr{V}(\mathrm{ann}\, \mathscr{B})$.
2. $R(a_1, \ldots, a_n)$ has less than full column rank.
3. $\mathscr{B}$ contains a non-zero exponential trajectory of frequency $(a_1, \ldots, a_n)$.

The points in $\mathscr{V}(\mathscr{B})$ are called the *characteristic points* of $\mathscr{B}$.

Note that if $\mathscr{B}$ contains a non-zero polynomial exponential trajectory of pure frequency $(a_1, \ldots, a_n)$ then by repeated differentiation it also contains a non-zero exponential trajectory of the same frequency.

The next result provides a characterization of the characteristic variety of a factor behavior $\mathscr{B}/\mathscr{B}'$.

Theorem 2 [15]. *Let $\mathscr{B}' \subseteq \mathscr{B}$ be behaviors and $(a_1, \ldots, a_n) \in \mathbb{C}^n$. Then the following are equivalent*:

1. *$(a_1, \dots, a_n)$ is a characteristic point of $\mathscr{B}/\mathscr{B}'$.*
2. *There exists a polynomial vector x such that $xw' = 0$ for all $w' \in \mathscr{B}'$ but xw equals a non-zero exponential trajectory of frequency $(a_1, \dots, a_n)$ for some $w \in \mathscr{B}$.*
3. *There exists a polynomial exponential trajectory of pure frequency $(a_1, \dots, a_n)$ in $\mathscr{B}\backslash\mathscr{B}'$.*

2.3. Generalized characteristic varieties

In order to define the zeros of a behavior, it is necessary to consider the rank loss points of the representation matrix of $\mathscr{B}$. By rank loss points we mean those values of $(a_1, \dots, a_n) \in \mathbb{C}^n$ such that the representation matrix of the behavior loses rank. For example, as we shall see, the invariant zeros of the system given in Eq. (3) are determined by the rank loss points of $P(z)$. Consider then the matrix $R \in \mathscr{R}^{g,q}$ of rank $r \leq q$, and let $I_r(R) \subset \mathscr{R}$ denote the ideal generated by the order r minors of R. The rank loss points of R are given by the elements of the variety of the ideal $I_r(R)$, that is by $V(I_r(R))$.

Definition 3 [4, 20.4]. Let $M = \text{Coker}_{\mathscr{R}} R$. Then for any positive integer i define the ith Fitting invariant of M denoted by $\text{Fitt}_i\, M$, as

$$\text{Fitt}_i\, M = I_{q-i}(R), \tag{19}$$

where q is the number of columns of R.

The following theorem, the first part of which is a well-known result, enables us to work exclusively with the module M when considering the rank loss points of the representation matrix R.

Theorem 3 [1, Proposition 1.5]. *For any finitely generated module M over a commutative domain*, $\text{rad}(\text{ann}\, M) = \text{rad}(\text{Fitt}_0\, M)$. *Furthermore for any $n \geqslant 0$*

$$\text{rad}(\text{Fitt}_n\, M) = \text{rad}\left(\text{ann}\left(\bigwedge^{n+1}\right) M\right). \tag{20}$$

Since ideals with the same radical have the same variety, from Definition 3 and Theorem 3 we see:

$$V(I_r(R)) = V(\text{Fitt}_{q-r}(M)) = V\left(\text{ann}\left(\bigwedge^{q-r+1}\right) M\right), \tag{21}$$

where $m(\mathscr{B}) = q - r$, and $\wedge$ denotes the wedge product. Since the rank loss points of R are given by Eq. (21), the rank loss points are in fact independent of the choice of representation matrix R. We therefore speak of the rank loss points of $\mathscr{B}$ or M.

We have the following definition which generalizes the concept of the characteristic variety:

Definition 4. Let $\mathscr{B} = D(M)$ be an nD behavior and define the *generalized characteristic variety* of $\mathscr{B}$ to be the variety, $\mathscr{V}(\mathscr{B})$,

$$\mathscr{V}(\mathscr{B}) := \operatorname{ann}\left(\bigwedge^{m(\mathscr{B})+1}\right) M \tag{22}$$

and the *generalized characteristic points* to be the elements of $\mathscr{V}(\mathscr{B})$.

We then have:

Corollary 1. *The rank loss points of any kernel representation of* $\mathscr{B}$ *are precisely the generalized characteristic points of* $\mathscr{B}$. *Moreover, if* $\mathscr{B}$ *is autonomous (i.e., M is a torsion module) then* $\mathscr{V}(\mathscr{B}) = V(\operatorname{ann} M)$.

The above corollary therefore states that for an autonomous system the rank loss points of $\mathscr{B}$ are precisely the characteristic points of $\mathscr{B}$. The following non-trivial theorem is central to the development of zeros for nD systems.

Theorem 4. *For nD differential/difference behaviors* $\mathscr{B}' \subseteq \mathscr{B}$ *such that the sequence* $0 \longrightarrow \mathscr{B}' \longrightarrow \mathscr{B} \longrightarrow \mathscr{B}/\mathscr{B}' \longrightarrow 0$ *is exact, then*:

(i) *In general* $\mathscr{V}(\mathscr{B}) \subseteq \mathscr{V}(\mathscr{B}') \cup \mathscr{V}(\mathscr{B}/\mathscr{B}')$.
(ii) *Specifically in the case that* $\mathscr{B}/\mathscr{B}'$ *is autonomous we have that* $\mathscr{V}(\mathscr{B}) = \mathscr{V}(\mathscr{B}') \cup \mathscr{V}(\mathscr{B}/\mathscr{B}')$.

Proof. See Appendix A. □

As with the characteristic variety, we have an interpretation of the generalized characteristic points in terms of exponential trajectories and rank loss points.

Theorem 5. *The following are equivalent for a behavior* $\mathscr{B} = \operatorname{Ker}_{\mathscr{A}}(R)$ *and* $(a_1, \ldots, a_n) \in \mathbb{C}^n$:

1. $(a_1, \ldots, a_n) \in \mathscr{V}(\mathscr{B})$.
2. *The rank of* $R(a_1, \ldots, a_n)$ *is less than the rank of* $R(z_1, \ldots, z_n)$.
3. *For any of up to* $m(\mathscr{B})$ *variables* ω_i, $\mathscr{B}$ *contains a non-zero exponential trajectory of frequency* $(a_1, \ldots, a_n)$ *which is zero in the specified components*.

Proof. See [15]. □

3. Invariant zeros of nD behaviors

For 1D systems described by Eqs. (1), the invariant zeros are the rank loss points of the Rosenbrock system matrix $P(s)$. From Eq. (5) we see that the behavior $\mathscr{B}_{x,u,0}$ corresponds to the matrix $P(s)$, and therefore the invariant zeros are given by the generalized characteristic points of the behavior $\mathscr{B}_{x,u,0}$. That is

$$\{\text{invariant zeros}\} = \mathscr{V}(\mathscr{B}_{x,u,0}),$$

where we shall term $\mathscr{V}(\mathscr{B}_{x,u,0})$, the invariant zero variety. We can easily generalise this to nD systems, and for any nD behavior $\mathscr{B}_{x,u,y}$, we can define the invariant zero points to be the elements of the variety $\mathscr{V}(\mathscr{B}_{x,u,0})$. As expected we can extend this concept very easily to define controllable and uncontrollable invariant zeros etc. to develop a zero structure—the structure $\mathscr{B}_{x,u,y}$ itself provides a map for this. The following pair of exact commutative diagrams demonstrate the structure of the behavior $\mathscr{B}_{x,u,y}$:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{B}^c_{x,u,0} & \longrightarrow & \mathscr{B}^c_{x,u,y} & \longrightarrow & \mathscr{B}^c_{y} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{B}_{x,u,0} & \longrightarrow & \mathscr{B}_{x,u,y} & \longrightarrow & \mathscr{B}_{y} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{\mathscr{B}_{x,u,0}}{\mathscr{B}^c_{x,u,0}} & \longrightarrow & \frac{\mathscr{B}_{x,u,y}}{\mathscr{B}^c_{x,u,y}} & \longrightarrow & \frac{\mathscr{B}_{y}}{\mathscr{B}^c_{y}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{23}$$

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{B}^c_{x,0,0} & \longrightarrow & \mathscr{B}^c_{x,u,0} & \longrightarrow & \mathscr{B}^c_{u,0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathscr{B}_{x,0,0} & \longrightarrow & \mathscr{B}_{x,u,0} & \longrightarrow & \mathscr{B}_{u,0} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{\mathscr{B}_{x,0,0}}{\mathscr{B}^c_{x,0,0}} & \longrightarrow & \frac{\mathscr{B}_{x,u,0}}{\mathscr{B}^c_{x,u,0}} & \longrightarrow & \frac{\mathscr{B}_{u,0}}{\mathscr{B}^c_{u,0}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{24}$$

where

$$\mathscr{B}_y := \left\{ y \in \mathscr{A}^p \;\middle|\; \exists \begin{pmatrix} x \\ u \end{pmatrix} \in \mathscr{A}^{l+m};\ \begin{pmatrix} x \\ u \\ y \end{pmatrix} \in \mathscr{B}_{x,u,y} \right\},$$

$$\mathscr{B}_{x,0,0} := \left\{ \begin{pmatrix} x \\ u \\ y \end{pmatrix} \in \mathscr{B}_{x,u,y} | u = y = 0 \right\},$$

$$\mathscr{B}_{u,0} := \left\{ \begin{pmatrix} u \\ 0 \end{pmatrix} \in \mathscr{A}^{m+p} \;\middle|\; \exists x \in \mathscr{A}^l;\ \begin{pmatrix} x \\ u \\ 0 \end{pmatrix} \in \mathscr{B}_{x,u,y} \right\}.$$

We make the following definitions:

Definition 5. For the behavior $\mathscr{B}_{x,u,y}$ with zero output sub-behavior $\mathscr{B}_{0,u,0}$ and invariant sub-behavior $\mathscr{B}_{x,u,0}$, we have the following:

(i) The invariant [invariant controllable] zero variety is defined to be $\mathscr{V}(\mathscr{B}_{x,u,0})[\mathscr{V}(\mathscr{B}^c_{x,u,0})]$ and the invariant [invariant controllable] zero points as the elements of $\mathscr{V}(\mathscr{B}_{x,u,0})$ $[\mathscr{V}(\mathscr{B}^c_{x,u,0})]$.

(ii) The invariant uncontrollable zero variety is defined to be $\mathscr{V}(\mathscr{B}_{x,u,0}/\mathscr{B}^c_{x,u,0})$ and the invariant uncontrollable zero points as the elements of $\mathscr{V}(\mathscr{B}_{x,u,0}/\mathscr{B}^c_{x,u,0})$.

(iii) The observable [observable controllable] zero variety is defined to be $\mathscr{V}(\mathscr{B}_{u,0})[\mathscr{V}(\mathscr{B}^c_{u,0})]$ and the observable [observable controllable] zero points as the elements of $\mathscr{V}(\mathscr{B}_{u,0})$ $[\mathscr{V}(\mathscr{B}^c_{u,0})]$.

(iv) The observable uncontrollable zero variety is defined to be $\mathscr{V}(\mathscr{B}_{u,0}/\mathscr{B}^c_{u,0})$ and the observable uncontrollable zero points as the elements of $\mathscr{V}(\mathscr{B}_{u,0}/\mathscr{B}^c_{u,0})$.

From [13] the uncontrollable pole points are defined to be the elements of the variety $\mathscr{V}(\mathscr{B}_{x,u,y}/\mathscr{B}^c_{x,u,y})$.

We have the following relations:

Theorem 6. *For the behavior $\mathscr{B}_{x,u,y}$ with zero output sub-behavior $\mathscr{B}_{0,u,0}$ and invariant sub-behavior $\mathscr{B}_{x,u,0}$ we have*

(i) *The invariant zero points are precisely the union of the invariant controllable and invariant uncontrollable zero points. That is*

$$\mathscr{V}(\mathscr{B}_{x,u,0}) = \mathscr{V}(\mathscr{B}^c_{x,u,0}) \cup \mathscr{V}(\mathscr{B}_{x,u,0}/\mathscr{B}^c_{x,u,0}).$$

(ii) *The observable zero points are precisely the union of the observable controllable and observable uncontrollable zero points. That is*

$$\mathscr{V}(\mathscr{B}_{u,0}) = \mathscr{V}(\mathscr{B}^c_{u,0}) \cup \mathscr{V}(\mathscr{B}_{u,0}/\mathscr{B}^c_{u,0}).$$

(iii) *The invariant uncontrollable zero points are precisely the union of the unobservable uncontrollable and observable uncontrollable zero points. That is*

$$\mathscr{V}(\mathscr{B}_{x,u,0}/\mathscr{B}^c_{x,u,0}) = \mathscr{V}(\mathscr{B}_{x,0,0}/\mathscr{B}^c_{x,0,0}) \cup \mathscr{V}(\mathscr{B}_{u,0}/\mathscr{B}^c_{u,0}).$$

(iv) *The invariant uncontrollable zero points are contained in the uncontrollable pole points. In general we have*

$$\mathscr{V}(\mathscr{B}_{x,u,0}/\mathscr{B}^c_{x,u,0}) = \mathscr{V}(\mathscr{B}_{x,u,y}/\mathscr{B}^c_{x,u,y}) \cup \mathscr{V}(\mathscr{B}_{y}/\mathscr{B}^c_{y}).$$

(v) *The invariant* [*controllable invariant*] *zero points are contained in the union of the unobservable* [*controllable unobservable*] *and observable* [*controllable observable*] *zeros. That is*

$$\mathscr{V}(\mathscr{B}_{x,u,0}) \subset \mathscr{V}(\mathscr{B}_{x,0,0}) \cup \mathscr{V}(\mathscr{B}_{u,0}),$$
$$\mathscr{V}(\mathscr{B}^c_{x,u,0}) \subset \mathscr{V}(\mathscr{B}^c_{x,0,0}) \cup \mathscr{V}(\mathscr{B}^c_{u,0}).$$

Proof. Note that the behaviors in the bottom row of diagram (24) are dual to torsion modules and are therefore autonomous. The proof of (i), (ii), (iii) and (v) then follows from a direct application of Theorem 4 to (24). The proof of (iv) is a direct application of Theorem 4 to the third row of diagram (23). □

In fact we can now show that the transmission zeros are not only contained in the invariant zeros but in the controllable invariant zeros—a subset of the invariant zeros. We need the following result:

Lemma 1. *For any 1D differential/difference behavior $\mathscr{B} = D(M)$, and any submodule $L \subset M$, we have for $\mathscr{B}' = D(L)$, that $\mathscr{V}(\mathscr{B}') \subset \mathscr{V}(\mathscr{B})$.*

Proof. See Appendix B. □

As we have already noted the invariant zeros in the classical framework correspond to the invariant zeros in the behavioral framework. Similarly the transmission zeros correspond to the observable controllable zeros. Therefore applying Lemma 1 to the exact commutative diagram (24), we have the following results for the 1D case:

(i) The observable zero variety is contained in the invariant zero variety. That is
$\mathscr{V}(\mathscr{B}_{u,0}) \subset \mathscr{V}(\mathscr{B}_{x,u,0})$.

(ii) The observable controllable zero variety is contained in the invariant controllable zero variety. That is
$\mathscr{V}(\mathscr{B}^c_{u,0}) \subset \mathscr{V}(\mathscr{B}^c_{x,u,0})$.

From (ii), we therefore see in the 1D case that the transmission zeros (observable controllable zeros) are certainly contained in the invariant zeros (since the invariant zeros are the union of the invariant controllable and invariant uncontrollable zeros). More precisely, we see that they are in fact contained in the invariant controllable zeros.

We have the following physical characterization of invariant zeros in terms of exponential and polynomial exponential trajectories.

Proposition 1 [14]. *Let $\mathscr{B}^c_{x,u,0} \subset \mathscr{B}_{x,u,0}$ where always $m' = m(\mathscr{B}^c_{x,u,0}) = m(\mathscr{B}_{x,u,0})$. Then we have the following*:

(i) *The point $\zeta \in \mathbb{C}^n$ is an invariant* [*resp. controllable*] *zero point of $\mathscr{B}$ if and only if for any choice of up to m' free* (*input*) *variables*, *there exists a non-zero exponential trajectory of frequency ζ contained in $\mathscr{B}_{x,u,0}$* [*resp. $\mathscr{B}^c_{x,u,0}$*] *with given choice of variables set to zero.*

(ii) *The point $\zeta \in \mathbb{C}^n$ is an invariant uncontrollable zero point of $\mathscr{B}$ if and only if there exists a non-zero polynomial exponential trajectory of frequency ζ contained in $\mathscr{B}_{x,u,0}$ but not in $\mathscr{B}^c_{x,u,0}$.*

Proof. The proof of (i) is a direct application of Theorem 5. The proof of (ii) is a direct application of Theorem 2. □

3.1. Application to 1D zero structure

From Definition 5(i), the invariant zeros correspond to the invariant zeros in the classical 1D case. Moreover it is shown in [14] that the transmission zeros correspond to the observable controllable zero points in Definition 5. Including the results in [13], we have the following correspondences between the classical and behavioral approach:

Classical	Behavioral
Poles	$\mathscr{B}_{x,0,y}$
Output decoupling zeros	$\mathscr{B}_{x,0,0}$
Input decoupling zeros	$\mathscr{B}_{x,u,y}/\mathscr{B}^c_{x,u,y}$
Input–output decoupling zeros	$\mathscr{B}_{x,0,0}/B^c_{x,0,0}$
Transmission poles	$\mathscr{B}^c_{0,y}$
Transmission zeros	$\mathscr{B}^c_{u,0}$
Invariant zeros	$\mathscr{B}_{x,u,0}$

Many relations between the different types of poles and zeros have been reported in the literature for the 1D case [6,3,2].

The definitions of zeros in [2] were shown to be equivalent to the classical definitions. We now give the interpretation of these zeros in terms of the generalized characteristic variety.

The *system dynamics*, defined in [2] is precisely the module of observables $M = \mathscr{R}^{1,q}/B^{\perp}$ defined in Theorem 1. The system dynamics in [2] is denoted by $\mathscr{R}$, and so to avoid confusion with notation we shall denote the system dynamics as $\varDelta$. The systems considered in [2] are 1D systems and the module of the system dynamics $\varDelta$ is a $k[z]$-module, that is a module over a principal ideal domain. Using the duality theory of Oberst [7] we are able to define unique modules that are the duals of behaviors, and so in this way we demonstrate the equivalence of the two approaches given in this paper and in [2]. For example given $\mathscr{B}_{x,u,y} = D(M)$ we have that

$$\mathscr{B}^{c}_{x,u,y} = D(M/tM) \quad \text{and} \quad \mathscr{B}_{x,u,y}/\mathscr{B}^{c}_{x,u,y} = D(tM),$$

where tM is the torsion submodule of $\mathscr{B}$. Similarly we have that

$$\mathscr{B}_{x,u,0} = D(M/\Omega_2), \quad \mathscr{B}_{x,0,0} = D(M/\Omega)\text{and} \quad \mathscr{B}_{u,y} = D(\Omega)$$

where $\Omega = \Omega_2 + \varLambda$, where Ω_2 and $\varLambda$ are given by (11)–(13). Therefore we see that

$$0 \longrightarrow \mathscr{B}_{x,0,0} \longrightarrow \mathscr{B}_{x,u,y} \longrightarrow \mathscr{B}_{u,y} \longrightarrow 0$$

is the dual of

$$0 \longrightarrow \Omega \longrightarrow M \longrightarrow M/\Omega \longrightarrow 0.$$

Using this result for $M_1, M_2 \subset M$,

$$D(M/(M_1 + M_2)) = D(M/M_1) \cap D(M/M_2)$$

we see that for $\mathscr{B}^{c}_{x,u,o} := \mathscr{B}^{c}_{x,u,y} \cap \mathscr{B}_{x,u,0}$, we have

$$\mathscr{B}^{c}_{x,u,0} = D(M/(tM + \Omega_2)).$$

Similarly for $\mathscr{B}_{u,y} = D(\Omega)$ and $\mathscr{B}_{y} = D(\Omega_2)$. Each behavior is the dual of a corresponding module; diagram (23) for example is the dual of

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & t\Omega_2 & \longrightarrow & tM & \longrightarrow & \frac{tM}{t\Omega_2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega_2 & \longrightarrow & M & \longrightarrow & M/\Omega_2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \frac{\Omega_2}{t\Omega_2} & \longrightarrow & \frac{M}{tM} & \longrightarrow & \frac{M}{tM+\Omega_2} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{25}$$

Using the module duality we show that the modules defined in [2] indeed correspond to behaviors using the duality of Oberst.

The torsion (for a module, N, over a PID, we have that $N = tN \oplus N_1$ where $N_1 \cong N/tN$) module T of $\varDelta$ corresponds to the torison submodule tM of M and the module $\varPhi$ in [2] corresponds to the module M/tM.

For the behavior

$$\mathscr{B}_{x,u,y} = D(\Omega_1 + \varLambda + \Omega_2)$$

we see that in the 1D case Ω_1, Λ, Ω_2 correspond to the submodules $[\zeta]$, $[u]$, $[y]$ of Δ defined in [2], where of course $[u]$ is free.

The modules of the form $\Phi \cap [y]$ are projections of $[y]$ onto Φ. Therefore the module $\Phi \cap [y]$ corresponds to the projection of Ω_2 onto M/tM, i.e., the module $\Omega_2/(\Omega_2 \cap tM)$. Similarly, the modules $\Delta/[y]$ for example correspond to M/Ω_2. In this way we see that all the modules described in [2] correspond to modules in the 1D behavioral approach.

We now show that the Smith zeros of a module (system) described in [2] are equivalent to the generalized characteristic points of the module. For any finitely generated module, M, over a PID $\mathscr{D}$,

$$M \cong \mathscr{R}^s \oplus \mathscr{R}/\alpha_1\mathscr{R} \oplus \cdots \oplus \mathscr{R}/\alpha_r\mathscr{R}, \tag{26}$$

where the *invariant factors* $\alpha_i \in \mathscr{R}$ are non-zero units and

$$\alpha_i | \alpha_{i+1}, \quad i = 1, \ldots, r-1, \tag{27}$$

where the decomposition (26), subject to (27) is unique, up to isomorphism. Define $\alpha(s) = \alpha_1(s) \ldots \alpha_n(s)$. In [2], the *Smith zeros* of a module M, which we shall denote by $S_M(M)$, are defined to be the roots of $\alpha(s)$, that is

$$S_M(M) = \{\zeta \in \mathbb{C} | \alpha(\zeta) = 0\} = V(\alpha\mathscr{R}). \tag{28}$$

We show that in the 1D case, the Smith zeros are precisely the generalized characteristic points of this module.

Proposition 2. *For any $k[z]$-module M, we have that $S_M(M) = \mathscr{V}(D(M))$. That is for a 1D behavior $\mathscr{B} = D(M)$, the generalized characteristic points of $\mathscr{B}$ are precisely the Smith zeros of M.*

Proof. For a behavior $\mathscr{B} = D(M)$ we have that $M = \text{Coker}_{\mathscr{R}} R$, where $R \in \mathscr{R}^{g,q}$ is of rank r, the rank loss points of $\mathscr{B} = D(\text{Coker}_{\mathscr{R}} R)$ are the elements of the variety $V(I_r(R))$. Consider then the exact sequence

$$\mathscr{R}^g \xrightarrow{R} \mathscr{R}^q \longrightarrow M \longrightarrow 0,$$

where we see that $M = \text{Coker}_{\mathscr{R}} R$. For a matrix R of rank r over a PID, there exist two invertible matrices T, U over $\mathscr{R}$ such that

$$TRU = \text{diag}(\alpha'_1, \ldots, \alpha'_r, 0, \ldots, 0),$$

where $\alpha'_i | \alpha'_{i+1}$ for $i = 1, \ldots, r-1$. It then follows that

$$M \cong \mathscr{R}^s \oplus \mathscr{R}/\alpha'_1\mathscr{R} \oplus \cdots \oplus \mathscr{R}/\alpha'_r\mathscr{R}, \tag{29}$$

where it may happen that some of the invariant factors of R are units, in which case the summand is simply zero. Since the decomposition is invariant up to isomorphism, (26) and (29) are isomorphic, that is $\alpha'_i = \alpha_i u_i$ for some unit u_i, then we see that for $\alpha'(s) = \alpha'_1 \ldots \alpha'_r$, that

$$S_M(M) = V(\alpha'(s)) = V(\alpha(s)).$$

Finally, for T, U invertible, and as above, we have

$$I_r(R) = I_r(TRU) = (\alpha'_1 \ldots \alpha'_r)\mathscr{R}.$$

Therefore the rank loss points of M are precisely the elements of the variety

$$V(I_r(R)) = V(\alpha'_1 \dots \alpha'_r \mathscr{R}) = V(\alpha(s)) = S_M(M).$$

We have therefore shown that the definition of the Smith zeros of a module M in [2] are precisely the rank singularities of the module in the 1D case. □

The invariant zeros in [2] are defined to be the Smith zeros of the module $\Delta/[y]$. In terms of the behavioral approach this is equivalent to the rank singularities of the module M/Ω_2. We see that $D(M/\Omega_2) = \mathscr{B}_{x,u,0}$ and the rank singularities of M/Ω_2 are precisely the elements of $\mathscr{V}(\mathscr{B}_{x,u,o})$ which we have defined to be the invariant zeros in the behavioral approach. Therefore in the 1D case, the invariant zeros defined in Definition 5(i), are precisely the invariant zeros defined by [2]. Similarly, for the other classes of poles and zeros, the definitions are equivalent in the 1D case. In this way, all the results on the zero structure in 1D presented in [2] are obtainable using the behavioral approach. In particular, the following important result from [2] holds equally well for the corresponding behavioral definitions in the 1D case [14]:

Proposition 3 [2]. *Let $\mathscr{B}_{x,u,y}$ be the 1D behavior as described above with input to output transfer matrix G of rank r. Then*

(i) *The transmission and input output decoupling zeros are contained in the invariant zeros. That is*

$$\{\text{transmission zeros}\} + \{\text{i.o.d.z}\} \subset \{\text{invariant zeros}\}. \tag{30}$$

(ii) *Assume that $r = p$, i.e. G is right-invertible; then the transmission zeros and input decoupling zeros are contained in the invariant zeros. That is*

$$\{\text{transmission zeros}\} + \{\text{i.d.z}\} \subset \{\text{invariant zeros}\}. \tag{31}$$

(iii) *Assume that $r = m$, i.e. G is left-invertible; then the transmission zeros and output decoupling zeros are contained in the invariant zeros. That is*

$$\{\text{transmission zeros}\} + \{\text{o.d.z}\} \subset \{\text{invariant zeros}\}. \tag{32}$$

Proof. (i) The invariant zeros are the generalized characteristic points of the behavior $\mathscr{B}_{x,u,0}$. The transmission zeros are the generalized characteristic points of the behavior $\mathscr{B}_{u,0}$, (the observable controllable points), the input output decoupling zeros are the unobservable uncontrollable pole points. From the commutative diagram (24), since $\mathscr{B}_{x,u,0}/\mathscr{B}^c_{x,u,0}$ is autonomous,

$$\mathscr{V}(\mathscr{B}_{x,0,0}/\mathscr{B}^c_{x,0,0}) \subset \mathscr{V}(\mathscr{B}_{x,u,0}/\mathscr{B}^c_{x,u,0}) \subset \mathscr{V}(\mathscr{B}_{x,u,0}). \tag{33}$$

Therefore the invariant zeros always contain the i.o.d zeros. Since the system is 1D, by Lemma 1, we see from the exact sequence

$$0 \longrightarrow \mathscr{B}^c_{x,0,0} \longrightarrow \mathscr{B}^c_{x,u,0} \longrightarrow \mathscr{B}^c_{u,0} \longrightarrow 0$$

that

$$\mathscr{V}(\mathscr{B}^c_{u,0}) \subset \mathscr{V}(\mathscr{B}^c_{x,u,0}) \subset \mathscr{V}(\mathscr{B}_{x,u,0}). \tag{34}$$

Therefore, combining (33) and (34), we have (30).

(ii) When $r = p$, we see that for $G = D^{-1}N$, where D and N are left coprime, that since D is full column rank and invertible, then $\operatorname{rank} G = \operatorname{rank} N = p$. Therefore for the behavior $\mathscr{B}^c_{u,y} = \operatorname{Ker}_{\mathscr{A}}(-ND)$, since we are in 1D, $\operatorname{rank}(-ND) = p$. Note also that $\mathscr{B}^c_{u,0} \cong \operatorname{rank}_{\mathscr{A}} N$, and so from the sequence

$$0 \longrightarrow \mathscr{B}^c_{u,0} \longrightarrow \mathscr{B}^c_{u,y} \longrightarrow \mathscr{B}^c_y \longrightarrow 0$$

by the additivity of $m(\cdot)$;

$$m(\mathscr{B}^c_y) = m(\mathscr{B}^c_{u,y}) - m(\mathscr{B}^c_{u,0}) = m(\mathscr{B}_{x,u,y}) - (m(\mathscr{B}_{x,u,y}) - p) = p.$$

We then have that $\mathscr{B}_y = D(\Omega_2) \cong \mathscr{A}^p$, hence Ω_2 is free. Consider therefore the exact sequence

$$0 \longrightarrow \mathscr{B}^c_{x,u,0} \longrightarrow \mathscr{B}^c_{x,u,y} \longrightarrow \mathscr{B}^c_y \longrightarrow 0$$

is the dual of the exact sequence

$$0 \longrightarrow (tM + \Omega_2)/tM \longrightarrow M/tM \longrightarrow M/(tM + \Omega_2) \longrightarrow 0,$$

where we see that since Ω_2 is free, $(tM + \Omega_2)/tM \cong \Omega_2$. That is $\mathscr{B}^c_y \cong \mathscr{B}_y$, and so we have the exact commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & \frac{\mathscr{B}_{x,u,0}}{\mathscr{B}^c_{x,u,0}} & = & \frac{\mathscr{B}_{x,u,y}}{\mathscr{B}^c_{x,u,y}} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathscr{B}_{x,u,0} & \longrightarrow & \mathscr{B} & \longrightarrow & \mathscr{B}_y & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \mathscr{B}^c_{x,u,0} & \longrightarrow & \mathscr{B}^c & \longrightarrow & \mathscr{B}^c_y & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array} \tag{35}$$

The i.d zeros are given by the variety $\mathscr{V}(\mathscr{B}/\mathscr{B}^c) = \mathscr{V}(\mathscr{B}_{x,u,0}/\mathscr{B}^c_{x,u,0})$, and so we see that

$$\mathscr{V}(\mathscr{B}/\mathscr{B}^c) \subset \mathscr{V}(\mathscr{B}_{x,u,0}). \tag{36}$$

Combining Eqs. (36) and (34), we obtain (31).

(iii) When $r = m$, N is full column rank and so we see that $\mathscr{B}^c_{u,0} \cong \operatorname{Ker}_{\mathscr{A}} N$ is autonomous. Since $\mathscr{B}^c_{x,0,0}$ is also autonomous we have that $\mathscr{B}_{x,u,0} = D(M/\Omega_2)$ is autonomous, i.e. M/Ω_2 is torsion. Recall that the o.d zeros are given by the characteristic variety $\mathscr{V}(B_{x,0,0})$. Since $\mathscr{B}_{x,u,0}$ is autonomous, from the exact sequence

$$0 \longrightarrow \mathscr{B}_{x,0,0} \longrightarrow \mathscr{B}_{x,0,0} \longrightarrow \mathscr{B}_{x,0} \longrightarrow 0$$

and by additively, $\mathscr{B}_{u,0}$ is autonomous and therefore by Theorem 4 we have $\mathscr{V}(\mathscr{B}_{x,0,0}) \subset \mathscr{V}(\mathscr{B}_{x,u,0})$. Together with (34) we have (32). □

In the proof of (ii) above we note that for $r = p$, then

$$\{\text{input decoupling zeros}\} = \{\text{uncontrollable zeros}\}.$$

From [15], all observable uncontrollable zero points are observable uncontrollable pole points which are contained in the uncontrollable pole points, that is input decoupling zeros. We therefore have the following result that is also true in the nD case.

Corollary 2. *For the* 1D *behavior described in Proposition* 3, *we always have that the subset of input decoupling zeros which are observable uncontrollable zero points are always contained in the invariant zeros.*

Proof. The proof of (i) is a direct application of Theorem 4 to the commutative diagram (24), where

$$\mathscr{V}\left(\frac{\mathscr{B}_{u,0}}{\mathscr{B}^c_{u,0}}\right) \cup \mathscr{V}\left(\frac{\mathscr{B}_{x,0,0}}{\mathscr{B}^c_{x,0,0}}\right) = \mathscr{V}\left(\frac{\mathscr{B}_{x,u,0}}{\mathscr{B}^c_{x,u,0}}\right) \subset \mathscr{V}(\mathscr{B}_{x,u,0})$$

and so we see that

$$\mathscr{V}\left(\frac{\mathscr{B}_{u,0}}{\mathscr{B}^c_{u,0}}\right) \subset \mathscr{V}(\mathscr{B}_{x,u,0})$$

and the result follows. □

Proposition 3(ii) states for $r = p$, that the input decoupling zeros are contained in the invariant zeros, and Proposition 3(iii) states, for $r = m$, that the output decoupling zeros are contained in the invariant zeros. This agrees with [6] Sections 5.1 and 5.2 where it is observed in Section 5.1 that for a system with more outputs than inputs some of the invariant zeros are output decoupling zeros, which confirms with (iii) above. Similarly, in Section 5.2 of [6] for a system with more inputs than outputs some of the invariant zeros are input decoupling zeros, which confirms with (ii) above.

3.2. The nD zero structure

We now consider the case when the behavior $\mathscr{B}_{x,u,y}$ is an nD behavior. We shall refer to the unobservable pole points as output decoupling zeros, and the unobservable uncontrollable pole points as the input–output decoupling zeros, and the observable controllable zero points as the transmission zeros. We have:

Proposition 4. *Let $\mathscr{B}_{x,u,y}$ be an nD behavior as described above. Then*

(i) *One always has that the uncontrollable observable zeros and input output decoupling zeros are contained in the invariant zeros. That is*

$$\{\text{i.o.d.z}\} \cup \{\text{unc. obs. zeros}\} \subset \{\text{invariant zeros}\}.$$

(ii) *If $\mathscr{B}_{x,u,y}$ is such that the outputs y are free, then the input decoupling zeros are contained in the invariant zeros. That is*

$$\{\text{input dec. zeros}\} \subset \{\text{invariant zeros}\}.$$

(iii) *If $\mathscr{B}_{x,u,0}$ is autonomous then the transmission zeros and output decoupling zeros are contained in the invariant zeros. That is*

$$\{\text{output dec. zeros}\} \subset \{\text{invariant zeros}\},$$
$$\{\text{transmission zeros}\} \subset \{\text{invariant zeros}\}.$$

Proof. The proof of (i) is a direct application of Theorem 4 to the commutative diagram (24). Similarly, for (ii) when Ω_2 is free, diagram (23) becomes

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \frac{\mathscr{B}_{x,u,0}}{\mathscr{B}^c_{x,u,0}} & = & \frac{\mathscr{B}_{x,u,y}}{\mathscr{B}^c_{x,u,y}} & & & & \\
 & & \uparrow & & \uparrow & & & & \\
0 & \longrightarrow & \mathscr{B}_{x,u,0} & \longrightarrow & \mathscr{B}_{x,u,y} & \longrightarrow & \mathscr{B}_y & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \| & & \\
0 & \longrightarrow & \mathscr{B}^c_{x,u,0} & \longrightarrow & \mathscr{B}^c_{x,u,y} & \longrightarrow & \mathscr{B}^c_y & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
\tag{37}
$$

where $\mathscr{B} = \mathscr{B}_{x,u,y}$. Recall that the i.d zeros are the characteristic points of $\mathscr{B}/\mathscr{B}^c$ and so the result follows from applying Theorem 4(ii) to the first column of (35). For (iii), we have that $\mathscr{B}_{x,u,0}$ is autonomous, and so all behaviors in (24) are autonomous, and so a direct application of Theorem 4(ii) gives the result. □

Finally, we give an example to illustrate the results in this paper.

Example. Consider a system described by the Rosenbrock model

$$
\begin{aligned}
&A(z)x = B(z)u,\\
&y = C(z)x + D(z)u.
\end{aligned}
$$

The kernel representation of the behavior in this case is given by

$$\mathscr{B}_{x,u,y} = \mathrm{Ker}_{\mathscr{A}} \begin{pmatrix} A & -B & 0 \\ C & D & -I \end{pmatrix} \subseteq \mathscr{A}^q.$$

Given the kernel representation of a behavior, we can find the kernel representation of its controllable part. Suppose that

$$\mathscr{B}^c_{x,u,y} = \mathrm{Ker}_{\mathscr{A}}(R^c_x \ R^c_u \ R^c_y) \subseteq \mathscr{A}^q.$$

Then we have

$$
\begin{aligned}
&\mathscr{B}_{x,0,0} = \mathrm{Ker}_{\mathscr{A}} \begin{pmatrix} A \\ C \end{pmatrix},\\
&\mathscr{B}^c_{x,0,0} = \mathrm{Ker}_{\mathscr{A}}(R^c_x).
\end{aligned}
$$

Similarly, by eliminating the latent variables x we get

$$
\begin{aligned}
&\mathscr{B}_{u,y} = \mathrm{Ker}_{\mathscr{A}}(R^m_u \ \ R^m_y),\\
&\mathscr{B}^c_{u,y} = \mathrm{Ker}_{\mathscr{A}}(R^{mc}_u \ \ R^{mc}_y),\\
&\mathscr{B}^c_{0,y} = \mathrm{Ker}_{\mathscr{A}}(R^{mc}_y),\\
&\mathscr{B}^c_{u,0} = \mathrm{Ker}_{\mathscr{A}}(R^{mc}_u).
\end{aligned}
$$

Now, the output decoupling zeros are the rank loss points of $\begin{pmatrix} A \\ C \end{pmatrix}$, transmission zeros are the rank loss points of R^{mc}_u and invariant zeros are the rank loss points of $\begin{pmatrix} A & -B \\ C & D \end{pmatrix}$. For the case of input decoupling zeros and input–output decoupling zeros, it is necessary to find a kernel representation of the corresponding behaviors. Then the rank loss points of the kernel representations give the corresponding zeros.

4. Conclusion

In this paper we have developed a framework within the behavioral theory that allows us to define various classes of zeros. The definition of invariant zeros and transmission zeros for nD systems proposed in this paper have been shown to correspond to the 1D definitions of invariant zeros and transmission zeros in the literature in the special case of state space models defined by Eq. (2). We have extended the relationship between rank loss points and varieties by introducing the generalized characteristic variety which is a more general form of the characteristic variety commonly used in nD systems theory. Using the exterior product of the module corresponding to the behavior we have shown how we can generalize the connection between column rank loss points and characteristic varieties to arbitrary rank loss points and the generalized characteristic varieties. By identifying classes of zeros with the generalized characteristic varieties of behaviors we have built up a comprehensive nD zero structure that we have shown incorporates the 1D pole-zero structure. We have also shown how the module approach suggested by Bourles is equivalent in the 1D case to the approach we have developed. The results themselves are special cases of more general results on zeros developed in [14] and build upon the notions of controllable and uncontrollable zeros of nD zeros given in [15,16].

Appendix A. Proof of Theorem 4

Let P be any prime ideal of $\mathscr{R}$, and let $\mathscr{R}_P$ be the local ring with unique maximal ideal $P\mathscr{R}_P = P_P$. The residue class field of the local ring $(\mathscr{R}_p, P_P)$ is defined to be $\mathscr{R}(P)$, where

$$\mathscr{R}(P) := \mathscr{R}_P/P_P \cong (\mathscr{R}/P)_P. \tag{38}$$

The field $(\mathscr{R}/P)_P$ is the field of fractions of the integral domain $\mathscr{R}/P$. For the $\mathscr{R}$-module M we have the $\mathscr{R}_p$-module M_P given by

$$M_P = \mathscr{R}_P \otimes_{\mathscr{R}} M.$$

The field $\mathscr{R}(P)$ induces a finite-dimensional vector space $M(P)$ over $\mathscr{R}(P)$ where $M(P)$ is also a $\mathscr{R}(P)$-module defined as

$$M(P) := \mathscr{R}(P) \otimes_{\mathscr{R}_P} M_P = \mathscr{R}(P) \otimes_{\mathscr{R}} M = M_P/P_P M_P.$$

For the $\mathscr{R}$-module $M(P)$ with representation matrix R, the matrix $R(P)$ is defined as

$$R := (r_{ij}) \in \mathscr{R}^{g,q} \mapsto R(P) := (r_{ij} + P_P) \in \mathscr{R}(P)^{g,q}.$$

The $\mathscr{R}(P)$-module $M(P)$ has representation matrix $R(P)$:

$$M(P) = \mathrm{Coker}_{\mathscr{R}(P)} R(P) = \mathscr{R}(P)^{1,q}/\mathscr{R}^{1,g} R(P). \tag{39}$$

Define the rank of the matrix $R(P)$ to be the maximal number of $\mathscr{R}(P)$ linearly independent rows or columns of $R(P)$. That is we consider all possible $\mathscr{R}(P)$-linear combinations. This definition of rank agrees with the usual definition of the rank of the matrix R as the rank taken over the quotient field $\mathscr{R}(0) = \mathrm{Quo}(\mathscr{R})$ of $\mathscr{R}$. By this we mean that we can view the matrix R as being the matrix $R(0)$, where we consider the prime ideal and the matrix $R = (r_{ij})$ is $R(0) = (r_{ij} + 0) = R$. Define the dimension of the vector space $M(P)$ as

$$\dim_{\mathscr{R}} M(P) = \dim_{\mathscr{R}(P)}(\mathscr{R}(P) \otimes_{\mathscr{R}} M) = q - \operatorname{rank} R(P).$$

The rank of $R(P)$ and the dimension of the vector space $M(P)$ are related as expected; we see that

$$\dim_{\mathscr{R}(P)}(M(P)) = q - \operatorname{rank} R(P).$$

We now make the following definition.

Definition 6 [7, Corollary 5.81]. For a finitely generated $\mathscr{R}$-module M define the *rank singularities* to be the set of prime ideals $P \subseteq \operatorname{Spec} \mathscr{R}$ such that the localization M_P is not free. Denote the rank singularities

$$RS(M) = \{P \in \operatorname{Spec} \mathscr{R} | M_P \text{ not free}\} \quad (40)$$
$$= \{P \in \operatorname{Spec} \mathscr{R} | \operatorname{rank} R(P) < \operatorname{rank} R\}. \quad (41)$$

We have the following important property.

Corollary 3. *If M is torsion then $RS(M) = \operatorname{supp}(M)$.*

Proof. See [7, Corollary 5.71(ii)]. □

For any $k = \mathbb{C}$, any maximal ideal P is in one to one correspondence with a point $a \in \mathbb{C}^n$, and so $P = I(a)$. The matrix $R(P) = R(I(a))$ is then simply given by $R(a)$, the matrix R evaluated at the point $a \in \mathbb{C}^n$. Therefore $P \in RS(M)$ if and only if rank $R(a) <$ rank R. It is this particular subset of maximal ideals of $RS(M)$ that we will be interested in, since they correspond to the set of rank loss points of R.

Theorem 7. *For a finitely generated $\mathscr{R}$-module M and submodule N, such that the sequence*

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

is exact, we have the following:

(i) $RS(tN) \subseteq RS(M)$, *where tN is the torsion submodule of N.*
(ii) $RS(M) \subseteq RS(N) \cup RS(M/N)$.
(iii) *When N is torsion $RS(M) = RS(N) \cup RS(M/N)$.*

Proof. (i) Let $tN \subset N$ be the torsion submodule of N. By Corollary 3, $RS(tN) = \operatorname{Supp}(tN)$ and so for any $P \in \operatorname{Supp}(tN)$ we have $(tN)_P = t(N)_P \neq 0$. Since $\mathscr{R}$ is Noetherian the module $\mathscr{R}_P$ is flat and therefore since $N_P = \mathscr{R}_P \otimes_{\mathscr{R}} N$ we have that

$$0 \longrightarrow N_P \longrightarrow M_P \longrightarrow (M/N)_P \longrightarrow 0 \quad (42)$$

is exact, where $tN_P \subset N_P$. By injectivity, M_P contains the non-zero torsion submodule tN_P and therefore cannot be a free module, and so $P \in RS(M)$.

(ii) Let $L = M/N$ and assume that $P \notin RS(N) \cup RS(L)$ and so L_P and N_P are free $\mathscr{R}_P$-modules. Since M_P is a finitely generated $\mathscr{R}_P$-module and $L_P \cong (\mathscr{R}_P)^r$ for some r, then

$$0 \longrightarrow \operatorname{Ker} \phi \longrightarrow M_P \xrightarrow{\phi} (\mathscr{R}_P)^r \longrightarrow 0. \quad (43)$$

Let $e_1, \ldots, e_r$ be a basis of $(\mathscr{R}_P)^r$ and choose $m_i \in M_P$ such that $\phi(m_i) = e_i$ $(1 \leq i \leq r)$. Then

$$M_p = \operatorname{Ker}\phi \oplus \langle m_1, \ldots, m_n\rangle$$

is exact. Clearly since ϕ is a homomorphism the elements $m_1, \ldots, m_n$ are linearly independent and therefore $\langle m_1, \ldots, m_n\rangle$ is a free module. Further Ker ϕ is isomorphic to N_P, which is a free module. Hence M_P is a free module. Hence $P \notin RS(M)$. This is sufficient to prove that if $P \in RS(M)$ then $P \in RS(N) \cup RS(M/N)$, that is $RS(M) \subseteq RS(N) \cup RS(M/N)$.

(iii) Assuming N is torsion and therefore $RS(N) = \operatorname{Supp}(N)$, then (ii) above shows that the inclusion $RS(M) \subseteq RS(N) \cup RS(M/N)$ holds. Now we need to prove that the opposite inclusion is true, that is $RS(M) \supseteq RS(N) \cup RS(M/N)$. Clearly if $P \in RS(N) \cup RS(M/N)$ then either $P \in RS(N)$ or $P \in RS(M/N)$. If the former case holds, part (i) shows that $P \in RS(M)$. Let $L = M/N$. If $P \in RS(L)\backslash RS(N)$ then $N_P = 0$ and we have that

$$0 \longrightarrow M_P \longrightarrow L_P \longrightarrow 0$$

Therefore $RS(M) \supseteq RS(N) \cup RS(M/N)$. □

Definition 7. We define the *maximal rank singularities* of a $\mathscr{R}$-module M, denoted $\Re_{\mathfrak{M}}(M)$, to be the set of maximal ideals contained in the set of rank singularities of M.

Theorem 8. *For a $\mathscr{R}$-module M and submodule N with*

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

we have that

(i) $\Re_{\mathfrak{M}}(tN) \subset \Re_{\mathfrak{M}}(M)$, *where tN is the torsion submodule of N.*
(ii) $\Re_{\mathfrak{M}}(M) \subset \Re_{\mathfrak{M}}(M/N)$.
(iii) *When N is torsion* $\Re_{\mathfrak{M}}(M) = \Re_{\mathfrak{M}}(N) \cup \Re_{\mathfrak{M}}(M/N)$.

Proof. (i) Let $P \in RS(tN) \subseteq RS(M)$ be maximal. Then necessarily $P \in \Re_{\mathfrak{M}}(M)$.

(ii) From Theorem 7, $P \in \Re_{\mathfrak{M}}(M) \subset RS(N) \cup RS(M/N)$, and since $\Re_{\mathfrak{M}}(M) \subset RS(M)$, this implies that $P \in RS(N) \cup RS(M/N)$. However, since $P \in \text{Max-Spec}(\mathscr{R})$ this implies that $P \in \Re_{\mathfrak{M}}(N) \cup \Re_{\mathfrak{M}}(M/N)$.

(iii) We always have $\Re_{\mathfrak{M}}(M) \subset \Re_{\mathfrak{M}}(N) \cup \Re_{\mathfrak{M}}(M/N)$. To prove the reverse we recall that for N torsion $RS(M) = RS(N) \cup RS(M/N)$. Therefore if $P \in \Re_{\mathfrak{M}}(N) \cup \Re_{\mathfrak{M}}(M/N)$ then $P \in RS(N) \cup RS(M/N) = RS(M)$. Since $P \in \text{Max-Spec}(\mathscr{R})$ then necessarily $P \in \Re_{\mathfrak{M}}(M)$. □

In order to fully understand the role of the maximal ideals we must realize the Galois connection between the maximal ideals and the points in $\mathbb{C}^n$. For example, we know that over $\mathbb{C}$ any maximal ideal corresponds to a unique point—this is not true for ideals over $\mathbb{R}$—in this case maximal ideals correspond to unique pairs of conjugate points in $\mathbb{C}^n$. Next we give a summary of the relevant parts of this connection.

Given a field k and its algebraic closure $\bar{k}$ we define the Galois group of $\bar{k}$ over k to be the set of k-linear automorphisms of $\bar{k}$ that leave k fixed. We denote this group by

$$\Gamma := \operatorname{Gal}(\bar{k}/k).$$

For example $\operatorname{Gal}(\mathbb{C}/\mathbb{R}) = \{id, \sigma\}$ where σ is complex conjugation. For $\gamma \in \Gamma$ and $a \in \bar{k}^n$, we define the group action

$$\Gamma \times \bar{k}^n \to \bar{k}^n, \quad \gamma(a) = (\gamma(a_1), \ldots, \gamma(a_n)), \tag{44}$$

which induces a partitioning of $\bar{k}^n$ given be the orbits $\Gamma(a) = \{\gamma(a) \text{ for all } \gamma \in \Gamma\}$, and we define $\Gamma(\bar{k}^n) = \{\Gamma(a) \text{ for all } a \in \bar{k}^n\}$. For example, when $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{R})$ then

$$\Gamma_{sp}(\mathbb{C}) = \{\{\alpha\} | \alpha \in \mathbb{R}\} \cup \{\{\beta, \bar{\beta}\} | \beta \in \mathbb{C}\backslash\mathbb{R}\}$$

and when $\Gamma = \mathrm{Gal}(\mathbb{C}/\mathbb{C})$, $\Gamma_{sp}(\mathbb{C}) = \{\{\alpha\}\} | \alpha \in \mathbb{C}\}$. We have

Theorem 9 [8, Lemma 5.5]. *Let $\bar{k}$ be the algebraic closure of k and let $\Gamma = \mathrm{Gal}(\bar{k}/k)$. Then*

$$\Gamma(\bar{k}^n) \mapsto \text{Max-Spec}\, k[x_1, \ldots, x_n],$$

where for each $a \in \mathbb{C}^n$, we identify $\Gamma(a)$ with $\mathbb{M}(a)$.

We denote the maximal ideal in $\mathbb{R}[x_1, \ldots, x_n]$ corresponding to the point $a \in \mathbb{C}^n$ by $\mathfrak{M}_{\mathscr{R}}(a)$. We consider the two cases when a is in $\mathbb{R}^n$ and $\mathbb{C}^n\backslash\mathbb{R}^n$. When $a \in \mathbb{R}^n$ then we have that

$$\mathfrak{M}_{\mathbb{R}}(a) = (x_1 - a_1, \ldots, x_n - a_n). \tag{45}$$

When $a \notin \mathbb{R}^n$ then

$$\mathfrak{M}_{\mathbb{R}}(a) = \mathbb{R}[x_1, \ldots, x_n] \cap (\mathfrak{M}_{\mathbb{C}}(a)\mathfrak{M}_{\mathbb{C}}(\bar{a})), \tag{46}$$

where $\bar{a}$ is the conjugate of a. It is clear from (46) *that $\mathfrak{M}_{\mathbb{R}}(a) = \mathfrak{M}_{\mathbb{R}}(\bar{a})$.*

Corollary 4. *For $\mathbb{R} \subset \mathbb{C}$ let $\Gamma_1 = \mathrm{Gal}(\mathbb{C}/\mathbb{C}) = \{id\}$ and $\Gamma_2 = \mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \{id, \sigma\}$. Then we have that*

(i) $\Gamma_{1sp}(\mathbb{C}^n) \subseteq \mathbb{C}^n \subseteq \text{Max-Spec}\mathbb{C}[x_1, \ldots, x_n]$ *given by* $\{a\} \mapsto \mathfrak{M}_{\mathbb{C}}(a)$.
(ii) $\Gamma_{2sp}(\mathbb{C}^n) \subseteq \text{Max-Spec}\mathbb{R}[x_1, \ldots, x_n]$ *given by* $\{a, \bar{a}\} \mapsto \mathfrak{M}_{\mathbb{R}}(a)$.

This important corollary tells us that all the maximal ideals in $\mathbb{C}^n$ correspond to points in $\mathbb{C}^n[x_1, \ldots, x_n]$ and all the maximal ideals in $\mathbb{R}^n[x_1, \ldots, x_n]$ can be identified with pairs of complex conjugate points in $\mathbb{C}^n$ or single real points. The following result establishes a link between maximal ideals in $\mathscr{R}$ and the rank singularities.

Proposition 5. *Let $a \in \mathbb{C}^n$ and let $P = I(a) = \mathfrak{M}(a)$. Then $\mathscr{R}(P) \subseteq k(a)$ is a field.*

Proof. Let $\psi_a : \mathscr{R}_P \mapsto k(a)$ be the evaluation mapping defined by $\psi : f/g \mapsto f(a)/g(a)$ with $\mathrm{Ker}\, \psi_a = P_P$. By the first isomorphism theorem [4] we have that $\mathscr{R}(P) := \mathscr{R}_P/P_P \subseteq k(a)$. In order to show that $\mathscr{R}(P)$ and $k(a)$ are isomorphic as fields we need to consider the isomorphism $\phi : \mathscr{R}(P) \mapsto k(a)$ defined by $\phi([f/g]) := f(a)/g(a)$ for $f/g(\mathrm{mod}\, P_P) := [f/g] \in \mathscr{R}(P)$. Then the exact commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & P_P & \longrightarrow & \mathscr{R}_P & \xrightarrow{\psi} & k(a) & \longrightarrow & 0 \\ & & & & \pi \downarrow & & \| & & \\ & & 0 & \longrightarrow & \mathscr{R}(P) & \xrightarrow{\phi} & k(a) & \longrightarrow & 0 \end{array}$$

where π is the canonical mapping $\pi(f/g) = f/g(\mathrm{mod}\, P_P) = [f/g]$ shows that $\mathscr{R}(P) \subseteq k(a)$ is a field. $\square$

In this section we investigate the relation between the generalized characteristic points and the rank singularities over the ring $k[x_1, \ldots, x_n]$. The results are central to the development of the general theory of zeros.

Lemma 2. *Let $a \in \mathbb{C}^n$ be a rank loss point of $R \in \mathscr{R}^{g,q}$. Then for $k = \mathbb{R}$ or $\mathbb{C}$ any $a' \in \Gamma(a)$ is also a rank loss point of R.*

Proof. The result is trivial for $k = \mathbb{C}$. When $k = \mathbb{R}$ then $\Gamma(a) = \{a\}$ or $\{a, \bar{a}\}$. It is clear that rank $R(a) =$ rank $R(\bar{a})$. □

The following theorem is the result which we really require here.

Theorem 10. *Let $\mathscr{R}$ be a polynomial ring over $k = \mathbb{R}$ or $\mathbb{C}$, and let $M = \text{Coker}_{\mathscr{R}} R$, $R \in \mathscr{R}^{g,q}$, with $r =$ rank R. Then $a \in \mathscr{V}(M)$ if, and only if, $I(a) \in \mathfrak{R}_{\mathfrak{M}}(M)$. That is*

$$\mathscr{V}(M) = \bigcup_{P \in \mathfrak{R}_{\mathfrak{M}}(M)} V(P). \tag{47}$$

Proof. We claim that there exists a one to one correspondence between the elements of $\Gamma_{sp}(\mathscr{V}(M))$ and the maximal ideals in $\mathfrak{R}_{\mathfrak{M}}(M)$. Let $a \in \mathscr{V}(M)$. Then by Lemma 2, $\Gamma(a) \subset \mathscr{V}(M)$ and rank $R(a') <$ rank R for any $a' \in \Gamma(a)$. From Theorem 9 identify $\Gamma(a)$ with the maximal ideal $P := \mathfrak{M}(a)$ in $\mathscr{R}$. Let $R = (r_{ij}) \in \mathscr{R}^{g,q}$. Then for $P \in$ Max-Spec $\mathscr{R}$ we have $R(P) \in \mathscr{R}(P)^{g,q}$ given by

$$R(P) = (r_{ij} + P_P). \tag{48}$$

By Proposition 5, $\mathscr{R}(P) \subseteq k(a)$, therefore $R(P) \in \mathscr{D}^{g,q}$ maps isomorphically to $R(a)$ by

$$R(P) = (r_{ij} + P_P) \mapsto (r_{ij}(a)) =: R(a).$$

Since $R(P) \subseteq R(a)$, then rank $R(P) =$ rank $R(a) <$ rank R, therefore $P \in \mathfrak{R}_{\mathfrak{M}}(M)$.

Conversely, suppose that $P \in$ Max-Spec $\mathscr{R}$ is such that rank $R(a) <$ rank R. Then again by Theorem 9 we identify $P = \mathfrak{M}(a')$ with $\Gamma(a') \in \Gamma_{sp}(\mathbb{C}^n)$ for some $a' \in \mathbb{C}^n$. We observe that $P = \mathfrak{M}(a') = \mathfrak{M}(a)$ for any $a \in \Gamma(a')$. Therefore select any $a \in \Gamma(a')$. By Proposition 5 we have $\mathscr{R}(P) \cong k(a)$ and so rank $R(a) <$ rank $R(P) <$ rank R. Therefore since this is true for any $a \in \Gamma(a')$ we see that $\Gamma(a') \subset \mathscr{V}(M)$ and so we identify $P \in \mathfrak{R}_{\mathfrak{M}}(M)$ with $\Gamma(a') \in \Gamma(\mathscr{V}(M))$.

Using the above claim we can now prove (47). For any $P \in \mathfrak{R}_{\mathfrak{M}}(M)$, by the above claim we identify P with $\Gamma(a)$, for some $a \in \mathscr{V}(M)$. Note that $P = \mathfrak{M}(a')$ for all $a' \in \Gamma(a)$, and so $V(P) = V(\mathfrak{M}(a')) = \Gamma(a)$. □

When we consider the ring $\mathbb{C}[x_1, \ldots, x_n]$, Corollary 4 tells us there is a one-to-one correspondence between the maximal ideals $P \in \mathfrak{R}_{\mathfrak{M}}(M)$ and the points $a \in \mathscr{V}(M)$. This is given by $a \in \mathscr{V}(M)$ if, and only if, rank $R(a) <$ rank R if, and only if, $P = (z_1 - a_1, \ldots, z_n - a_n) = I(a) \in \mathfrak{R}_{\mathfrak{M}}(M)$. Since $V(z_1 - a_1, \ldots, z_n - a_n)) = \{a\}$ we see that Theorem 10 simply states that the set of rank loss points are precisely the set of points given by the varieties of the maximal rank singularities.

When $k = \mathbb{R}$ the relationship between the maximal rank singularities and rank loss points over the ring $\mathbb{R}[x_1, \ldots, x_n]$ is not the same as for $\mathbb{C}$. We note that $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R})$ and so $\Gamma_{sp}(\mathscr{V}(M))$ consists of singleton sets of real valued $a \in \mathbb{R}^n$ or conjugate pairs of complex valued rank loss points. That is for $a \in \mathscr{V}$ we have $\Gamma(a) = \{a, \bar{a}\}$. In this case Theorem 10 shows that such a pair of conjugate points, $\{a, \bar{a}\}$ correspond to a single maximal ideal $P = \mathfrak{M}(a) \in \mathfrak{R}_{\mathfrak{M}}(M)$, given by (46). Conversely any $P \in \mathfrak{R}_{\mathfrak{M}}(M)$ corresponds to a pair of conjugate points that are identified in $\Gamma(\mathscr{V}(M))$. In Theorem 10, we see that $V(P) = \{a, \bar{a}\}$, as opposed to the singleton set $\{a\}$, as was the case when $k = \mathbb{C}$, and the set of rank loss points is indeed the union of all such sets of pairs.

Recall now Theorem 4. Then the proof of this result is a direct consequence of Theorem 8 and Theorem 10.

Appendix B. Proof of Lemma 1

For any module M over a principal ideal domain, M is free if, and only if, M is torsion-free. We have the following result for the special case when $\mathscr{R}$ is the principal ideal domain $k[\underline{z}]$.

Lemma 3. *For any $k[\underline{z}]$-module M and submodule $L \subset M$, it always holds that $RS(L) \subset RS(M)$.*

Proof. For $L \subset M$ we have the following exact sequence

$$0 \longrightarrow L \longrightarrow M \longrightarrow M/L \longrightarrow 0.$$

Suppose $P \in RS(L)$ and therefore L_P is not free. Since L_P is a module over a PID, L_P is not free if, and only if, L_P is not torsion-free. Therefore L_P contains torsion elements since we have

$$0 \xrightarrow{\varphi} L_P \longrightarrow M_P \longrightarrow M_P/L_P \longrightarrow 0,$$

where φ is injective, then M_P contains torsion elements and is therefore non-torsion free and therefore not free. Hence $P \in RS(M)$. □

Corollary 5. *For any $k[\underline{z}]$-module M and submodule $L \subset M$, it always holds that $\Re_{\mathfrak{M}}(L) \subset \Re_{\mathfrak{M}}(M)$, and therefore*

$$V\left(\operatorname{ann} \bigwedge^{\mathscr{X}(L)+1} L\right) \subset V\left(\operatorname{ann} \bigwedge^{\mathscr{X}(M)+1} M\right)$$

that is the rank singularities of L are contained in the rank singularities of M.

Proof. Since $RS(L) \subset RS(M)$ it follows that for any maximal ideal $P \in RS(L)$, $P \in RS(M)$ and therefore $P \in \Re_{\mathfrak{M}}(M)$. The result now follows from a direct application of Theorem 10. □

We now we are in a position to prove Lemma 1. In particular, let $\mathscr{B} = D(M)$ and $\mathscr{B}'$ for some $L \subset M$. We have that

$$\mathscr{V}(\mathscr{B}') := V\left(\operatorname{ann} \bigwedge^{m(L)+1} L\right) \quad \text{and} \quad \mathscr{V}(\mathscr{B}) := V\left(\operatorname{ann} \bigwedge^{m(M)+1} M\right)$$

From Corollary 5, we see that $\mathscr{B}' \subset \mathscr{B}$ as required.

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