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$\mathscr{H}_\infty$ and guaranteed cost control of discrete linear repetitive processes

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Abstract

Repetitive processes are a distinct class of 2D systems (i.e. information propagation in two independent directions) of both systems theoretic and applications interest. In general, they cannot be controlled by direct extension of existing techniques from either standard (termed 1D here) or 2D systems theory. Here first we give major new results on the design of control laws using an $\mathscr{H}_\infty$ setting and including the possibility of uncertainty in the process model. Then we give the first ever results on guaranteed cost control, i.e. including a performance criterion in the design. The designs in both cases can be computed using linear matrix inequalities. These results are for so-called discrete linear repetitive processes which arise in applications areas such as iterative learning control.

Keywords: Linear repetitive processes; $\mathscr{H}_\infty$ control; Guaranteed cost control; Linear matrix inequalities

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1. Introduction

Repetitive processes are a distinct class of 2D systems of both system theoretic and applications interest. The essential unique characteristic of such a process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem for these processes in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction.

To introduce a formal definition, let $\alpha < +\infty$ denote the pass length (assumed constant). Then in a repetitive process the pass profile $y_k(p)$, $0 \leqslant p \leqslant \alpha$, generated on pass k acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile $y_{k+1}(p)$, $0 \leqslant p \leqslant \alpha$, $k \geqslant 0$.

Physical examples of repetitive processes include long-wall coal cutting and metal rolling operations (see, for example, [16]). Also in recent years applications have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples of these so-called algorithmic applications include classes of iterative learning control (ILC) schemes [13] and iterative algorithms for solving nonlinear dynamic optimal control problems based on the maximum principle [14]. In the case of iterative learning control for the linear dynamics case, the stability theory for so-called differential and discrete linear repetitive processes is the essential basis for a rigorous stability/convergence theory of a powerful class of such algorithms. For the nonlinear optimal control algorithm, the repetitive process analysis has provided the essential basis for the development of highly reliable iterative solution algorithms.

Attempts to control these processes using standard (or 1D) systems theory/algorithms fail (except in a few very restrictive special cases) precisely because such an approach ignores their inherent 2D systems structure, i.e. information propagation occurs from pass-to-pass and along a given pass and also the initial conditions are reset before the start of each new pass. In seeking a rigorous foundation on which to develop a control theory for these processes, it is natural to attempt to exploit structural links which exist between, in particular, so-called discrete linear repetitive processes and 2D linear systems described by the extensively studied Roesser [15] or Fornasini–Marchesini [6] state space models.

The fact that the pass length is finite (and hence information in this direction only occurs over a finite duration) is the key difference with other classes of 2D discrete linear systems. This means that large parts of established systems theory for 2D discrete linear systems described by the Roesser and Fornasini–Marchesini state space models either cannot be applied at all or only after appropriate modification. Hence there is a need to develop a systems theory for these processes for onward translation, where appropriate, into numerically reliable design algorithms.

A rigorous stability theory for linear constant pass length repetitive processes has been developed. This theory [16] is based on an abstract model in a Banach space setting which includes all such processes as special cases. Also the results of applying this theory to a wide range of cases have been reported, including the sub-class considered here. This has resulted in stability tests that can, if desired, be implemented by direct application of well known 1D linear systems tests.

One feature of repetitive processes (which is not always present in other classes of 2D systems) is that it is possible to define physically meaningful control laws for them. For example, in the ILC application, one such family of control laws is composed of state feedback control action on the current pass combined with information ‘feedforward’ from the previous pass (or trial in the ILC context) which, of course, has already been generated and is therefore available for use. In the general case of repetitive processes it is clearly highly desirable to have an analysis setting where such control laws can be designed for stability and/or guaranteed performance. Moreover, previous work has shown that an LMI re-formulation of the stability conditions for discrete linear repetitive processes leads naturally to design algorithms for control laws to ensure stability along the pass under control action—see, for example, [8].

The $\mathscr{H}_\infty$ setting for the control related analysis of 1D linear systems is now a very mature area and it is natural to ask if such an approach can be extended to 2D linear systems/linear repetitive processes. In the case of 2D linear systems described by the Roesser and Fornasini–Marchesini state space models, there has been a substantial volume of work on stabilizing control law design, including the case when there is uncertainty in the model structure—see, for example, [4]. In the case of discrete linear repetitive processes, little or no work has yet been reported. Hence $\mathscr{H}_\infty$ controller design should be very profitable approach with onward translation to, for example, the ILC area (where the problem of what is meant by robustness of such schemes is still a largely open general question).

In this paper, we first give new results on the control of discrete linear repetitive processes which formulate and solve the fundamental problem of finding an admissible controller such that (as one interpretation) the $\mathscr{H}_\infty$ norm of a transfer function (matrix) satisfies a scalar magnitude constraint. The solutions here include control laws which are activated by pass profile information only and hence the assumption that the complete current pass state vector is available or can be reconstructed by an observer is not required. This alone is a major advance alone over existing 2D systems results with the added bonus that these control laws have a sound physical basis.

By optimizing the controller over the scalar magnitude constraint, we get as close as required to the minimal $\mathscr{H}_\infty$ norm. Also it is shown that the $\mathscr{H}_\infty$ control problem here can, in computational terms, be solved using linear matrix inequalities (LMIs) [3]. Moreover, significant new results on the robust control of these processes are developed within this analysis setting.

In the final part of the paper, a solution to the guaranteed cost control problem for these processes is developed, where a quadratic cost function is included as part of the design task. This cost function is physically motivated and the results are among the first on control law design with performance for these processes. Where appropriate, we will highlight the connections and differences with the existing results for 2D discrete linear systems.

Throughout this paper, the null matrix and the identity matrix with appropriate dimensions are denoted by 0 and I, respectively. Moreover, $M > 0$ (< 0) denotes a real symmetric positive (negative) definite matrix. Also for square symmetric matrices U_1 and U_2 of the same dimensions we use $U_1 \geqslant U_2$ to denote the case when $U_1 - U_2$ is positive semi-definite. Finally, we use ($\bigstar$) to denote the transpose of matrix blocks in some of the LMIs employed (which are required to be symmetric).

Consider a $q \times 1$ vector sequence $\{w_i(j)\}$, defined over nonnegative integers i and j, i.e. $0 \leqslant i \leqslant \infty$ and $0 \leqslant j \leqslant \infty$ which is written as $\{[0, \infty], \ [0, \infty]\}$. Then the ℓ_2 norm of this vector sequence is given by

$$\|w\|_2 = \sqrt{\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} w_i^{\mathrm{T}}(j) w_i(j)}$$

and this sequence is said to be a member of $\ell_2^q\{[0, \infty], [0, \infty]\}$, or ℓ_2^q for short, if $||w||_2 < \infty$.

2. Background

The state space model of the discrete linear repetitive processes considered in this work has the following form over $0 \leqslant p \leqslant \alpha, k \geqslant 0$

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + B_0 y_k(p) + Bu_{k+1}(p) + B_{11} w_{k+1}(p), \\ y_{k+1}(p) &= Cx_{k+1}(p) + D_0 y_k(p) + Du_{k+1}(p) + B_{12} w_{k+1}(p). \end{aligned} \tag{1}$$

Here on pass k, $x_k(p)$ is the $n \times 1$ state vector, $y_k(p)$ is the $m \times 1$ pass profile vector, $u_k(p)$ is the $l \times 1$ vector of control inputs and $w_{k+1}(p)$ is the $r \times 1$ disturbance input vector which belongs to ℓ_2^r.

To complete the process description, it is necessary to specify the boundary conditions, i.e. the state initial vector on each pass and the initial pass profile. Here no loss of generality arises from assuming $x_{k+1}(0) = d_{k+1}$, $k \geqslant 0$, where d_{k+1} is an $n \times 1$ vector with known constant entries, and $y_0(p) = f(p)$, where $f(p)$ is an $m \times 1$ vector whose entries are known functions of p.

The stability theory [16] for linear repetitive processes consists of two distinct concepts, termed asymptotic stability and stability along the pass respectively. In effect, asymptotic stability is bounded input bounded output stability (defined in terms of the norm on the underlying function space) over the finite pass length, and for the processes considered here requires that all eigenvalues of D_0 have modulus strictly less than unity, i.e. $r(D_0) < 1$ where $r(\cdot)$ denotes the spectral radius of its argument. If this property holds, and the control input sequence applied $\{u_k\}_{k \geqslant 1}$ converges strongly to u_∞ as $k \to \infty$, then the resulting output pass profile sequence $\{y_k\}_{k \geqslant 1}$ converges strongly to y_∞—the so-called limit profile—which is described (with $D = 0$ for simplicity) by a 1D discrete linear systems state space model with state matrix $A_{lp} := A + B_0(I - D_0)^{-1}C$.

The fact that the pass length is finite means that the limit profile may not be stable as a 1D linear system, i.e. $r(A_{lp}) < 1$, e.g. $A = -0.5$, $B = 0$, $B_0 = 0.5 + b_0$, $C = 1$, $D = D_0 = 0$, and the real scalar b_0 is chosen such that $|b_0| \geqslant 1$. Stability along the pass prevents this from arising by demanding the bounded input bounded output property uniformly, i.e. independent of the pass length α. Mathematically, this can be analyzed by letting $\alpha \to +\infty$.

Before proceeding, it is instructive to briefly outline how the abstract model based stability theory for linear repetitive processes can be applied to one class of ILC schemes. We mostly follow [13] (which deals with so-called differential linear repetitive processes, where the current pass state updating is governed by a linear matrix differential equation, and for which (1) can be regarded as an approximation of the defining state space model under sampling).

Since the original work by Arimoto et. al. [1], the general area of ILC has been the subject of intense research effort both in terms of the underlying theory and 'real world' applications. Typical ILC algorithms construct the input to plant on a given trial from the input used on the last trial, or pass in repetitive process terminology, plus an additive increment which is typically a function of the past values of the observed output error, i.e. the difference between the achieved and desired plant output. Suppose that $u_k(t)$ denotes the input on the kth trial which is of duration T, i.e. $t \in [0, T]$. Suppose also that $e_k(t)$ denotes the difference between the desired trajectory $r(t)$ and the system output $y_k(t)$ on the same trial. Then the objective of constructing a sequence of input functions such that the performance is gradually improving with each successive trial can be refined to a convergence condition on the input and error

$$\lim_{k\to\infty} ||e_k|| = 0, \qquad \lim_{k\to\infty} ||u_k - u_\infty|| = 0,$$

where $||\cdot||$ is a signal norm in a chosen function space (e.g. $L_2^m[0, T]$) with a norm based topology.

This definition of convergent learning is, in effect, a stability problem on an infinite-dimensional two-dimensional (2D)-product space. As such, it places the analysis of ILC schemes firmly outside standard (or 1D) control theory (although it still has a significant role to play in certain cases of practical interest). Instead, ILC schemes must be seen in the context of fixed-point problems or, more precisely, repetitive processes.

Suppose now that the state space model of the plant to be controlled is assumed to be of the following form

$$\begin{aligned} \dot{x}_k(t) &= Ax_k(t) + Bu_k(t), \quad 0 \leqslant t \leqslant T, \\ y_k(t) &= Cx_k(t), \end{aligned}$$

where on trial k, $x_k(t)$ is the $n \times 1$ state vector, $y_k(t)$ is the $m \times 1$ output vector, and $u_k(t)$ is the $l \times 1$ vector of control inputs. If the signal to be tracked is denoted by $r(t)$ then $e_k(t) = r(t) - y_k(t)$ is the error on trial k. Also without loss of generality in this section (except where stated) we set $x_{k+1}(0) = 0$, $k \geqslant 0$. The class of ILC schemes considered here are of the following form, i.e. a (static and dynamic) combination of previous input vectors, the current trial error, and the errors on a finite number of previous trials. On trial $k + 1$ the control input is calculated using

$$u_{k+1}(t) = \sum_{i=1}^{N} \alpha_i u_{k+1-i}(t) + \sum_{i=1}^{N} K_i[e_{k+1-i}](t) + K_0[e_{k+1}](t).$$

In addition to the 'memory' N, the design parameters in this control law are the static scalars α_i, $1 \leqslant i \leqslant N$, the linear operator $K_0[\cdot](t)$ which describes the current trial error contribution and the linear operator $K_i[\cdot](t)$, $1 \leqslant i \leqslant N$, which describes the contribution of the error on trial $k + 1 - i$. Next we show how the controlled system can be written as a special case of the general model of linear constant pass length repetitive processes.

First note that the open loop error dynamics can be written in convolution form as

$$e_{k+1}(t) = r(t) - G[u_{k+1}](t), \quad 0 \leqslant t \leqslant T,$$

where

$$G[u](t) = C \int_0^t \mathrm{e}^{A(t-\tau)} Bu(\tau)\, \mathrm{d}\tau.$$

Using this description, it is easily shown that the controlled system error dynamics on trial $k + 1$ can be written over $0 \leqslant t \leqslant T$ as

$$e_{k+1}(t) = (I + GK_0)^{-1} \left\{ \sum_{i=1}^{N} (\alpha_i I - GK_i)[e_{k+1-i}](t) + \left(1 - \sum_{i=1}^{N} \alpha_i\right) r(t) \right\}$$

or, equivalently,

$$\hat{e}_{k+1} = L_{\mathrm{T}} \hat{e}_k + b,$$

where

$$\hat{e}_k(t) = \begin{bmatrix} e_{k+1-N}^{\mathrm{T}}(t) & \cdots & e_k^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}$$

is the so-called error super-vector, and

$$L_{\mathrm{T}} = \begin{bmatrix} 0 & I & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & I \\ E_0 E_N & \cdots & E_0 E_2 & E_0 E_1 \end{bmatrix}$$

with

$$E_0[y](t) = (I + GK_0)^{-1}[y](t),$$
$$E_i[y](t) = (\alpha_i I - GK_i)[y](t), \quad 1 \leqslant i \leqslant N$$

and

$$b = \begin{bmatrix} 0 & 0 & \cdots & \left(1 - \sum_{i=1}^{N} \alpha_i\right) r^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}}.$$

Suppose now that $\hat{e}_k \in E_{\mathrm{T}}$, where E_{T} is a suitably chosen Banach space, and $b \in W_{\mathrm{T}}$, where W_{T} is a linear subspace of E_{T}. Then in this setting, the bounded linear operator L_{T} maps E_{T} into itself, the term $L_{\mathrm{T}}\hat{e}_k$ describes the contributions of the errors on the previous N trials to the current one, and b, termed the disturbance vector, describes the contribution from external sources on the current trial. Note also that the theory which now follows applies to any ILC scheme which can be written in the abstract form on which the repetitive process stability theory is based. It is also routine to argue that convergence of the controlled ILC scheme as $k \to \infty$ is equivalent to stability of its linear repetitive process interpretation.

Consider the application of the stability theory to processes described by (1). Then it is easy to establish that this can be studied by deleting the disturbance terms. Moreover, numerous equivalent sets of necessary and sufficient conditions for stability along the pass are known, but here the essential starting point is based on the so-called 2D characteristic polynomial for these processes given next.

Define the shift operators z_1, z_2 in the along the pass (p) and pass-to-pass (k) directions acting e.g. on $x_k(p+1)$ and $y_{k+1}(p)$ respectively as

$$x_k(p) := z_1 x_k(p+1), \quad y_k(p) := z_2 y_{k+1}(p).$$

Then the 2D characteristic polynomial for processes described by (1) is defined as

$$\mathscr{C}(z_1, z_2) = \det\left(\begin{bmatrix} I - z_1 A & -z_1 B_0 \\ -z_2 C & I - z_2 D_0 \end{bmatrix}\right)$$

and it can be shown [16] that stability along the pass holds if, and only if,

$$\mathscr{C}(z_1, z_2) \neq 0 \quad \text{in } \overline{U}^2,$$

where $\overline{U}^2 = \{(z_1, z_2) : |z_1| \leqslant 1,\ |z_2| \leqslant 1\}$. Note that stability along the pass can also be expressed in the form

$$\mathscr{C}(z_1, z_2) = \det(I - z_1 A_1 - z_2 A_2) \neq 0 \quad \text{in } \overline{U}^2,$$

where

$$A_1 = \begin{bmatrix} A & B_0 \\ 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0 \\ C & D_0 \end{bmatrix}. \tag{2}$$

This in turn has led to the development of LMI based conditions for stability along the pass which are sufficient but not necessary.

Theorem 1 [8]. *A discrete linear repetitive process described by* (1) *is stable along the pass if there exists a block-diagonal matrix* $P = \mathrm{diag}\{P_1, P_2\} > 0$ *such that the following LMI holds*

$$\Phi^{\mathrm{T}} P \Phi - P < 0,$$

where

$$\Phi = \begin{bmatrix} A & B_0 \\ C & D_0 \end{bmatrix}$$

is the so-called augmented plant matrix.

Note also that the sufficient but not necessary basis of this result is offset by the fact that it easily allows the design of control laws. This topic is returned to in the next section.

In this paper, we wish to address the problem of control law design for stability along the pass and performance. In this latter respect, two areas are treated, the first of which is disturbance, or noise, attenuation which is defined as follows.

Definition 1. A discrete linear repetitive process described by (1) is said to have $\mathscr{H}_\infty$ disturbance attenuation γ if it is stable along the pass and

$$\sup_{0 \neq w \in \ell_2^r} \frac{\|y\|_2}{\|w\|_2} < \gamma. \tag{3}$$

The relevance of control law design to reject the effects of disturbances on measurements (and subsequent computations) of variables is well founded physically by noting the conditions in which physical examples have to operate, e.g. long-wall coal cutting and iterative learning control applications such as using a gantry robot to synchronously place objects on a chain conveyor [2].

Consider now the 2D transfer function matrix coupling the disturbance and current pass profile vectors which is given by

$$G_{yw}(z_1, z_2) = \begin{bmatrix} 0 & I \end{bmatrix} \begin{bmatrix} I - z_1 A & -z_1 B_0 \\ -z_2 C & I - z_2 D_0 \end{bmatrix}^{-1} \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}.$$

Then the 2D Parseval theorem [12], which states that (3) is equivalent to the requirement that $\|G_{yw}(z_1, z_2)\|_\infty < \gamma$, leads to

$$\|G_{yw}(z_1, z_2)\|_\infty = \sup_{\omega_1, \omega_2 \in [0, 2\pi]} \overline{\sigma}[G_{yw}(\mathrm{e}^{\mathrm{j}\omega_1}, \mathrm{e}^{\mathrm{j}\omega_2})],$$

where $\overline{\sigma}(\cdot)$ denotes the maximum singular value.

Introduce now the following Lyapunov function for the processes considered here

$$V(k, p) = x_{k+1}^{\mathrm{T}}(p) P_1 x_{k+1}(p) + y_k^{\mathrm{T}}(p) P_2 y_k(p) \tag{4}$$

with associated increment $\Delta V(k, p)$

$$\begin{aligned} \Delta V(k, p) = {} & x_{k+1}^{\mathrm{T}}(p+1) P_1 x_{k+1}(p+1) - x_{k+1}^{\mathrm{T}}(p) P_1 x_{k+1}(p) \\ & + y_{k+1}^{\mathrm{T}}(p) P_2 y_{k+1}(p) - y_k^{\mathrm{T}}(p) P_2 y_k(p), \end{aligned}$$

where $P_1 > 0$ and $P_2 > 0$. Then we have the following result.

Lemma 1. *A discrete linear repetitive process described by* (1) *is stable along the pass if*

$$\Delta V(k, p) < 0.$$

Proof. Introduce the vector

$$\zeta_k(p) = \begin{bmatrix} x_{k+1}(p) \\ y_k(p) \end{bmatrix} \tag{5}$$

and then the matrices defined in (2) can be used to rewrite the state-space model (1) (in the absence of the input and disturbances) as

$$\begin{bmatrix} x_{k+1}(p+1) \\ y_{k+1}(p) \end{bmatrix} = (A_1 + A_2)\zeta_k(p).$$

Hence

$$\Delta V(k, p) = \zeta_k^{\mathrm{T}}(p)\left(A_1^{\mathrm{T}} P A_1 + A_2^{\mathrm{T}} P A_2 - P\right)\zeta_k(p),$$

where $P = \mathrm{diag}\{P_1, P_2\}$. Now (for any $\begin{bmatrix}\zeta_k^{\mathrm{T}}(p) & w_{k+1}^{\mathrm{T}}(p)\end{bmatrix}^{\mathrm{T}} \neq 0$) $\Delta V(k, p) < 0$ requires that

$$A_1^{\mathrm{T}} P A_1 + A_2^{\mathrm{T}} P A_2 - P = \Phi^{\mathrm{T}} P \Phi - P < 0$$

and the proof is completed by using Theorem 1. □

Theorem 2. *A discrete linear repetitive process described by* (1) *is stable along the pass and has* $\mathscr{H}_\infty$ *disturbance attenuation* $\gamma > 0$ *if there exist matrices* $P_1 > 0$ *and* $P_2 > 0$ *such that the following LMI with* $P = \mathrm{diag}\{P_1, P_2\}$ *holds*

$$\begin{bmatrix} -P & P\Phi & PB_1 & 0 \\ \Phi^{\mathrm{T}} P & -P & 0 & C_2^{\mathrm{T}} \\ B_1^{\mathrm{T}} P & 0 & -\gamma^2 I & 0 \\ 0 & C_2 & 0 & -I \end{bmatrix} < 0,$$

where

$$C_2 = \begin{bmatrix} 0 & I \end{bmatrix}, \quad B_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix}.$$

Proof. It is easily shown that the $\mathscr{H}_\infty$ disturbance attenuation γ holds if the associated Hamiltonian defined by

$$H(k, p) = \Delta V(k, p) + y_{k+1}^{\mathrm{T}}(p) y_{k+1}(p) - \gamma^2 w_{k+1}^{\mathrm{T}}(p) w_{k+1}(p)$$

satisfies

$$H(k, p) < 0. \tag{6}$$

This requires that $\Delta V(k, p) < 0$ and hence by Lemma 1 stability along the pass.

Using the process state space model (1) with no input terms, it is easily shown that

$$H(k,p)=\begin{bmatrix}\zeta_k^{\mathrm T}(p) & w_{k+1}^{\mathrm T}(p)\end{bmatrix}\begin{bmatrix}\Phi^{\mathrm T}P\Phi-P+C_2^{\mathrm T}C_2 & \Phi^{\mathrm T}PB_1\\ B_1^{\mathrm T}P\Phi & B_1^{\mathrm T}PB_1-\gamma^2 I\end{bmatrix}\times\begin{bmatrix}\zeta_k(p)\\ w_{k+1}(p)\end{bmatrix}$$

and (6) holds (for any $\begin{bmatrix}\zeta_k^{\mathrm T}(p) & w_{k+1}^{\mathrm T}(p)\end{bmatrix}^{\mathrm T}\neq 0$) provided

$$\begin{bmatrix}\Phi^{\mathrm T}P\Phi-P+C_2^{\mathrm T}C_2 & \Phi^{\mathrm T}PB_1\\ B_1^{\mathrm T}P\Phi & B_1^{\mathrm T}PB_1-\gamma^2 I\end{bmatrix}<0.$$

Finally, an obvious application of the Schur's complement formula shows that this last condition is equivalent to (2) and the proof is complete. □

Remark 1. Consider the Roesser model with augmented plant matrix Φ. Then it is known that bounded-input bounded-output stability of this model is equivalent to stability along the pass of discrete linear repetitive processes described by (1) (in the disturbance free case). Hence an alternative proof of this last result is to follow the method in [4].

The next section of this paper will solve the disturbance rejection or attenuation problem which can be summarized as finding an implementable control law which will ensure stability along the pass of the controlled process together with a prescribed degree of disturbance rejection, including the case when there is uncertainty in the model structure—this problem has not been formulated or solved for 2D linear systems.

We will make extensive use of the following well known results throughout this paper.

Lemma 2 [11]. *Let Σ_1, Σ_2 be real matrices of compatible dimensions. Then for any matrix $\mathscr{F}$ satisfying $\mathscr{F}^{\mathrm T}\mathscr{F}\leqslant I$ and scalar $\epsilon>0$*

$$\Sigma_1\mathscr{F}\Sigma_2+\Sigma_2^{\mathrm T}\mathscr{F}^{\mathrm T}\Sigma_1^{\mathrm T}\leqslant\epsilon^{-1}\Sigma_1\Sigma_1^{\mathrm T}+\epsilon\Sigma_2^{\mathrm T}\Sigma_2.$$

Lemma 3 [7]. *Let Ψ be a $q\times q$ symmetric matrix. Also let P and Q be real matrices of dimensions $s\times q$ and $r\times q$ respectively. Then there exists an $r\times s$ matrix Θ such that*

$$\Psi+P^{\mathrm T}\Theta^{\mathrm T}Q+Q^{\mathrm T}\Theta P<0$$

if, and only if, the inequalities

$$\mathscr{N}_p^{\mathrm T}\Psi\mathscr{N}_p<0\quad\text{and}\quad\mathscr{N}_q^{\mathrm T}\Psi\mathscr{N}_q<0$$

hold, where $\mathscr{N}_p\in\ker(P)$ and $\mathscr{N}_q\in\ker(Q)$.

Lemma 4 [5]. *Suppose that the* $n \times n$ *matrices* $\Sigma > 0$ *and* $\Gamma > 0$ *are given and* n_c *is a positive integer. Then there exists* $n \times n_c$ *matrices* Σ_2, Γ_2 *and* $n_c \times n_c$ *symmetric matrices* Σ_3, *and* Γ_3, *such that*

$$\begin{bmatrix} \Sigma & \Sigma_2 \\ \Sigma_2^{\mathrm{T}} & \Sigma_3 \end{bmatrix} > 0 \quad \textit{and} \quad \begin{bmatrix} \Sigma & \Sigma_2 \\ \Sigma_2^{\mathrm{T}} & \Sigma_3 \end{bmatrix}^{-1} = \begin{bmatrix} \Gamma & \Gamma_2 \\ \Gamma_2^{\mathrm{T}} & \Gamma_3 \end{bmatrix}$$

if, and only if,

$$\begin{bmatrix} \Sigma & I \\ I & \Gamma \end{bmatrix} \geqslant 0.$$

3. $\mathscr{H}_\infty$ control of discrete repetitive processes

Their physical basis means that it is possible to define the current pass error for the processes considered here as the difference, at each point along the pass, between a specified reference trajectory for that pass, which in most cases will be the same on each pass, and the actual pass profile produced. Then it is possible to define a so-called current pass error actuated controller which uses the generated error vector to construct the current pass control input vector. In which context, preliminary work, see, for example, [16], has shown that, except in a few very restrictive special cases, the controller used must be actuated by a combination of current pass information and feedforward' information from the previous pass to guarantee even stability along the pass. Note also here that in the ILC application area the previous pass output vector (or trial in ILC terminology) is an obvious signal to use as feedforward action. Moreover, simple structure (proportional plus integral) control laws based on this approach have already been practically implemented on an experimental tested with highly promising results, e.g. [2]. Here we aim to provide control law design algorithms in a general setting with extension to the case of uncertainty in the model structure.

As summarized in the previous section, it is already known [8] that an LMI reformulation of the stability along the pass property enables the design of physically based control laws to be undertaken for stability along the pass. The control law considered in this previous work has the following form over $0 \leqslant p \leqslant \alpha$, $k \geqslant 0$

$$u_{k+1}(p) = K_1 x_{k+1}(p) + K_2 y_k(p) =: K \begin{bmatrix} x_{k+1}(p) \\ y_k(p) \end{bmatrix}, \tag{7}$$

where K_1 and K_2 are appropriately dimensioned matrices to be designed. In effect, this control law uses feedback of the current state vector (which is assumed to be available for use) and ‘feedforward’ of the previous pass profile vector. Note that in repetitive processes the term ‘feedforward’ is used to describe the case where state or pass profile information from the previous pass (or passes) is used as (part of) the input to a control law applied on the current pass, i.e. to information which is propagated in the pass-to-pass (k) direction.

The following result enables the control law (7) to be designed to give stability along the pass with a prescribed $\mathscr{H}_\infty$ disturbance attenuation level (γ).

Theorem 3. *Suppose that a control law of the form* (7) *is applied to a discrete linear repetitive process described by* (1). *Then the resulting process is stable along the pass with prescribed $\mathscr{H}_\infty$ disturbance attenuation $\gamma > 0$ if there exists matrices $W_1 > 0$, $W_2 > 0$, N_1 and N_2 such that the following LMI holds*

$$\begin{bmatrix} -W_1 & (\star) & (\star) & (\star) & (\star) & (\star) \\ 0 & -W_2 & (\star) & (\star) & (\star) & (\star) \\ W_1A^{\mathrm{T}} + N_1^{\mathrm{T}}B^{\mathrm{T}} & W_1C^{\mathrm{T}} + N_1^{\mathrm{T}}D^{\mathrm{T}} & -W_1 & (\star) & (\star) & (\star) \\ W_2B_0^{\mathrm{T}} + N_2^{\mathrm{T}}B^{\mathrm{T}} & W_2D_0^{\mathrm{T}} + N_2^{\mathrm{T}}D^{\mathrm{T}} & 0 & -W_2 & (\star) & (\star) \\ B_{11}^{\mathrm{T}} & B_{12}^{\mathrm{T}} & 0 & 0 & -\gamma^2 I & (\star) \\ 0 & 0 & 0 & W_2 & 0 & -I \end{bmatrix} < 0. \tag{8}$$

If this condition holds, the matrices in the control law (7) *are given by*

$$K_1 = N_1W_1^{-1}, \quad K_2 = N_2W_2^{-1}. \tag{9}$$

Proof. Applying the LMI of Theorem 2 to the resulting state space model, it follows immediately that stability along the pass with the control law applied holds if there exists matrices $P_1 > 0$ and $P_2 > 0$ such that

$$\begin{bmatrix} -P_1 & (\star) & (\star) & (\star) & (\star) & (\star) \\ 0 & -P_2 & (\star) & (\star) & (\star) & (\star) \\ A^{\mathrm{T}}P_1 + K_1^{\mathrm{T}}B^{\mathrm{T}}P_1 & C^{\mathrm{T}}P_2 + K_1^{\mathrm{T}}D^{\mathrm{T}}P_2 & -P_1 & (\star) & (\star) & (\star) \\ B_0^{\mathrm{T}}P_1 + K_2^{\mathrm{T}}B^{\mathrm{T}}P_1 & D_0^{\mathrm{T}}P_2 + K_2^{\mathrm{T}}D^{\mathrm{T}}P_2 & 0 & -P_2 & (\star) & (\star) \\ B_{11}^{\mathrm{T}}P_1 & B_{12}^{\mathrm{T}}P_2 & 0 & 0 & -\gamma^2 I & (\star) \\ 0 & 0 & 0 & I & 0 & -I \end{bmatrix} < 0.$$

This last inequality is not in LMI form because it is nonlinear with respect to its parameters. Consequently, set $P_1 = W_1^{-1}$, $P_2 = W_2^{-1}$ and then pre and post-multiply it by $\operatorname{diag}\{W_1, W_2, W_1, W_2, I, I\}$ to obtain

$$\begin{bmatrix} -W_1 & (\star) & (\star) & (\star) & (\star) & (\star) \\ 0 & -W_2 & (\star) & (\star) & (\star) & (\star) \\ W_1A^{\mathrm{T}} + W_1K_1^{\mathrm{T}}B^{\mathrm{T}} & W_1C^{\mathrm{T}} + W_1K_1^{\mathrm{T}}D^{\mathrm{T}} & -W_1 & (\star) & (\star) & (\star) \\ W_2B_0^{\mathrm{T}} + W_2K_2^{\mathrm{T}}B^{\mathrm{T}} & W_2D_0^{\mathrm{T}} + W_2K_2^{\mathrm{T}}D^{\mathrm{T}} & 0 & -W_2 & (\star) & (\star) \\ B_{11}^{\mathrm{T}} & B_{12}^{\mathrm{T}} & 0 & 0 & -\gamma^2 I & (\star) \\ 0 & 0 & 0 & W_2 & 0 & -I \end{bmatrix} < 0.$$

Now set $N_1 = K_1W_1$ and $N_2 = K_2W_2$ in this last expression to obtain the LMI of (8) and the proof is complete. □

Note that the $\mathscr{H}_\infty$ disturbance attenuation here can be minimized using the following EVP procedure (see, for example, [3]) which leads to minimization of the effects of the disturbance vector.

$$\min_{W_1>0, W_2>0, N_1, N_2} \mu, \quad \text{subject to (8) with } \mu = \gamma^2.$$

Next we extend the above analysis to the case of robust $\mathscr{H}_\infty$ control.

Consider a linear repetitive process of the form (1) with uncertainty modelled as additive perturbations to the nominal model matrices, resulting in the state space model

$$\begin{aligned} x_{k+1}(p+1) &= (A+\Delta A)x_{k+1}(p) + (B+\Delta B)u_{k+1}(p) \\ &\quad + (B_0+\Delta B_0)y_k(p) + (B_{11}+\Delta B_{11})w_{k+1}(p), \\ y_{k+1}(p) &= (C+\Delta C)x_{k+1}(p) + (D+\Delta D)u_{k+1}(p) \\ &\quad + (D_0+\Delta D_0)y_k(p) + (B_{12}+\Delta B_{12})w_{k+1}(p). \end{aligned} \tag{10}$$

The matrices ΔA, ΔB, ΔB_0, ΔB_{11}, ΔC, ΔD, ΔD_0, ΔB_{12} represent admissible uncertainties which are assumed to satisfy

$$\begin{bmatrix} \Delta A & \Delta B_0 & \Delta B_{11} & \Delta B \\ \Delta C & \Delta D_0 & \Delta B_{12} & \Delta D \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \mathscr{F} \begin{bmatrix} E_1 & E_2 & E_3 & E_4 \end{bmatrix}, \tag{11}$$

where H_1, H_2, E_1, E_2, E_3, E_4 are some known constant matrices with compatible dimensions and $\mathscr{F}$ is an unknown constant matrix which satisfies

$$\mathscr{F}^{\mathrm{T}}\mathscr{F} \leqslant I. \tag{12}$$

Now we have the following result.

Theorem 4. *Suppose that a control law defined by* (7) *is applied to discrete linear repetitive process described by* (10) *with the uncertainty structure satisfying* (11) *and* (12). *Then the resulting process is stable along the pass with the prescribed* $\mathscr{H}_\infty$ *disturbance attenuation* $\gamma > 0$ *if there exists a scalar* $\epsilon > 0$ *and matrices* $W_1 > 0$, $W_2 > 0$, *and* N_1, N_2 *such that the following LMI holds*

$$\left[\begin{matrix} -W_1 + 3\epsilon H_1 H_1^{\mathrm{T}} & (\star) & (\star) & (\star) \\ 3\epsilon H_1 H_2^{\mathrm{T}} & -W_2 + 3\epsilon H_2 H_2^{\mathrm{T}} & (\star) & (\star) \\ W_1 A^{\mathrm{T}} + N_1^{\mathrm{T}} B^{\mathrm{T}} & W_1 C^{\mathrm{T}} + N_1^{\mathrm{T}} D^{\mathrm{T}} & -W_1 & (\star) \\ W_2 B_0^{\mathrm{T}} + N_2^{\mathrm{T}} B^{\mathrm{T}} & W_2 D_0^{\mathrm{T}} + N_2^{\mathrm{T}} D^{\mathrm{T}} & 0 & -W_2 \\ B_{11}^{\mathrm{T}} & B_{12}^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & 0 & W_2 \\ 0 & 0 & 0 & W_1 E_1^{\mathrm{T}} + N_1^{\mathrm{T}} E_4^{\mathrm{T}} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{matrix}\right.$$

$$\left.\begin{matrix} (\star) & (\star) & (\star) & (\star) & (\star) \\ (\star) & (\star) & (\star) & (\star) & (\star) \\ (\star) & (\star) & (\star) & (\star) & (\star) \\ (\star) & (\star) & (\star) & (\star) & (\star) \\ -\gamma^2 I & (\star) & (\star) & (\star) & (\star) \\ 0 & -I & (\star) & (\star) & (\star) \\ 0 & 0 & -\epsilon I & (\star) & (\star) \\ W_2E_2^{\mathrm{T}} + N_2^{\mathrm{T}}E_4^{\mathrm{T}} & 0 & 0 & -\epsilon I & (\star) \\ 0 & E_3^{\mathrm{T}} & 0 & 0 & -\epsilon I \end{matrix}\right] < 0. \tag{13}$$

If this condition holds, the corresponding control law matrices are given by (9).

Proof. With the control law applied, stability along the pass can be expressed as the requirement that

$$\Gamma + \widetilde{H}\widetilde{F}\widetilde{E} + \widetilde{E}^{\mathrm{T}}\widetilde{F}^{\mathrm{T}}\widetilde{H}^{\mathrm{T}} < 0,$$

where

$$\Gamma = \begin{bmatrix} -W_1 & (\star) & (\star) & (\star) & (\star) & (\star) \\ 0 & -W_2 & (\star) & (\star) & (\star) & (\star) \\ W_1A^{\mathrm{T}} + N_1^{\mathrm{T}}B^{\mathrm{T}} & W_1C^{\mathrm{T}} + N_1^{\mathrm{T}}D^{\mathrm{T}} & -W_1 & (\star) & (\star) & (\star) \\ W_2B_0^{\mathrm{T}} + N_2^{\mathrm{T}}B^{\mathrm{T}} & W_2D_0^{\mathrm{T}} + N_2^{\mathrm{T}}D^{\mathrm{T}} & 0 & -W_2 & (\star) & (\star) \\ B_{11}^{\mathrm{T}} & B_{12}^{\mathrm{T}} & 0 & 0 & -\gamma^2 I & (\star) \\ 0 & 0 & 0 & W_2 & 0 & -I \end{bmatrix},$$

$$\widetilde{H} = \begin{bmatrix} 0 & 0 & H_1 & H_1 & H_1 & 0 \\ 0 & 0 & H_2 & H_2 & H_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\widetilde{F} = \mathrm{diag}\{\mathscr{F}, \mathscr{F}, \mathscr{F}, \mathscr{F}, \mathscr{F}, \mathscr{F}\},$$
$$\widetilde{E} = \mathrm{diag}\{0, 0, E_1W_1 + E_4N_1, E_2W_2 + E_4N_2, E_3, 0\}.$$

The LMI of (13) is now obtained by an application of the inequality of Lemma 2 followed by an obvious application of the Schur's complement formula. □

To reduce the effects of the disturbance vector, the following linear objective minimization problem can be used

$$\min_{W_1>0,\, W_2>0,\, \epsilon>0,\, N_1,\, N_2} \mu,$$
subject to (13) with $\mu = \gamma^2$.

Consider now the case when the uncertainty in the process state space model is of the additive structure defined above but the disturbance terms are absent. Then on applying the control law (7), the resulting process can be written in the form

$$\begin{bmatrix} x_{k+1}(p+1) \\ y_{k+1}(p) \end{bmatrix} = \overline{A} \begin{bmatrix} x_{k+1}(p) \\ y_k(p) \end{bmatrix}, \tag{14}$$

where

$$\overline{A} = \begin{bmatrix} A + BK_1 & B_0 + BK_2 \\ C + DK_1 & D_0 + DK_2 \end{bmatrix} + \begin{bmatrix} \Delta A + \Delta BK_1 & \Delta B_0 + \Delta BK_2 \\ \Delta C + \Delta DK_1 & \Delta D_0 + \Delta DK_2 \end{bmatrix}.$$

Suppose also that the matrices describing the uncertainty in this last model can be written in the form

$$\begin{aligned} &\begin{bmatrix} \Delta A + \Delta BK_1 & \Delta B_0 + \Delta BK_2 \\ \Delta C + \Delta DK_1 & \Delta D_0 + \Delta DK_2 \end{bmatrix} \\ &\quad = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \gamma^{-1} \mathscr{F} \begin{bmatrix} E_1 + E_4K_1 & E_2 + E_4K_2 \end{bmatrix} = \gamma^{-1} H \mathscr{F} E, \end{aligned} \tag{15}$$

where the matrices H_1, H_2, E_1, E_2, E_4 have known constant entries, $\gamma > 0$ is a given scalar, and the matrix $\mathscr{F}$ satisfies (12). Moreover, the design parameter γ can be used to attenuate the effects of the uncertainty via following result.

Theorem 5. *Suppose that a control law defined by* (7) *is applied to a discrete linear repetitive process described by* (10) *with the uncertainty structure satisfying* (15) *and* (12). *Then the resulting process is stable along the pass if there exist matrices* $W_1 > 0$, $W_2 > 0$ *and* N_1, N_2 *such that the following LMI holds*

$$\begin{bmatrix} -W_1 & (\star) & (\star) & (\star) & (\star) & (\star) \\ 0 & -W_2 & (\star) & (\star) & (\star) & (\star) \\ W_1A^{\mathrm{T}} + N_1^{\mathrm{T}}B^{\mathrm{T}} & W_1C^{\mathrm{T}} + N_1^{\mathrm{T}}D^{\mathrm{T}} & -W_1 & (\star) & (\star) & (\star) \\ W_2B_0^{\mathrm{T}} + N_2^{\mathrm{T}}B^{\mathrm{T}} & W_2D_0^{\mathrm{T}} + N_2^{\mathrm{T}}D^{\mathrm{T}} & 0 & -W_2 & (\star) & (\star) \\ H_1^{\mathrm{T}} & H_2^{\mathrm{T}} & 0 & 0 & -\gamma^2 I & (\star) \\ 0 & 0 & E_1W_1 + E_4N_1 & E_2W_2 + E_4N_2 & 0 & -I \end{bmatrix} < 0. \tag{16}$$

If this condition holds then the stabilizing matrices K_1 *and* K_2 *in the control law* (7) *are again given by* (9).

Proof. Applying the result of Theorem 1 to the state space model resulting from application of the control law, it follows immediately that stability along the pass holds if there exists a block-diagonal matrix $P = \operatorname{diag}\{P_1, P_2\} > 0$ such that the following LMI holds

$$\overline{A}^{\mathrm{T}} P \overline{A} - P < 0.$$

An obvious application of the Schur's complement formula now yields

$$\begin{bmatrix} -P^{-1} & (\Omega + \gamma^{-1} HFE) \\ (\Omega + \gamma^{-1} HFE)^{\mathrm{T}} & -P \end{bmatrix} < 0,$$

where

$$\Omega = \begin{bmatrix} A + BK_1 & B_0 + BK_2 \\ C + DK_1 & D_0 + DK_2 \end{bmatrix}.$$

Applying the result of Lemma 2 to this last condition and then pre- and post-multiplying the result by $\operatorname{diag}\{\epsilon^{-\frac{1}{2}} P, \epsilon^{-\frac{1}{2}} I\}$ yields

$$\begin{bmatrix} -\overline{P} + \overline{P}\gamma^{-2} HH^{\mathrm{T}}\overline{P} & \overline{P}\Omega \\ \Omega^{\mathrm{T}}\overline{P} & -\overline{P} + E^{\mathrm{T}}E \end{bmatrix} < 0,$$

where $\overline{P} = \epsilon^{-1} P$. Finally another obvious application of the Schur's complement formula gives the following LMI

$$\begin{bmatrix} -\overline{P} & \overline{P}\Omega & \overline{P}H & 0 \\ \Omega^{\mathrm{T}}\overline{P} & -\overline{P} & 0 & E^{\mathrm{T}} \\ H^{\mathrm{T}}\overline{P} & 0 & -\gamma^2 I & 0 \\ 0 & E & 0 & -I \end{bmatrix} < 0.$$

The proof is now completed in an identical manner to that of Theorem 2. □

Note also that the parameter γ in this last result can be minimized using the following linear objective minimization procedure and leads to increased robustness.

$$\min_{W_1>0, W_2>0, N_1, N_2} \mu,$$
$$\text{subject to (16)} \quad \text{with } \mu = \gamma^2.$$

At this stage, some comments on the relationship with Roesser model analysis can be made. The first point is that for repetitive processes the static state control law applied here is well defined physically as at least the pass profile vector, which can be considered as a vertically transmitted state sub-vector in the Roesser model interpretation of the process dynamics, is also a process output and hence can be directly measured. Hence this static control law has structure for discrete linear repetitive processes alone which, as the analysis here shows, can be exploited to powerful effect. As in 1D systems theory, there will be cases when all elements in the current pass state vector cannot be directly measured. If this is the case then one option is to use the dynamic output controller of the next section, where again the structure of the process dynamics (and, in particular, the 2D transfer function matrix $G_{yw}(z_1, z_2)$, which arises directly from the underlying dynamics of these processes (as opposed to an assumption made)) allows us to obtain, relative to Roesser model analysis, simpler and hence more effective results. Note also that it should be possible to replace the current pass state vector in the control law here with the current pass profile vector—see [17] where this problem is solved for the problem of computing a control law to ensure stability along the pass with the control law applied.

4. $\mathscr{H}_\infty$ control with a full dynamic pass profile controller

The control law of the previous section requires that the complete current pass state vector is available for measurement. If this is not the case then one option is to use an observer to reconstruct it. In this section, we consider an alternative of controlling processes described by (1) through use of a so-called full dynamic pass profile controller (with state dimension $n_c = n + m$) defined as

$$\begin{aligned} \begin{bmatrix} x^c_{k+1}(p+1) \\ y^c_{k+1}(p) \end{bmatrix} &= \begin{bmatrix} A_{c11} & A_{c12} \\ A_{c21} & A_{c22} \end{bmatrix} \begin{bmatrix} x^c_{k+1}(p) \\ y^c_k(p) \end{bmatrix} + \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix} y_{k+1}(p), \\ u_{k+1}(p) &= \begin{bmatrix} C_{c1} & C_{c2} \end{bmatrix} \begin{bmatrix} x^c_{k+1}(p) \\ y^c_k(p) \end{bmatrix} + D_c y_{k+1}(p), \end{aligned} \tag{17}$$

where $x^c_k(p)$ and $y^c_k(p)$ denote state vectors for the controller.

To obtain the state space model describing the result of applying the controller, introduce the extra notation

$$B_2 = \begin{bmatrix} B \\ D \end{bmatrix}, \quad A_c = \begin{bmatrix} A_{c11} & A_{c12} \\ A_{c21} & A_{c22} \end{bmatrix}, \quad B_c = \begin{bmatrix} B_{c1} \\ B_{c2} \end{bmatrix}, \quad C_c = \begin{bmatrix} C_{c1} & C_{c2} \end{bmatrix}.$$

Also introduce the so-called augmented state and pass profile vectors as

$$\bar{x}_{k+1}(p) = \begin{bmatrix} x_{k+1}(p) \\ x^c_{k+1}(p) \end{bmatrix}, \quad \bar{y}_k(p) = \begin{bmatrix} y_k(p) \\ y^c_k(p) \end{bmatrix}.$$

Then we have

$$\begin{aligned} \begin{bmatrix} \bar{x}_{k+1}(p+1) \\ \bar{y}_{k+1}(p) \end{bmatrix} &= \overline{A} \begin{bmatrix} \bar{x}_{k+1}(p) \\ \bar{y}_k(p) \end{bmatrix} + \overline{B} w_k(p), \\ y_{k+1}(p) &= \overline{C} \begin{bmatrix} \bar{x}_{k+1}(p) \\ \bar{y}_k(p) \end{bmatrix}, \end{aligned}$$

where

$$\overline{A} = \Pi \begin{bmatrix} \Phi + B_2 D_c C_2 & B_2 C_c \\ B_c C_2 & A_c \end{bmatrix} \Pi, \quad \overline{B} = \Pi \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \overline{C} = \begin{bmatrix} C_2 & 0 \end{bmatrix} \Pi,$$

$$\Pi = \begin{bmatrix} I & 0 & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & 0 & I \end{bmatrix}$$

and also $\Pi = \Pi^{\mathrm{T}} = \Pi^{-1}$.

Introduce the so-called matrix of controller data as

$$\Theta = \begin{bmatrix} D_c & C_c \\ B_c & A_c \end{bmatrix}$$

and

$$\mathscr{A} = \begin{bmatrix} \Phi & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathscr{B}_2 = \begin{bmatrix} B_2 & 0 \\ 0 & I \end{bmatrix}, \quad \mathscr{C}_2 = \begin{bmatrix} C_2 & 0 \\ 0 & I \end{bmatrix},$$
$$\mathscr{C} = \begin{bmatrix} C_2 & 0 \end{bmatrix}, \quad \mathscr{B} = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}.$$

Then the state space model matrices considered here can be written in the following form which is affine in the controller data matrix Θ

$$\overline{A} = \Pi[\mathscr{A} + \mathscr{B}_2\Theta\mathscr{C}_2]\Pi, \quad \overline{C} = \mathscr{C}\Pi, \quad \overline{B} = \Pi\mathscr{B} \tag{18}$$

Now we have the following result which gives an existence condition for the controller matrices A_c, B_c, C_c, D_c to ensure stability along the pass and then enables controller design.

Theorem 6. *Suppose that a full dynamic pass profile controller defined by* (17) *is applied to a discrete linear repetitive process described by* (1). *Suppose also that there exist matrices* $P_{11} > 0$, $(P_{11} = \mathrm{diag}\{P_{h11}, P_{v11}\})$ *and* $R_{11} > 0$, $(R_{11} = \mathrm{diag}\{R_{h11}, R_{v11}\})$ *such that the LMIs defined by* (19)–(21) *below hold. Then the resulting process is stable along the pass and has* $\mathscr{H}_\infty$ *disturbance attenuation* $\gamma > 0$.

$$\begin{bmatrix} \mathscr{N}_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Phi R_{11}\Phi^{\mathrm{T}} - R_{11} & B_1 & \Phi R_{11} C_2^{\mathrm{T}} \\ B_1^{\mathrm{T}} & -\gamma^2 I & 0 \\ C_2 R_{11}\Phi^{\mathrm{T}} & 0 & -I + C_2 R_{11} C_2^{\mathrm{T}} \end{bmatrix}$$
$$\times \begin{bmatrix} \mathscr{N}_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0, \tag{19}$$

$$\begin{bmatrix} \mathscr{N}_2 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Phi^{\mathrm{T}} P_{11}\Phi - P_{11} & \Phi^{\mathrm{T}} P_{11} B_1 & C_2^{\mathrm{T}} \\ B_1^{\mathrm{T}} P_{11}\Phi & B_1^{\mathrm{T}} P_{11} B_1 - \gamma^2 I & 0 \\ C_2 & 0 & -I \end{bmatrix}$$
$$\times \begin{bmatrix} \mathscr{N}_2 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0, \tag{20}$$

$$\begin{bmatrix} P_{h11} & I \\ I & R_{h11} \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} P_{v11} & I \\ I & R_{v11} \end{bmatrix} \geqslant 0, \tag{21}$$

where $\mathscr{N}_1$ *and* $\mathscr{N}_2$ *are full column rank matrices whose images satisfy* $\mathrm{Im}\mathscr{N}_1 = \ker(B_2^{\mathrm{T}})$ *and* $\mathrm{Im}\mathscr{N}_2 = \ker(C_2)$ *respectively.*

Proof. Interpreting the result of Theorem 2 in terms of the matrices given in (18) yields

$$\begin{bmatrix} -P & (\bigstar) & (\bigstar) & (\bigstar) \\ \Pi\mathscr{A}^{\mathrm{T}}\Pi P + \Pi\mathscr{C}_2^{\mathrm{T}}\Theta^{\mathrm{T}}\mathscr{B}_2^{\mathrm{T}}\Pi P & -P & (\bigstar) & (\bigstar) \\ \mathscr{B}^{\mathrm{T}}\Pi P & 0 & -\gamma^2 I & (\bigstar) \\ 0 & \mathscr{C}\Pi & 0 & -I \end{bmatrix} < 0, \tag{22}$$

where $P = \mathrm{diag}\{P_h, P_v\}$. Next, pre and post-multiply (22) by $\mathrm{diag}\{\Pi, \Pi, I, I\}$ and then set $R = \Pi P \Pi$ to obtain

$$\begin{bmatrix} -R & (\bigstar) & (\bigstar) & (\bigstar) \\ \mathscr{A}^{\mathrm{T}}R + \mathscr{C}_2^{\mathrm{T}}\Theta^{\mathrm{T}}\mathscr{B}_2^{\mathrm{T}}R & -R & (\bigstar) & (\bigstar) \\ \mathscr{B}^{\mathrm{T}}R & 0 & -\gamma^2 I & (\bigstar) \\ 0 & \mathscr{C} & 0 & -I \end{bmatrix} < 0. \tag{23}$$

Now, define the matrices

$$\Psi = \begin{bmatrix} -R & R\mathscr{A} & R\mathscr{B} & 0 \\ \mathscr{A}^{\mathrm{T}}R & -R & 0 & \mathscr{C}^{\mathrm{T}} \\ \mathscr{B}^{\mathrm{T}}R & 0 & -\gamma^2 I & 0 \\ 0 & \mathscr{C} & 0 & -I \end{bmatrix}, \quad M^{\mathrm{T}} = \begin{bmatrix} R\mathscr{B}_2 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$
$$N = \begin{bmatrix} 0 & \mathscr{C}_2 & 0 & 0 \end{bmatrix}$$

to re-write (23) in the form

$$\Psi + M^{\mathrm{T}}\Theta N + N^{\mathrm{T}}\Theta^{\mathrm{T}} M < 0. \tag{24}$$

Use of Lemma 3 now yields the following two matrix inequalities which are equivalent to (24)

$$W_M^{\mathrm{T}}\Psi W_M < 0 \quad \text{and} \quad W_N^{\mathrm{T}}\Psi W_N < 0,$$

where

$$W_M \in \ker(M), \quad W_N \in \ker(N) \tag{25}$$

and

$$M = M_n S = \begin{bmatrix} \mathscr{B}_2^{\mathrm{T}} & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R & 0 & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

Note now that

$$\ker(M_n S) = S^{-1}\ker(M_n)$$

and using (25) yields

$$W_M = S^{-1} W_{M_n}.$$

Therefore

$$W_M^{\mathrm{T}} \Psi W_M < 0 \Leftrightarrow W_{M_n}^{\mathrm{T}} S^{-\mathrm{T}} \Psi S^{-1} W_{M_n} < 0$$

and also

$$\widetilde{\Phi} = S^{-\mathrm{T}} \Psi S^{-1} = \begin{bmatrix} -R^{-1} & \mathscr{A} & \mathscr{B} & 0 \\ \mathscr{A}^{\mathrm{T}} & -R & 0 & \mathscr{C}^{\mathrm{T}} \\ \mathscr{B}^{\mathrm{T}} & 0 & -\gamma^2 I & 0 \\ 0 & \mathscr{C} & 0 & -I \end{bmatrix}$$

and we have the following two inequalities which are not in LMI form

$$W_{M_n}^{\mathrm{T}} \widetilde{\Phi} W_{M_n} < 0 \quad \text{and} \quad W_N^{\mathrm{T}} \Psi W_N < 0$$

To obtain these inequalities in the required LMI form, first write the defining matrices out in full, i.e.

$$M_n = \begin{bmatrix} \mathscr{B}_2^{\mathrm{T}} & 0 & 0 & 0 \end{bmatrix} = \left[\begin{array}{cc|cc|c|c} B_2^{\mathrm{T}} & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \end{array}\right],$$

$$N = \begin{bmatrix} 0 & \mathscr{C}_2 & 0 & 0 \end{bmatrix} = \left[\begin{array}{cc|cc|c|c} 0 & 0 & C_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \end{array}\right].$$

Also it is easily seen that the kernels of M_n and N are images of

$$W_{M_n} = \begin{bmatrix} \mathscr{N}_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix}, \quad W_N = \begin{bmatrix} I & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & \mathscr{N}_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

where $\mathscr{N}_1 \in \ker(B_2^{\mathrm{T}})$ and $\mathscr{N}_2 \in \ker(C_2)$. Now, rewrite the matrix R as

$$R = \Pi P \Pi = \left[\begin{array}{cc|cc} P_{h_{11}} & 0 & P_{h_{12}} & 0 \\ 0 & P_{v_{11}} & 0 & P_{v_{12}} \\ \hline P_{h_{12}}^{\mathrm{T}} & 0 & P_{h_{22}} & 0 \\ 0 & P_{v_{12}}^{\mathrm{T}} & 0 & P_{v_{22}} \end{array}\right] = \left[\begin{array}{c|c} P_{11} & P_{12} \\ \hline P_{12}^{\mathrm{T}} & P_{22} \end{array}\right], \tag{26}$$

where

$$P_h = \begin{bmatrix} P_{h_{11}} & P_{h_{12}} \\ P_{h_{12}}^{\mathrm{T}} & P_{h_{22}} \end{bmatrix}, \quad P_v = \begin{bmatrix} P_{v_{11}} & P_{v_{12}} \\ P_{v_{12}}^{\mathrm{T}} & P_{v_{22}} \end{bmatrix}$$

and note also that

$$P_h^{-1} = \begin{bmatrix} R_{h_{11}} & R_{h_{12}} \\ R_{h_{12}}^{\mathrm{T}} & R_{h_{22}} \end{bmatrix}, \quad P_v^{-1} = \begin{bmatrix} R_{v_{11}} & R_{v_{12}} \\ R_{v_{12}}^{\mathrm{T}} & R_{v_{22}} \end{bmatrix}$$

and

$$R^{-1} = \Pi P^{-1}\Pi = \left[\begin{array}{cc|cc} R_{h_{11}} & 0 & R_{h_{12}} & 0 \\ 0 & R_{v_{11}} & 0 & R_{v_{12}} \\ \hline R_{h_{12}}^{\mathrm{T}} & 0 & R_{h_{22}} & 0 \\ 0 & R_{v_{12}}^{\mathrm{T}} & 0 & R_{v_{22}} \end{array}\right] = \left[\begin{array}{c|c} R_{11} & R_{12} \\ \hline R_{12}^{\mathrm{T}} & R_{22} \end{array}\right].$$

Hence the matrix $\widetilde{\Phi}$ can be rewritten as

$$\widetilde{\Phi} = \begin{bmatrix} -R_{11} & -R_{12} & \Phi & 0 & B_1 & 0 \\ -R_{12}^{\mathrm{T}} & -R_{12} & 0 & 0 & 0 & 0 \\ \Phi^{\mathrm{T}} & 0 & -P_{11} & -P_{12} & 0 & C_2^{\mathrm{T}} \\ 0 & 0 & -P_{12}^{\mathrm{T}} & -P_{22} & 0 & 0 \\ B_1^{\mathrm{T}} & 0 & 0 & 0 & -\gamma^2 I & 0 \\ 0 & 0 & C_2 & 0 & 0 & -I \end{bmatrix}$$

and on using the inequality

$$W_{M_n}^{\mathrm{T}}\widetilde{\Phi}W_{M_n} < 0,$$

we have (note that the second block row of W_{M_n} is zero)

$$\Upsilon^{\mathrm{T}} \left[\begin{array}{c|cc|cc} -R_{11} & \Phi & 0 & B_1 & 0 \\ \hline \Phi^{\mathrm{T}} & -P_{11} & -P_{12} & 0 & C_2^{\mathrm{T}} \\ 0 & -P_{12}^{\mathrm{T}} & -P_{22} & 0 & 0 \\ \hline B_1^{\mathrm{T}} & 0 & 0 & -\gamma^2 I & 0 \\ 0 & C_2 & 0 & 0 & -I \end{array}\right] \Upsilon < 0,$$

where $\Upsilon = \mathrm{diag}\{\mathcal{N}_1, I, I, I, I\}$. Next, by an obvious application of the Schur's complement formula, the LMI of (19) is obtained.

In order to obtain the inequality (20), rewrite the matrix Ψ as

$$\Psi = \begin{bmatrix} -P_{11} & (\star) & (\star) & (\star) & (\star) & (\star) \\ -P_{12}^{\mathrm{T}} & -P_{22} & (\star) & (\star) & (\star) & (\star) \\ \Phi^{\mathrm{T}}P_{11} & \Phi^{\mathrm{T}}P_{12} & -P_{11} & (\star) & (\star) & (\star) \\ 0 & 0 & -P_{12}^{\mathrm{T}} & -P_{22} & (\star) & (\star) \\ B_1^{\mathrm{T}}P_{11} & B_1^{\mathrm{T}}P_{12} & 0 & 0 & -\gamma^2 I & (\star) \\ 0 & 0 & C_2 & 0 & 0 & -I \end{bmatrix}.$$

By an obvious application of the Schur's complement formula, the inequality

$$W_N^{\mathrm{T}}\Psi W_N < 0$$

becomes

$$\begin{bmatrix} \mathcal{N}_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Phi^{\mathrm{T}}P_{11}\Phi - P_{11} & -P_{12} & \Phi^{\mathrm{T}}P_{11}B_1 & C_2^{\mathrm{T}} \\ -P_{12}^{\mathrm{T}} & -P_{22} & 0 & 0 \\ B_1^{\mathrm{T}}P_{11}\Phi & 0 & B_1^{\mathrm{T}}P_{11}B_1 - \gamma^2 I & 0 \\ C_2 & 0 & 0 & -I \end{bmatrix} \times \begin{bmatrix} \mathcal{N}_2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0,$$

which is equivalent to (20).

The last requirement here is to obtain the conditions which allow us to find the matrix P and its inverse. To begin, first note again that $P = \text{diag}\{P_h, P_v\}$ and that only P_{11} and R_{11} appear in the first two LMIs to be satisfied. Application of Lemma 4 now gives the required conditions. □

If this last result holds then the stabilizing control law can be designed using the following algorithm

1. Compute the matrices P_{h12}, P_{v12} using
$$P_{h11} - R_{h11}^{-1} = P_{h12}P_{h22}^{-1}P_{h12}^{\mathrm{T}},$$
$$P_{v11} - R_{v11}^{-1} = P_{v12}P_{v22}^{-1}P_{v12}^{\mathrm{T}},$$
where it is assumed that $P_{h22} = I$ and $P_{v22} = I$.
2. Construct $P_h > 0$ and $P_v > 0$ and then the matrix $P = \text{diag}\{P_h, P_v\}$.
3. Compute the matrices M, N and Ψ.
4. Solve the LMI (24) (where Θ is the unknown matrix) and hence obtain the controller state space model matrices.

The attenuation level γ can be minimized using the following optimization procedure

$$\min_{P_{11}>0, R_{11}>0} \mu,$$
subject to (19)–(21) with $\mu = \gamma^2$.

As a numerical example, consider the process whose state space model is defined by the following matrices

$$A = \begin{bmatrix} 0.4 & 0.2 \\ 1.1 & 0.1 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0.3 & 0.3 \\ 0.4 & 0.9 \end{bmatrix},$$
$$C = \begin{bmatrix} 0.7 & 0.4 \\ 0.2 & 0.1 \end{bmatrix}, \quad D_0 = \begin{bmatrix} 0.6 & 0.6 \\ 0.9 & 0.9 \end{bmatrix},$$

$$B = \begin{bmatrix} 1.2 \\ 1.1 \end{bmatrix}, \quad D = \begin{bmatrix} 3.0 \\ 1.7 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 0.1 \\ 0.2 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0.2 \\ 0.3 \end{bmatrix}.$$

This example is easily shown to be unstable along the pass—as confirmed by the simulation results of Fig. 1, where the left hand plot corresponds to the first entry in the pass profile vector and that on the right the second, with the following boundary conditions

$$x_{k+1}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad y_0(p) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad 0 \leqslant p \leqslant 20.$$

Using the design procedure of the last result gives an $\mathscr{H}_\infty$ disturbance attenuation $\gamma = 2.2153$ and the matrices

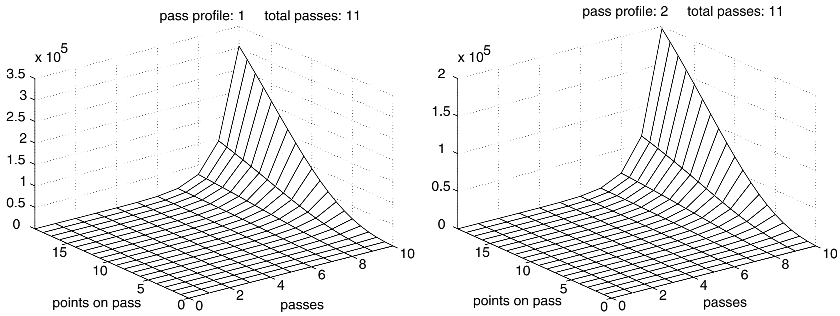

Fig. 1. The open loop response.

$$P_h = 10^3 \times \begin{bmatrix} 1.7653 & -1.0543 & 0.0420 & 0 \\ -1.0543 & 0.6343 & -0.0251 & 0.0019 \\ 0.0420 & -0.0251 & 0.0010 & 0 \\ 0 & 0.0019 & 0 & 0.0010 \end{bmatrix},$$

$$P_v = \begin{bmatrix} 4.5436 & -7.4567 & 1.8072 & 0 \\ -7.4567 & 19.6458 & -4.2781 & 0.0837 \\ 1.8072 & -4.2781 & 1.0000 & 0 \\ 0 & 0.0837 & 0 & 1.0000 \end{bmatrix}.$$

Hence the full dynamic pass profile controller is defined by the matrices

$$A_c = \begin{bmatrix} -0.3693 & -3.0691 & 0.0420 & -0.0084 \\ 0.0305 & 0.3044 & 0.0074 & -0.0014 \\ 0.0145 & 0.4187 & 0.0222 & -0.0044 \\ -0.0003 & -0.0081 & -0.0008 & 0.0001 \end{bmatrix},$$

$$B_c = \begin{bmatrix} 4.1204 & 16.8363 \\ -0.1022 & -1.0908 \\ 2.3073 & 2.1374 \\ -0.0357 & -0.0283 \end{bmatrix},$$

$$C_c = \begin{bmatrix} 0.0050 & 0.1238 & -0.0014 & 0.0003 \end{bmatrix},$$
$$D_c = \begin{bmatrix} -0.2938 & -0.3040 \end{bmatrix}.$$

The plots in Fig. 2 (where the left hand plot corresponds to the first entry in the pass profile vector and that on the right the second) confirm that the controlled process is stable along the pass. Suppose also that the interest is in the level of disturbance rejection. Then one means of studying this is to examine the 2D frequency response (recall the discussion of the 2D transfer function matrix in Section 2) between the disturbance and pass profile with the control law applied and Fig. 3 (1st channel on

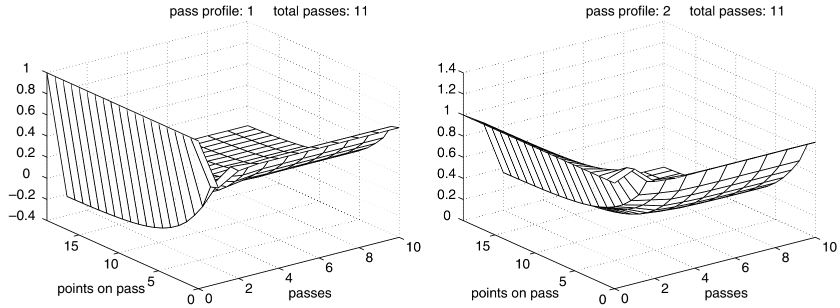

Fig. 2. The controlled response.

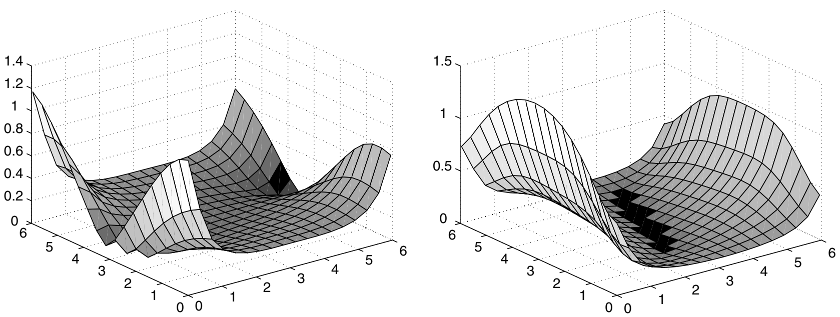

Fig. 3. The 2D frequency responses.

the left, 2nd on the right) shows the resulting plots under zero boundary conditions. The maximum values are 1.4746 and 1.2013 respectively, which are both below the computed $\mathscr{H}_\infty$ attenuation level.

To design a full dynamic pass profile controller in the presence of the uncertainty structure of the previous section, consider the defining state space model written in the form

$$\begin{bmatrix} x_{k+1}(p+1) \\ y_{k+1}(p) \end{bmatrix} = \left(\begin{bmatrix} A & B_0 \\ C & D_0 \end{bmatrix} + \begin{bmatrix} \Delta A & \Delta B_0 \\ \Delta C & \Delta D_0 \end{bmatrix} \right) \begin{bmatrix} x_{k+1}(p) \\ y_k(p) \end{bmatrix} + \left(\begin{bmatrix} B \\ D \end{bmatrix} + \begin{bmatrix} \Delta B \\ \Delta D \end{bmatrix} \right) u_{k+1}(p). \tag{27}$$

To simplify notation, introduce the so-called uncertain augmented process and input matrices respectively for this model as

$$\Delta\Phi = \begin{bmatrix} \Delta A & \Delta B_0 \\ \Delta C & \Delta D_0 \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \gamma^{-1} \mathscr{F} \begin{bmatrix} E_1 & E_2 \end{bmatrix},$$

$$\Delta\Lambda = \begin{bmatrix} \Delta B \\ \Delta D \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \gamma^{-1} \mathscr{F} E_4.$$

The matrices H_1, H_2, E_1, E_2, E_4 are known and constant and a scalar $\gamma > 0$ is given, hence they are defined in the same form as in (11) and the matrix $\mathscr{F}$ satisfies (12). In the case when the full dynamic pass profile controller is applied, the controlled process state space model can be written as

$$\begin{aligned} \begin{bmatrix} \bar{x}_{k+1}(p+1) \\ \bar{y}_{k+1}(p) \end{bmatrix} &= (\overline{A} + \Delta\overline{A}) \begin{bmatrix} \bar{x}_{k+1}(p) \\ \bar{y}_k(p) \end{bmatrix} \\ y_{k+1}(p) &= \overline{C} \begin{bmatrix} \bar{x}_{k+1}(p) \\ \bar{y}_k(p) \end{bmatrix}, \end{aligned} \tag{28}$$

with

$$\begin{aligned} \overline{A} + \Delta\overline{A} &= \Pi \begin{bmatrix} \Phi + B_2 D_c C_2 & B_2 C_c \\ B_c C_2 & A_c \end{bmatrix} \Pi \\ &\quad + \Pi \begin{bmatrix} \Delta\Phi + \Delta\Lambda D_c C_2 & \Delta\Lambda C_c \\ 0 & 0 \end{bmatrix} \Pi \\ &= \Pi \begin{bmatrix} \Phi + B_2 D_c C_2 & B_2 C_c \\ B_c C_2 & A_c \end{bmatrix} \Pi \\ &\quad + \Pi \begin{bmatrix} \gamma^{-1} H \\ 0 \end{bmatrix} \mathscr{F} \begin{bmatrix} E + E_4 D_c C_2 & E_4 C_c \end{bmatrix} \Pi \\ &= \overline{A} + \overline{H} \mathscr{F} \overline{E}, \end{aligned}$$

where the matrices Φ, B_2, C_2 are as before and

$$H = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}, \quad E = \begin{bmatrix} E_1 & E_2 \end{bmatrix}.$$

Now we have the following result.

Theorem 7. *Suppose that a full dynamic pass profile controller defined by* (17) *is applied to a discrete linear repetitive process described by* (27) *with associated uncertainty structure. Then the resulting process* (28) *is stable along the pass holds if there exist matrices* $P_{11} > 0$, $(P_{11} = \text{diag}\{P_{h11}, P_{v11}\})$, $R_{11} > 0$, $(R_{11} = \text{diag}\{R_{h11}, R_{v11}\})$ *and a scalar* $\gamma > 0$ *such that the following LMIs hold*

$$\begin{bmatrix} \mathscr{N}_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Phi^{\mathrm{T}} P_{11} \Phi - P_{11} & \Phi^{\mathrm{T}} P_{11} H & E^{\mathrm{T}} \\ H^{\mathrm{T}} P_{11} \Phi & H^{\mathrm{T}} P_{11} H - \gamma^2 I & 0 \\ E & 0 & -I \end{bmatrix} \times \begin{bmatrix} \mathscr{N}_1 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0, \tag{29}$$

$$\begin{bmatrix} \mathcal{N}_2 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \Phi R_{11}\Phi^{\mathrm{T}} - R_{11} & \Phi R_{11} E^{\mathrm{T}} & H \\ E R_{11}\Phi^{\mathrm{T}} & -I + E R_{11} E^{\mathrm{T}} & 0 \\ H^{\mathrm{T}} & 0 & -\gamma^2 I \end{bmatrix} \times \begin{bmatrix} \mathcal{N}_2 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} < 0, \tag{30}$$

$$\begin{bmatrix} P_{h11} & I \\ I & R_{h11} \end{bmatrix} \geqslant 0, \quad \begin{bmatrix} P_{v11} & I \\ I & R_{v11} \end{bmatrix} \geqslant 0, \tag{31}$$

where $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *are full column rank matrices whose images satisfy* $\mathrm{Im}\mathcal{N}_1 = \ker(C_2^{\mathrm{T}})$ *and* $\mathrm{Im}\mathcal{N}_2 = \ker([B_2^{\mathrm{T}} E_4^{\mathrm{T}}])$ *respectively.*

Proof. Based on the proof of Theorem 5 it can be shown that stability along the pass with the controller applied holds in this case if

$$\begin{bmatrix} -\overline{P} & \overline{PA} & \overline{PH} & 0 \\ \overline{A}^{\mathrm{T}}\overline{P} & -\overline{P} & 0 & \overline{E}^{\mathrm{T}} \\ \overline{H}^{\mathrm{T}}\overline{P} & 0 & -\gamma^2 I & 0 \\ 0 & \overline{E} & 0 & -I \end{bmatrix} < 0,$$

where $\overline{A}$, $\overline{H}$, $\overline{E}$ are defined as before. Next, apply similar transformations to those used in the proof of previous result to obtain (29)–(31) and the proof is complete. □

To increase robustness, the term γ in the LMIs of (29)–(30) has to be minimized. This can be achieved by using linear objective minimization procedure

$$\min_{P_{11}>0,\, R_{11}>0} \mu,$$
$$\text{subject to } (29)-(31) \text{ with } \mu = \gamma^2.$$

5. Guaranteed cost control

Many applications will require a controller or control law which not only guarantees stability along the pass but also meets specified performance criteria. This is an area for which relatively little work has yet been reported in the general 2D systems area [10]. Here we give a comprehensive treatment for one aspect of this general problem for discrete linear repetitive processes and, in particular, those described by (27) and associated uncertainty structure.

The problem is to obtain a control law which simultaneously robustly stabilizes such a process and guarantees that the associated cost function defined by

$$J = \sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\left(u_{k+1}^{\mathrm{T}}(p)\Psi u_{k+1}(p)\right) + \sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\left(\begin{bmatrix} x_{k+1}(p) \\ y_k(p) \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}\begin{bmatrix} x_{k+1}(p) \\ y_k(p) \end{bmatrix}\right), \tag{32}$$

where $\Psi > 0$, $Q_1 > 0$ and $Q_2 > 0$ are design matrices to be specified, is bounded for all admissible uncertainties. In physical terms this cost function can be interpreted as the sum of quadratic costs on the input, state and pass profile vectors on each pass.

Remark 2. Repetitive processes are defined over the finite pass length α and only a finite number of passes, say s, will ever be executed. Hence the corresponding cost function used should be modified to

$$J = \sum_{k=0}^{s}\sum_{p=0}^{\alpha}\left(u_{k+1}^{\mathrm{T}}(p)\Psi u_{k+1}(p)\right) + \sum_{k=0}^{s}\sum_{p=0}^{\alpha}\left(\begin{bmatrix} x_{k+1}(p) \\ y_k(p) \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}\begin{bmatrix} x_{k+1}(p) \\ y_k(p) \end{bmatrix}\right).$$

However, it is routine to argue that the signals involved can be extended from $[0, \alpha]$ to the infinite interval in such a way that projection of the infinite interval solution onto the finite interval is possible. An identical argument holds in the pass-to-pass direction and hence we will work with (32).

The approach taken in this section is as follows: we first derive a sufficient condition which guarantees that the unforced (the control input terms are deleted) process is stable along the pass with an associated cost function which is bounded for all admissible uncertainties and then this result is extended to design a guaranteed cost controller for both forms of control action considered in this paper.

5.1. Guaranteed cost bound

Since the process is assumed to be unforced (i.e. $u_{k+1}(p) = 0$) then the process model (27) is rewritten as

$$\begin{bmatrix} x_{k+1}(p+1) \\ y_{k+1}(p) \end{bmatrix} = \left(\begin{bmatrix} A & B_0 \\ C & D_0 \end{bmatrix} + \begin{bmatrix} \Delta A & \Delta B_0 \\ \Delta C & \Delta D_0 \end{bmatrix}\right)\begin{bmatrix} x_{k+1}(p) \\ y_k(p) \end{bmatrix} \tag{33}$$

and the associated cost function (32) becomes

$$J_0 = \sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\left(\begin{bmatrix} x_{k+1}(p) \\ y_k(p) \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}\begin{bmatrix} x_{k+1}(p) \\ y_k(p) \end{bmatrix}\right). \tag{34}$$

The following theorem gives a sufficient condition for stability along the pass with guaranteed cost.

Theorem 8. *An unforced discrete linear repetitive process described by* (33) *is stable along the pass for all admissible uncertainties if there exist matrices* $P_1 > 0$, $P_2 > 0$ *and a scalar* $\epsilon > 0$ *such that the following LMI holds*

$$\begin{bmatrix} -P_1 & 0 & P_1A & P_1B_0 & P_1H_1 & P_1H_1 \\ 0 & -P_2 & P_2C & P_2D_0 & P_2H_2 & P_2H_2 \\ A^{\mathrm{T}}P_1 & C^{\mathrm{T}}P_2 & Q_1 - P_1 + \epsilon E_1^{\mathrm{T}}E_1 & 0 & 0 & 0 \\ B_0^{\mathrm{T}}P_1 & D_0^{\mathrm{T}}P_2 & 0 & Q_2 - P_2 + \epsilon E_2^{\mathrm{T}}E_2 & 0 & 0 \\ H_1^{\mathrm{T}}P_1 & H_2^{\mathrm{T}}P_2 & 0 & 0 & -\epsilon I & 0 \\ H_1^{\mathrm{T}}P_1 & H_2^{\mathrm{T}}P_2 & 0 & 0 & 0 & -\epsilon I \end{bmatrix} < 0. \tag{35}$$

Also if this condition holds, *the cost function* (34) *satisfies the upper bound*

$$J_0 \leqslant \sum_{k=1}^{\infty} x_{k+1}(0)P_1x_{k+1}(0) + \sum_{p=0}^{\infty} y_0^{\mathrm{T}}(p)P_2y_0(p). \tag{36}$$

Proof. Recall the vector $\zeta_k(p)$ of (5), the matrices A_1 and A_2 of (2) and introduce

$$\Delta A_1 = \begin{bmatrix} \Delta A & \Delta B_0 \\ 0 & 0 \end{bmatrix}, \quad \Delta A_2 = \begin{bmatrix} 0 & 0 \\ \Delta C & \Delta D_0 \end{bmatrix}.$$

Then we can rewrite (33) as

$$\begin{bmatrix} x_{k+1}(p+1) \\ y_{k+1}(p) \end{bmatrix} = ((A_1 + \Delta A_1) + (A_2 + \Delta A_2))\,\zeta_k(p)$$

and evaluating the Lyapunov function of (4) for the process state space model considered here gives

$$\begin{aligned} \Delta V(k,p) = \zeta_k^{\mathrm{T}}(p)[&(A_1 + \Delta A_1)^{\mathrm{T}}P(A_1 + \Delta A_1) \\ &+ (A_2 + \Delta A_2)P(A_2 + \Delta A_2) - P]\zeta_k(p), \end{aligned}$$

where $P = \mathrm{diag}\{P_1, P_2\}$ and stability along the pass holds if $\Delta V(k,p) < 0$. Moreover, the inequality

$$\Delta V(k,p) + \zeta_k^{\mathrm{T}}(p)Q\zeta_k(p) < 0$$

implies that (33) is stable along the pass where $Q = \mathrm{diag}\{Q_1, Q_2\} > 0$, and hence

$$(A_1 + \Delta A_1)^{\mathrm{T}}P(A_1 + \Delta A_1) + (A_2 + \Delta A_2)P(A_2 + \Delta A_2) - P + Q < 0. \tag{37}$$

Next, by an obvious application of, in turn, the Schur's complement formula and Lemma 2 to (37) yields

$$\begin{bmatrix} -P_1 & 0 & P_1A & P_1B_0 \\ 0 & -P_2 & P_2C & P_2D_0 \\ A^{\mathrm{T}}P_1 & C^{\mathrm{T}}P_2 & Q_1 - P_1 + \epsilon E_1^{\mathrm{T}}E_1 & 0 \\ B_0^{\mathrm{T}}P_1 & D_0^{\mathrm{T}}P_2 & 0 & Q_2 - P_2 + \epsilon E_2^{\mathrm{T}}E_2 \end{bmatrix}$$
$$+ \epsilon^{-1}\begin{bmatrix} 0 & 0 & P_1H_1 & P_1H_1 \\ 0 & 0 & P_2H_2 & P_2H_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ H_1^{\mathrm{T}}P_1 & H_2^{\mathrm{T}}P_2 & 0 & 0 \\ H_1^{\mathrm{T}}P_1 & H_2^{\mathrm{T}}P_2 & 0 & 0 \end{bmatrix} < 0.$$

Again using the Schur's complement formula, we find that the last inequality is equivalent to the LMI (35). Furthermore, noting that

$$\Upsilon = \sum_{k=0}^{\infty}\sum_{p=0}^{\infty} \zeta_k^{\mathrm{T}}(p)Q\zeta_k(p)$$

and, since the process is stable along the pass, we now have that

$$\begin{aligned} \Upsilon \leqslant & -\sum_{k=0}^{\infty}\left(\sum_{p=0}^{\infty} x_{k+1}(p+1)^{\mathrm{T}}P_1x_{k+1}(p+1) - x_{k+1}^{\mathrm{T}}(p)P_1x_{k+1}(p)\right) \\ & -\sum_{p=0}^{\infty}\left(\sum_{k=0}^{\infty} y_{k+1}^{\mathrm{T}}(p)P_2y_{k+1}(p) - y_k^{\mathrm{T}}(p)P_2y_k(p)\right) \\ & = \sum_{k=0}^{\infty} x_{k+1}^{\mathrm{T}}(0)P_1x_{k+1}(0) + \sum_{p=0}^{\infty} y_0^{\mathrm{T}}(p)P_2y_0(p), \end{aligned}$$

which ensures that (36) holds and the proof is complete. □

Note that it is possible to minimize the upper bound on the cost function (36) using the following optimization procedure

$$\begin{aligned} & \min_{P_1>0,P_2>0}\left[\sum_{k=0}^{\infty} x_{k+1}^{\mathrm{T}}(0)P_1x_{k+1}(0) + \sum_{p=0}^{\infty} y_0^{\mathrm{T}}(p)P_2y_0(p)\right] \\ & = \min_{P_1>0,P_2>0}\left[\sum_{k=0}^{\infty} \mathrm{trace}\left(P_1x_{k+1}(0)x_{k+1}^{\mathrm{T}}(0)\right)\right. \\ & \quad \left. + \,\mathrm{trace}\left(P_2\sum_{p=0}^{\infty} y_0(p)y_0^{\mathrm{T}}(p)\right)\right] \quad \text{subject to (35).} \end{aligned}$$

5.2. Guaranteed cost control analysis

Here, it is assumed that all elements in the current pass state vector can be measured and hence a control law of the form (7) can be applied to a process described by (27). In which case the associated cost function for the resulting process is given by

$$J=\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\left(\begin{bmatrix} x_{k+1}(p) \\ y_k(p) \end{bmatrix}^{\mathrm{T}}\begin{bmatrix} Q_1+K_1^{\mathrm{T}}\Psi K_1 & K_1^{\mathrm{T}}\Psi K_2 \\ K_2^{\mathrm{T}}\Psi K_1 & Q_2+K_2^{\mathrm{T}}\Psi K_2 \end{bmatrix}\begin{bmatrix} x_{k+1}(p) \\ y_k(p) \end{bmatrix}\right), \tag{38}$$

which is of the form of that in Theorem 8 and we have the following result.

Theorem 9. *Suppose that a control law of the form* (7) *is applied to a discrete linear repetitive process described by* (27) *with the associated uncertainty structure. Then the resulting process is stable along the pass for all admissible uncertainties if there exist matrices* $W_1>0$, $W_2>0$, N_1 *and* N_2 *and a scalar* $\epsilon>0$ *such that the following LMI holds*

$$\begin{bmatrix}
-W_1+2\epsilon H_1H_1^{\mathrm{T}} & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) \\
2\epsilon H_1H_2^{\mathrm{T}} & -W_2+2\epsilon H_2H_2^{\mathrm{T}} & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) \\
W_1A^{\mathrm{T}}+N_1^{\mathrm{T}}B^{\mathrm{T}} & W_1C^{\mathrm{T}}+N_1^{\mathrm{T}}D^{\mathrm{T}} & -W_1 & (\star) & (\star) & (\star) & (\star) & (\star) & (\star) \\
W_2B_0^{\mathrm{T}}+N_2^{\mathrm{T}}B^{\mathrm{T}} & W_2D_0^{\mathrm{T}}+N_2D^{\mathrm{T}} & 0 & -W_2 & (\star) & (\star) & (\star) & (\star) & (\star) \\
0 & 0 & E_1W_1+E_3N_1 & 0 & -\epsilon I & (\star) & (\star) & (\star) & (\star) \\
0 & 0 & 0 & E_2W_2+E_3N_2 & 0 & -\epsilon I & (\star) & (\star) & (\star) \\
0 & 0 & N_1 & N_2 & 0 & 0 & -\Psi^{-1} & (\star) & (\star) \\
0 & 0 & W_1 & 0 & 0 & 0 & 0 & -Q_1^{-1} & (\star) \\
0 & 0 & 0 & W_2 & 0 & 0 & 0 & 0 & -Q_2^{-1}
\end{bmatrix} < 0. \tag{39}$$

Also, if this condition holds the stabilizing control law matrices K_1, K_2 *are given by* (9) *and the cost function* (38) *of the controlled process satisfies the following upper bound*

$$J \leqslant \sum_{k=0}^{\infty} x_{k+1}^{\mathrm{T}}(0)W_1^{-1}x_{k+1}(0)+\sum_{p=0}^{\infty} y_0^{\mathrm{T}}(p)W_2^{-1}y_0(p). \tag{40}$$

Proof. Based on interpreting (35) in terms of its state space model, we conclude that the controlled process is robustly stabilized by the control law (7) if the following matrix inequality is satisfied

$$\begin{bmatrix} -P_1 & 0 & P_1A + P_1BK_1 & P_1B_0 + P_1BK_2 \\ 0 & -P_2 & P_2C + P_2DK_1 & P_2D_0 + P_2DK_2 \\ A^{\mathrm{T}}P_1 + K_1^{\mathrm{T}}B^{\mathrm{T}}P_1 & C^{\mathrm{T}}P_2 + K_1^{\mathrm{T}}D^{\mathrm{T}}P_2 & Q_1 - P_1 + K_1^{\mathrm{T}}\Psi K_1 & K_1^{\mathrm{T}}\Psi K_2 \\ B_0^{\mathrm{T}}P_1 + K_2^{\mathrm{T}}B^{\mathrm{T}}P_1 & D_0^{\mathrm{T}}P_2 + K_2D^{\mathrm{T}}P_1 & K_2^{\mathrm{T}}\Psi K_1 & Q_2 - P_2 + K_2^{\mathrm{T}}\Psi K_2 \end{bmatrix}$$
$$+ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E_1^{\mathrm{T}} + K_1^{\mathrm{T}}E_3^{\mathrm{T}} & 0 \\ 0 & 0 & 0 & E_2^{\mathrm{T}} + K_2^{\mathrm{T}}E_3^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \mathscr{F}^{\mathrm{T}} & 0 & 0 & 0 \\ 0 & \mathscr{F}^{\mathrm{T}} & 0 & 0 \\ 0 & 0 & \mathscr{F}^{\mathrm{T}} & 0 \\ 0 & 0 & 0 & \mathscr{F}^{\mathrm{T}} \end{bmatrix}$$
$$\times \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ H_1^{\mathrm{T}}P_1 & H_2^{\mathrm{T}}P_2 & 0 & 0 \\ H_1^{\mathrm{T}}P_1 & H_2^{\mathrm{T}}P_2 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & P_1H_1 & P_1H_1 \\ 0 & 0 & P_2H_2 & P_2H_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\times \begin{bmatrix} \mathscr{F} & 0 & 0 & 0 \\ 0 & \mathscr{F} & 0 & 0 \\ 0 & 0 & \mathscr{F} & 0 \\ 0 & 0 & 0 & \mathscr{F} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E_1 + E_3K_1 & 0 \\ 0 & 0 & 0 & E_2 + E_3K_2 \end{bmatrix} < 0.$$

Now set $W_1 = P_1^{-1}$, $W_2 = P_2^{-1}$, $U_1 = W_1Q_1W_1$ and $U_2 = W_2Q_2W_2$ and then pre- and post- multiply both sides of this last inequality by $\operatorname{diag}\{W_1, W_2, W_1, W_2\}$. Next, by an obvious application of the result of Lemma 2 we now obtain

$$\begin{bmatrix} -W_1 + 2\epsilon H_1H_1^{\mathrm{T}} & 2\epsilon H_2H_1^{\mathrm{T}} & AW_1 + BN_1 & B_0W_2 + BN_2 \\ 2\epsilon H_1H_2^{\mathrm{T}} & -W_2 + 2\epsilon H_2H_2^{\mathrm{T}} & CW_1 + DN_1 & D_0W_2 + DN_2 \\ W_1A^{\mathrm{T}} + N_1^{\mathrm{T}}B^{\mathrm{T}} & W_1C^{\mathrm{T}} + N_1^{\mathrm{T}}D^{\mathrm{T}} & U_1 - W_1 + N_1^{\mathrm{T}}\Psi N_1 & N_1^{\mathrm{T}}\Psi N_2 \\ W_2B_0^{\mathrm{T}} + N_2^{\mathrm{T}}B^{\mathrm{T}} & W_2D_0^{\mathrm{T}} + N_2D^{\mathrm{T}} & N_2^{\mathrm{T}}\Psi N_1 & U_2 - W_2 + N_2^{\mathrm{T}}\Psi N_2 \end{bmatrix}$$

$$
+\,\epsilon^{-1}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & W_1E_1^{\mathrm{T}}+N_1^{\mathrm{T}}E_3^{\mathrm{T}} & 0 \\ 0 & 0 & 0 & W_2E_2^{\mathrm{T}}+N_2^{\mathrm{T}}E_3^{\mathrm{T}} \end{bmatrix}
\times\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E_1W_1+E_3N_1 & 0 \\ 0 & 0 & 0 & E_2W_2+E_3N_2 \end{bmatrix} < 0,
$$

where $N_1 = K_1W_1$ and $N_2 = K_2W_2$. Finally, making an obvious application of the Schur's complement formula gives (39) and since $P_1 = W_1^{-1}$ and $P_2 = W_2^{-1}$,(39) is converted into (45). Finally, the bound on the cost function (40) can be established in an identical manner to that on J_0 in the previous result. Hence the details are omitted here. □

The presence of the nonlinear terms W_1^{-1} and W_2^{-1} in (40) means that it is not possible to apply a linear objective minimization procedure to minimize this cost function. However, a control law which minimizes the guaranteed cost can be achieved as follows. First note that

$$
\begin{aligned}
\sum_{k=0}^{s} x_{k+1}^{\mathrm{T}}(0)W_1^{-1}x_{k+1}(0) &= \sum_{k=0}^{s}\operatorname{trace}\left(x_{k+1}^{\mathrm{T}}(0)W_1^{-1}x_{k+1}(0)\right)\\
&= \sum_{k=0}^{s}\operatorname{trace}\left(W_1^{-1}x_{k+1}(0)x_{k+1}^{\mathrm{T}}(0)\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\sum_{p=0}^{\alpha} y_0^{\mathrm{T}}(p)W_2^{-1}y_0(p) &= \sum_{p=0}^{\alpha}\operatorname{trace}\left(y_0^{\mathrm{T}}(p)W_2^{-1}y_0(p)\right)\\
&= \sum_{p=0}^{\alpha}\operatorname{trace}\left(W_2^{-1}y_0(p)y_0^{\mathrm{T}}(p)\right).
\end{aligned}
$$

Next, recall that if a matrix $\widetilde{M}$ is symmetric and positive semi-definite i.e. $\widetilde{M} \geqslant 0$, then the eigenvalue decomposition of such a matrix gives $\widetilde{M} = V\varXi V^{\mathrm{T}}$, where V is some unitary matrix and $\varXi$ is a diagonal matrix with nonnegative diagonal entries. Therefore, the matrix square root of $\widetilde{M}$ can be defined as $\widetilde{M}^{\frac{1}{2}} = V\varXi^{\frac{1}{2}}V^{\mathrm{T}}$ and computed in a well conditioned manner [9]. Based on this, the matrices $\varSigma_1^{\frac{1}{2}}$ and $\varSigma_2^{\frac{1}{2}}$ can be obtained as

$$
\varSigma_1 = \varSigma_1^{\frac{1}{2}}\varSigma_1^{\frac{1}{2}} = \sum_{k=0}^{s} x_{k+1}^{\mathrm{T}}(0)x_{k+1}(0), \quad \varSigma_2 = \varSigma_2^{\frac{1}{2}}\varSigma_2^{\frac{1}{2}} = \sum_{p=0}^{\alpha} y_0^{\mathrm{T}}(p)y_0(p).
$$

Furthermore, introduce the symmetric matrices Ω_1 and Ω_2 which satisfy

$$\mathrm{trace}\left(\Sigma_1^{\frac{1}{2}} W_1^{-1} \Sigma_1^{\frac{1}{2}}\right) < \mathrm{trace}(\Omega_1) \quad \text{and} \quad \mathrm{trace}\left(\Sigma_2^{\frac{1}{2}} W_2^{-1} \Sigma_2^{\frac{1}{2}}\right) < \mathrm{trace}(\Omega_2),$$

respectively and hence we can write

$$\Sigma_1^{\frac{1}{2}} W_1^{-1} \Sigma_1^{\frac{1}{2}} < \Omega_1, \quad \Sigma_2^{\frac{1}{2}} W_2^{-1} \Sigma_2^{\frac{1}{2}} < \Omega_2.$$

Application of the Schur's complement formula now gives

$$\begin{bmatrix} -\Omega_1 & \Sigma_1^{\frac{1}{2}} \\ \Sigma_1^{\frac{1}{2}} & -W_1 \end{bmatrix} < 0 \quad \text{and} \quad \begin{bmatrix} -\Omega_2 & \Sigma_2^{\frac{1}{2}} \\ \Sigma_2^{\frac{1}{2}} & -W_2 \end{bmatrix} < 0. \tag{41}$$

Finally, the following minimization problem can be formulated:

$$\min_{W_1>0,\, W_2>0,\, N_1,\, N_2} \mathrm{trace}(\Omega_1 + \Omega_2),$$

subject to (39) and (41),

which gives a control law that guarantees that the cost function is minimized.

5.3. Guaranteed cost control with a full dynamic pass profile controller

In what follows, we assume that the current pass state vector is not available for control purposes and instead we consider the use of a full dynamic pass profile controller of the form (17) to ensure stability along the pass with a guaranteed bound on the associated cost function.

To simplify notation, the following matrices are introduced

$$\Delta\Phi = \begin{bmatrix} \Delta A & \Delta B_0 \\ \Delta C & \Delta D_0 \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \mathscr{F} \begin{bmatrix} E_1 & E_2 \end{bmatrix}, \quad \Delta B_2 = \begin{bmatrix} \Delta B \\ \Delta D \end{bmatrix} = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix} \mathscr{F} E_4,$$

where H_1, H_2, E_1, E_2, E_4 are known real matrices satisfying (11) and the matrix $\mathscr{F}$ satisfies (12).

With $D_c = 0$ for simplicity, the controlled process state space model can be written as

$$\begin{aligned} \begin{bmatrix} \bar{x}_{k+1}(p+1) \\ \bar{y}_{k+1}(p) \end{bmatrix} &= (\widetilde{A} + \Delta\widetilde{A}) \begin{bmatrix} \bar{x}_{k+1}(p) \\ \bar{y}_k(p) \end{bmatrix}, \\ y_{k+1}(p) &= \widetilde{C} \begin{bmatrix} \bar{x}_{k+1}(p) \\ \bar{y}_k(p) \end{bmatrix}, \end{aligned} \tag{42}$$

where

$$\begin{aligned} \widetilde{A} + \Delta\widetilde{A} &= \Pi \begin{bmatrix} \Phi & B_2 C_c \\ B_c C_2 & A_c \end{bmatrix} \Pi + \Pi \begin{bmatrix} \Delta\Phi & \Delta B_2 C_c \\ 0 & 0 \end{bmatrix} \Pi \\ &= \Pi \begin{bmatrix} \Phi & B_2 C_c \\ B_c C_2 & A_c \end{bmatrix} \Pi + \Pi \begin{bmatrix} H \\ 0 \end{bmatrix} \mathscr{F} \begin{bmatrix} E & E_4 C_c \end{bmatrix} \Pi \\ &= \widetilde{A} + \overline{H}\mathscr{F}\overline{E}, \\ \widetilde{C} &= \begin{bmatrix} C_2 & 0 \end{bmatrix} \Pi, \end{aligned}$$

and the matrices H and E are as before. The associated cost function is

$$J=\sum_{k=0}^{\infty}\sum_{p=0}^{\infty}\left(\begin{bmatrix}\bar{x}_{k+1}(p)\\ \bar{y}_k(p)\end{bmatrix}^{\mathrm{T}}\varPi\begin{bmatrix}Q & 0\\ 0 & Y\end{bmatrix}\varPi\begin{bmatrix}\bar{x}_{k+1}(p)\\ \bar{y}_k(p)\end{bmatrix}\right), \tag{43}$$

where $Q=\operatorname{diag}\{Q_1,Q_2\}$, $Y=C_c^{\mathrm{T}}\varPsi C_c$ and Q_1, Q_2, $\varPsi$ are given matrices in (32).

Now we have the following result which gives the existence condition for a guaranteed cost controller of the form (17) (with $D_c=0$).

Theorem 10. *Suppose that a full dynamic pass profile controller defined by* (17) *is applied to a discrete linear repetitive process described by* (27) *with the associated uncertainty structure. Then the resulting process is stable along the pass if for some prescribed* $\epsilon>0$ *there exist matrices* $P_{11}>0$, $(P_{11}=\operatorname{diag}\{P_{h11},P_{v11}\})$, $R_{11}>0$, $(R_{11}=\operatorname{diag}\{R_{h11},R_{v11}\})$ *such that the linear matrix inequalities defined by* (44)–(46) *below hold*

$$\begin{bmatrix}\mathcal{N}_1 & 0\\ 0 & I\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\varPhi R_{11}\varPhi^{\mathrm{T}}-R_{11} & \varPhi R_{11}E^{\mathrm{T}} & 0 & H & \varPhi R_{11}Q^{\frac{1}{2}}\\ ER_{11}\varPhi^{\mathrm{T}} & -\epsilon^{-1}I+ER_{11}E^{\mathrm{T}} & 0 & 0 & ER_{11}Q^{\frac{1}{2}}\\ 0 & 0 & -I & 0 & 0\\ H^{\mathrm{T}} & 0 & 0 & -\epsilon I & 0\\ Q^{\frac{1}{2}}R_{11}\varPhi^{\mathrm{T}} & Q^{\frac{1}{2}}R_{11}E^{\mathrm{T}} & 0 & 0 & -I+Q^{\frac{1}{2}}R_{11}Q^{\frac{1}{2}}\end{bmatrix}\begin{bmatrix}\mathcal{N}_1 & 0\\ 0 & I\end{bmatrix}<0, \tag{44}$$

$$\begin{bmatrix}\mathcal{N}_2 & 0 & 0 & 0\\ 0 & I & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I\end{bmatrix}^{\mathrm{T}}\begin{bmatrix}\varPhi^{\mathrm{T}}P_{11}\varPhi-P_{11} & \varPhi^{\mathrm{T}}P_{11}H & E^{\mathrm{T}} & Q^{\frac{1}{2}}\\ H^{\mathrm{T}}P_{11}\varPhi & H^{\mathrm{T}}P_{11}H-\epsilon I & 0 & 0\\ E & 0 & -\epsilon^{-1}I & 0\\ Q^{\frac{1}{2}} & 0 & 0 & -I\end{bmatrix}\begin{bmatrix}\mathcal{N}_2 & 0 & 0 & 0\\ 0 & I & 0 & 0\\ 0 & 0 & I & 0\\ 0 & 0 & 0 & I\end{bmatrix}<0, \tag{45}$$

$$\begin{bmatrix}P_{h11} & I\\ I & R_{h11}\end{bmatrix}\geqslant 0,\quad \begin{bmatrix}P_{v11} & I\\ I & R_{v11}\end{bmatrix}\geqslant 0, \tag{46}$$

where $\mathcal{N}_1$ *and* $\mathcal{N}_2$ *are full column rank matrices whose images satisfy* $\operatorname{Im}\mathcal{N}_1=\ker([B_2^{\mathrm{T}}E_4^{\mathrm{T}}\varPsi^{\frac{1}{2}}])$ *and* $\operatorname{Im}\mathcal{N}_2=\ker(C_2)$ *respectively.*

If these conditions hold, the cost function (43) *of the controlled process* (42) *satisfies the following upper bound*

$$J \leqslant \sum_{k=0}^{\infty} x_{k+1}^{\mathrm{T}}(0) P_{h11} x_{k+1}(0) + \sum_{p=0}^{\infty} y_0^{\mathrm{T}}(p) P_{v11} y_0(p).$$

Proof. Following the steps in the proof of Theorem 5 it follows that the stability along the pass condition for the uncertain process (42) can be written in the form

$$\begin{bmatrix} -P & P\widetilde{A} & P\overline{H} & 0 & 0 \\ \widetilde{A}^{\mathrm{T}} P & -P & 0 & \overline{E}^{\mathrm{T}} & \overline{S}^{\mathrm{T}} \\ \overline{H}^{\mathrm{T}} P & 0 & -\epsilon I & 0 & 0 \\ 0 & \overline{E} & 0 & -\epsilon^{-1} I & 0 \\ 0 & \overline{S} & 0 & 0 & -I \end{bmatrix} < 0, \tag{47}$$

where $\widetilde{A}$, $\overline{H}$, $\overline{E}$ are as before and

$$\begin{aligned} \overline{S} &= \begin{bmatrix} Q^{\frac{1}{2}} & 0 \\ 0 & \Psi^{\frac{1}{2}} C_c \end{bmatrix} \Pi = \left(\begin{bmatrix} Q^{\frac{1}{2}} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \Psi^{\frac{1}{2}} & 0 \end{bmatrix} \begin{bmatrix} 0 & C_c \\ B_c & A_c \end{bmatrix} \begin{bmatrix} C_2 & 0 \\ 0 & I \end{bmatrix} \right) \Pi \\ &= \widehat{Q}\Pi + \widehat{\Psi}\Theta\mathscr{C}_2\Pi \end{aligned}$$

and $\mathscr{C}_2$ and Θ (with $D_c = 0$) are also as before. Next, pre multiply (47) by diag $\{\Pi, \Pi, I, I, I\}$, post-multiply the result by the transpose of this last matrix, and then set $R = \Pi P \Pi$ (see (26)) to obtain

$$\Xi + M^{\mathrm{T}}\Theta N + N\Theta^{\mathrm{T}} M < 0,$$

where

$$\Xi = \begin{bmatrix} -R & R\mathscr{A} & R\mathscr{H} & 0 & 0 \\ \mathscr{A}^{\mathrm{T}} R & -R & 0 & \mathscr{E}^{\mathrm{T}} & \widehat{Q}^{\mathrm{T}} \\ \mathscr{H}^{\mathrm{T}} R & 0 & -\epsilon I & 0 & 0 \\ 0 & \mathscr{E} & 0 & -\epsilon^{-1} I & 0 \\ 0 & \widehat{Q} & 0 & 0 & -I \end{bmatrix},$$

$$M^{\mathrm{T}} = \begin{bmatrix} R\mathscr{B}_2 \\ 0 \\ 0 \\ \mathscr{E}_4 \\ \widehat{\Psi} \end{bmatrix}, \quad N = \begin{bmatrix} 0 & \mathscr{C}_2 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\mathscr{H} = \begin{bmatrix} H \\ 0 \end{bmatrix}, \quad \mathscr{E} = \begin{bmatrix} E & 0 \end{bmatrix}, \quad \mathscr{E}_4 = \begin{bmatrix} E_4 & 0 \end{bmatrix}.$$

Next, define the matrix variable $U = \mathrm{diag}\{R, I, I, I, I\}$ to write $M = M_n U$. Now, the matrices M_n and N can re-written as

$$M_n = \begin{bmatrix} \mathscr{B}_2^{\mathrm{T}} & 0 & 0 & \mathscr{E}_4^{\mathrm{T}} & \widehat{\Psi}^{\mathrm{T}} \end{bmatrix} = \left[\begin{array}{cc|cc|c|c|cc} B_2^{\mathrm{T}} & 0 & 0 & 0 & 0 & E_4^{\mathrm{T}} & 0 & \Psi^{\frac{1}{2}} \\ 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right],$$

$$N = \begin{bmatrix} 0 & \mathscr{C}_2 & 0 & 0 & 0 \end{bmatrix} = \left[\begin{array}{cc|cc|c|c|cc} 0 & 0 & C_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 & 0 \end{array}\right],$$

and hence the kernels of M_n and N are the images of

$$W_{M_n} = \begin{bmatrix} \mathscr{N}_{11} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 \\ \mathscr{N}_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I \\ \mathscr{N}_{13} & 0 & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$W_N = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathscr{N}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix},$$

where $\mathscr{N}_{11} = \ker(B_2^{\mathrm{T}})$, $\mathscr{N}_{12} = \ker(E_4^{\mathrm{T}})$, $\mathscr{N}_{13} = \ker(\Psi^{\frac{1}{2}})$ and $\mathscr{N}_2 = \ker(C_2)$. Now invoke Lemma 3 to obtain the following conditions which are equivalent to (47)

$$W_{M_n}^{\mathrm{T}} U^{-\mathrm{T}} \Xi U^{-1} W_{M_n} < 0 \quad \text{and } W_N^{\mathrm{T}} \Xi W_N < 0.$$

Since some rows of W_{M_n} and W_N are zero then

$$W_{M_n} = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \\ 0 & 0 & I & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \mathscr{N}_{11} & 0 & 0 & 0 & 0 \\ \mathscr{N}_{12} & 0 & 0 & 0 & 0 \\ \mathscr{N}_{13} & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & I \end{bmatrix},$$

$$W_N = \begin{bmatrix} I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathcal{N}_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I \end{bmatrix}$$

and routine matrix manipulations now yield (44)–(46). Finally, the cost function bound is established in an identical manner to that of the previous result and hence the details are omitted here. □

The guaranteed cost controller here can be computed as per the procedure given in Section 4.

Remark 3. Note that the parameter ϵ which appears in (44) and (45), has to be chosen before the LMI computations can be undertaken. Furthermore, the upper bound on the cost function depends on the value of this scalar. Hence by decreasing iteratively ϵ a lower upper bound can be obtained.

As a numerical example, return to the example in Section 4 and add

$$B = \begin{bmatrix} 1.2 & 0.5 \\ 1.1 & 0.8 \end{bmatrix}, \quad D = \begin{bmatrix} 3.0 & 1.2 \\ 1.7 & 0.9 \end{bmatrix}$$

and take the matrices defining the uncertainty model as

$$H = \begin{bmatrix} 0.0284 & 0.0583 \\ 0.0469 & 0.0423 \\ 0.0065 & 0.0516 \\ 0.0988 & 0.0334 \end{bmatrix},$$
$$E = \begin{bmatrix} 0.0433 & 0.0580 & 0.0530 & 0.0209 \\ 0.0226 & 0.0760 & 0.0641 & 0.0380 \end{bmatrix},$$
$$E_4 = \begin{bmatrix} 0.0783 & 0.0461 \\ 0.0681 & 0.0568 \end{bmatrix}$$

and the matrices Q_1, Q_2 and Ψ in the cost function (38) as

$$Q_1 = \text{diag}\{80, 80\}, \quad Q_2 = \text{diag}\{80, 80\}, \ \Psi = 40.$$

Using the design procedure of Theorem 10 for 10 passes and $\alpha = 20$ and choosing $\epsilon = 800$ the problem is solvable and the solution matrices are

$$P_{h11} = 10^5 \times \begin{bmatrix} 4.9134 & -3.0092 \\ -3.0092 & 2.1449 \end{bmatrix}, \quad P_{v11} = 10^4 \times \begin{bmatrix} 1.7237 & -2.5214 \\ -2.5214 & 4.9326 \end{bmatrix},$$

$$R_{h11} = \begin{bmatrix} 0.0050 & -0.0001 \\ -0.0001 & 0.0097 \end{bmatrix}, \quad R_{v11} = \begin{bmatrix} 0.0075 & -0.0018 \\ -0.0018 & 0.0064 \end{bmatrix}.$$

Hence the controller matrices are given by

$$A_c = \begin{bmatrix} -0.3144 & -0.4540 & -0.4368 & -1.0556 \\ 0.1726 & 0.3978 & -0.5687 & -1.8399 \\ 0.0789 & 0.3460 & -0.0312 & -0.0139 \\ -0.0440 & -0.1703 & -0.4553 & -1.4232 \end{bmatrix},$$

$$B_c = \begin{bmatrix} -38.7642 & 175.9122 \\ -70.6957 & -157.5573 \\ 95.4216 & 95.3948 \\ -98.5052 & -98.5710 \end{bmatrix},$$

$$C_c = \begin{bmatrix} 0.0001 & 0.0009 & 0.0011 & 0.0027 \\ 0.0006 & 0.0015 & 0.0025 & 0.0094 \end{bmatrix}$$

and the guaranteed cost of the uncertain controlled process satisfies $J < 1.3626 \cdot 10^6$.

6. Conclusions

This paper has produced substantial new results on the design of controllers, or control laws, for discrete linear repetitive processes. The first part develops an $\mathscr{H}_\infty$ setting for the design of a static control law which, noting their links to, in particular, ILC, makes such a control law much more powerful than in the 2D discrete linear systems case. This analysis has then been extended to the case when there is uncertainty in the process model. We also show that all these results extend to the use of a dynamic controller actuated by the previous pass profile which, by the process structure, is available for use. Here it should be noted that it is not possible to directly apply existing 1D robust control results and it was felt necessary to start with an additive uncertainty structure and the success of this approach provides a good basis on which to consider other uncertainty models. In the final part of this paper a guaranteed cost control problem has been solved. This is the first major result on control for performance of these processes and again the cost function used is well grounded in terms of the process dynamics and the requirements of industrial examples.

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References

[1] S. Arimoto, S. Kawamura, F. Miyazaki, Bettering operation of robots by learning, J. Robot. Syst. 1 (2) (1984) 123–140.
[2] A.D. Barton, P.L. Lewin, Experimental comparison of the performance of different chain conveyor controllers, Proc. Inst. Mech. Eng. Part I: J. Syst. Contr. Eng. 214 (5) (2000) 361–369.
[3] S. Boyd, L.E. Ghaoui, E. Feron, V. Balakrishnan, Linear Matrix Inequalities in System and Control Theory, SIAM Studies in Applied and Numerical Mathematics, vol. 15, SIAM, Philadelphia, USA, 1994.
[4] C. Du, L. Xie, H_∞ Control and Filtering of Two-dimensional Systems, Lecture Notes in Control and Information Sciences, vol. 278, Springer-Verlag, Berlin, Germany, 2002.
[5] G.E. Dullerud, F. Paganini, A Course in Robust Control Theory. A Convex Approach, Texts in Applied Mathematics, vol. 36, Springer-Verlag, New York, USA, 2000.
[6] E. Fornasini, G. Marchesini, Doubly indexed dynamical systems: state models and structural properties, Math. Syst. Theory 12 (1978) 59–72.
[7] P. Gahinet, P. Apkarian, A linear matrix inequality approach to H_∞ control, Int. J. Robust Nonlin. Contr. 4 (1994) 421–448.
[8] K. Gałkowski, E. Rogers, S. Xu, J. Lam, D.H. Owens, LMIs – a fundamental tool in analysis and controller design for discrete linear repetitive processes, IEEE Trans. Circuits Syst. I: Fundam. Theory Appl. 49 (6) (2002) 768–778.
[9] G.M. Golub, C.F. Van Loan, Matrix Computations, John Hopkins Series in the Mathematical Sciences, third ed., The John Hopkins University Press, Baltimore, USA, 1996.
[10] X. Guan, C. Long, G. Duan, Robust optimal guaranteed cost control for 2D discrete systems, IEE Proc.—Contr. Theory Appl. 148 (5) (2001) 355–361.
[11] P.P. Khargonekar, I.R. Petersen, K. Zhou, Robust stabilization of uncertain linear systems: Quadratic stabilizability and H_∞ control theory, IEEE Trans. Automat. Contr. 35 (3) (1990) 356–361.
[12] W.-S. Lu, A. Antoniou, Two-Dimensional Digital Filters, Electrical Engineering and Elecronics, vol. 80, Marcel Dekker, Inc, New York, USA, 1992.
[13] D.H. Owens, N. Amann, E. Rogers, M. French, Analysis of linear iterative learning control schemes—a 2D systems/repetitive processes approach, Multidimen. Syst. Signal Process. 11 (1–2) (2000) 125–177.
[14] P.D. Roberts, Two-dimensional analysis of an iterative nonlinear optimal control algorithm, IEEE Trans. Circuits Syst. I: Fundamen. Theory Appl. 49 (6) (2002) 872–878.
[15] R.P. Roesser, A discrete state-space model for linear image processing, IEEE Trans. Automat. Contr. 20 (1) (1975) 1–10.
[16] E. Rogers, D.H. Owens, Stability Analysis for Linear Repetitive Processes, Lecture Notes in Control and Information Sciences, vol. 175, Springer-Verlag, Berlin, Germany, 1992.
[17] B. Sulikowski, K. Gałkowski, E. Rogers, D.H. Owens, Output feedback control of discrete linear repetitive processes, Automatica 40 (12) (2004) 2167–2173.