Stability and stabilisation of acausal discrete linear repetitive processes

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Repetitive processes are a distinct class of two-dimensional systems (i.e. information propagation in two independent directions occurs) of both systems theoretic and applications interest. In this paper we introduce a new model for these processes in order to represent dynamics which arise in some applications areas and which are not included in those currently available. Then we proceed to define quadratic stability for this case, obtain conditions for its existence, and also use feedback control to solve a stabilization problem.

1 Introduction

The essential unique characteristic of a repetitive, or multipass, process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem for these processes in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass to pass direction.

In this paper, we first propose a new model for discrete linear repetitive processes which captures features of the dynamics which are excluded from the currently used models but which could well arise in a number of potential applications areas. For this new model, it is concluded that the methods of stabilization by feedback control action for either 2D linear systems described by Roesser/Fornasini Marchesini state space models cannot be applied. Hence the major overall outcome of this paper is the development of a new approach which removes this difficulty.

Throughout this paper, the null matrix and the identity matrix with the required dimensions are denoted by 0 and I, respectively. Moreover, $M > 0$ (< 0) denotes a real symmetric positive (negative) definite matrix.

2 Preliminaries and the new model

The most basic discrete linear repetitive process state space model is given in [1]. This model however cannot be used in many practically related situations. For example, it has been shown that the structure of the pass state initial vector sequence alone can cause instability. Other modifications/extensions include the so-called wave models and some intrinsically anti-causal models which were motivated by signal processing and robotics applications. In this paper we consider the following model over $k \geq 0$ and $0 \leq p \leq \alpha$, where k is the pass number, $\alpha < \infty$ is the pass length, $x_k(p) \in \mathbb{R}^n$ is the state vector, $y_k \in \mathbb{R}^m$ is the pass profile vector, and $u_k^i \in \mathbb{R}^l,\ i = 1, 2, 3$ are control input vectors

$$\begin{aligned} x_{k+1}(p) &= A_1 x_k(p-1) + A_2 x_k(p) + A_3 x_k(p+1) + B_1 u_k^1(p) + B_2 u_k^2(p) + B_3 u_k^3(p) \\ y_k(p) &= C x_k(p) \end{aligned} \tag{1}$$

with boundary conditions

$$\begin{aligned} x_0(p) &= g(p), && 0 \leq p \leq \alpha \\ x_k(-1) &= 0,\ x_k(\alpha) = g_k, && k > 0 \end{aligned} \tag{2}$$

where $g(p)$ is an $n \times 1$ vector whose entries are known functions of p and g_k is an $n \times 1$ vector with known constant entries and the sequence $\{g_k\}$ is bounded. In comparison to the standard model of discrete linear repetitive processes, the updating structure is radically different and also the boundary conditions differ in that now it is the final (vector) value on each pass which is specified. The model considered here is also intrinsically non-causal in the right-upper quadrant sense but note that the underlying dynamics can posses space, as opposed to temporal, characteristics and hence this property is not required here. This and more general situations also occur in signal processing and are related to so-called semi-causal and minimum neighbor systems. We assign the term 'wave' discrete linear repetitive process to dynamics described by (1) and (2).

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3 Quadratic stability and stabilization

First we define so-called quadratic stabilization for processes described by (1) and (2) and then show how this leads directly to implementable stability tests and control law design algorithms.

Hence introduce the so-called extended Lyapunov function $V(k) = \sum_{p=0}^{\infty} x_k^T(p)\widetilde{P}x_k(p)$ where $\widetilde{P} > 0$ and $V(0) < \infty$. Then we have the formal definition of quadratic stability.

Definition 3.1 A discrete linear repetitive process described by (1) and (2) is said to be quadratically stable if, and only if, there exists a matrix $\widetilde{P} > 0$ such that

$$V(k+1) < V(k), \quad \forall\, k \geq 0 \tag{3}$$

The following result gives an LMI based interpretation of this property which forms the basis of the analysis in the rest of this paper. Note that this condition is sufficient but not necessary and hence there is a degree of conservativeness associated with its use.

Theorem 3.2 *A discrete linear repetitive process described by* (1) *and* (2) *is quadratically stable if* $\sum_{p=0}^{\infty} \|x_0(p)\| < \infty$ *and there exist matrices* $P > 0$, $Q > 0$ *and* $Z > 0$ *such that the following LMI is feasible*

$$\begin{bmatrix} A_1^T(P+Q+Z)A_1 - P & A_1^T(P+Q+Z)A_2 & A_1^T(P+Q+Z)A_3 \\ A_2^T(P+Q+Z)A_1 & A_2^T(P+Q+Z)A_2 - Q & A_2^T(P+Q+Z)A_3 \\ A_3^T(P+Q+Z)A_1 & A_3^T(P+Q+Z)A_2 & A_3^T(P+Q+Z)A_3 - Z \end{bmatrix} < 0. \tag{4}$$

3.1 Stabilization by State Feedback

Here we produce the first results on the control of processes described by (1) and (2) under the action of the following control law over $k \geq 0$ and $0 \leq p \leq \alpha$

$$\begin{bmatrix} u_k^1(p) \\ u_k^2(p) \\ u_k^3(p) \end{bmatrix} = I_3 \otimes K \begin{bmatrix} x_k(p-1) \\ x_k(p) \\ x_k(p+1) \end{bmatrix} \tag{5}$$

where $\otimes$ denotes the Kronecker product.

The following result now gives a sufficient condition for the existence of a quadratically stabilizing control law of the form considered here.

Theorem 3.3 *A discrete linear repetitive process described by* (1) *and* (2) *is quadratically stabilizable by a control law of the form (5) if there exist matrices* N, $X > 0$, $Y > 0$, $J > 0$ *and* $M > 0$, *such that the following LMI is feasible*

$$\begin{bmatrix} -X & (*) & (*) & (*) \\ 0 & -Y & (*) & (*) \\ 0 & 0 & -J & (*) \\ A_1M + B_1N & A_2M + B_2N & A_3M + B_3N & -M \end{bmatrix} < 0 \tag{6}$$

where (*) in entry i, j *denotes fact that* $V_{ij} = V_{ji}^T$, *and*

$$M = X + Y + J. \tag{7}$$

If (6) holds, then a stabilizing K *in the control law (5) is given by*

$$K = NM^{-1}. \tag{8}$$

4 Conclusions

This paper has proposed a new model for discrete linear repetitive processes to include terms missing from the standard model but which can arise in applications. Moreover, we have developed substantial results on quadratic stabilization of this class of repetitive processes, based on using LMIs. These provide a very strong basis on which to develop a comprehensive systems theory for this new model for onward translation (where appropriate) into computational algorithms.

References

[1] E. Rogers and D. H. Owens. *Stability analysis for Linear Repetitive Processes*, volume 175 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag, 1992.