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Multi-machine operations modelled and controlled as switched linear repetitive processes

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Many industrial processes involve the processing of a single workpiece by successively passing it through a sequence of machines. The most common example is metal rolling where the metal strip of finite length is shaped by passing it through different sets of rolls and the output from one forms the input to the next and so on. In this paper, we develop a new approach to the analysis and overall control of such systems by first modelling them as a linear repetitive process with switched dynamics. The end result is control law design algorithms which can be implemented using LMI based computations.

Keywords: repetitive dynamics; multi-machine dynamics; control law design

1. Introduction

The unique characteristic of a repetitive process (also termed a multipass process in the early literature) can be illustrated by considering machining operations where the material or workpiece involved is processed by a series of sweeps, or passes, of the processing tool. Assuming the pass length $\alpha < +\infty$ to be constant, the output vector, or pass profile, $y_k(p)$, $p=0,1,\ldots,\alpha-1$, (p being the independent spatial or temporal variable), generated on pass k acts as a forcing function on, and hence contributes to, the dynamics of the new pass profile $y_{k+1}(p)$, $p=0,1,\ldots,\alpha-1$, $k=0, 1,\ldots$. This, in turn, leads to the unique control problem in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction, i.e. in the collection of pass profile vectors $\{y_k\}_k$.

Industrial examples include long-wall coal-cutting and metal rolling, see the original papers cited in, for example, Rogers, Galkowski and Owens (2007) for further details. A number of so-called algorithmic examples also exist where adopting a repetitive process setting for analysis has clear advantages over alternative approaches to systems related analysis. These include iterative learning control schemes, e.g., Moore, Chen and Bahl (2005) and iterative solution algorithms for dynamic non-linear optimal control problems based on the maximum principle, e.g., Roberts (2002). In the case of iterative learning control for the linear dynamics case, the stability theory for differential (and discrete) linear repetitive processes is one method which can be used to undertake a stability/convergence analysis of a powerful class of such algorithms and thereby produce vital design information concerning the trade-offs required between convergence and transient performance, see, e.g., Owens, Amann, Rogers and French (2000).

In many practical applications, the material or workpiece involved is of finite length (or thickness) and is processed (or operated on) by a sequence of operations each of which has its own dynamics. Examples here include metal rolling or operations using multiple operation robot arms. In the case of the former, a common requirement is that a metal strip of finite length is shaped by passing it through different sets of rolls where the output from one forms the input to the next and so on. More generally, a number of successive operations may be carried out under one regime (or dynamics) and then the dynamics change, or switch, to allow further processing to take place. This paper shows how such dynamics can be modelled as a discrete linear repetitive process with switching in the pass-to-pass direction. Then we develop new results

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on stability and control law design and give an illustrative numerical example.

Throughout this paper, the null matrix and the identity matrix with the required dimensions are denoted by 0 and I, respectively. Moreover, $M > 0$ ($M < 0$) denotes a real symmetric positive (negative) definite matrix. In the next section we introduce the required background results.

2. Background

Repetitive processes can exhibit many forms of dynamics but here the interest in so-called discrete linear repetitive processes where the dynamics along any pass are governed by a matrix difference equation. Moreover, the boundary conditions, i.e., the initial pass profile and the initial conditions, at the start of each new pass are of critical importance. The simplest possible model over $p = 0, 1, \dots, \alpha - 1$, $k \geq 0$ is

$$\begin{cases} x_{k+1}(p+1) = Ax_{k+1}(p) + Bu_{k+1}(p) + B_0 y_k(p) \\ y_{k+1}(p) = Cx_{k+1}(p) + Du_{k+1}(p) + D_0 y_k(p) \end{cases} \tag{1}$$

where on pass k $x_k(p) \in \mathbb{R}^n$ is the state vector, $u_k(p) \in \mathbb{R}^r$ is the control vector and $y_k(p) \in \mathbb{R}^m$ is the pass profile (or output) vector. The boundary conditions of interest here are of the form

$$x_{k+1}(0) = d_{k+1} \quad \text{and} \quad y_0(p) = f(p), \tag{2}$$

where $d_{k+1} \in \mathbb{R}^n$ has known constant entries and $f(p) \in \mathbb{R}^m$ is known function p. Note, however, that other forms for $x_{k+1}(0)$ arise and the structure of these alone can destroy stability — see Rogers et al. (2007) for a detailed treatment of this aspect.

Note also that these processes share many joint features with the so-called spatially interconnected systems, which have already found numerous important physical applications; see, for example, D'Andrea and Dullerud (2003) and references therein. This arises from the fact that some of the state-space models in this area can be rewritten as a discrete linear repetitive process state-space model (or its differential equivalent).

In this work, we are interested in a generalisation of the case when an example completes a pass and then before the start of the next pass the process dynamics switch to a model with the same structure but different state-space matrices. In particular, we consider the case of processes described by

$$\left.\begin{array}{l} \begin{cases} x_{l+1}(p+1) = A_{l+1}x_{l+1}(p) + B_{l+1}u_{l+1}(p) \\ \qquad + B_{0,l+1}y_l(p) \\ y_{l+1}(p) = C_{l+1}x_{l+1}(p) + D_{l+1}u_{l+1}(p) \\ \qquad + D_{0,l+1}y_l(p) \end{cases} \\ A_{l+1+\tau} = A_{l+1}, \quad B_{l+1+\tau} = B_{l+1}, \\ B_{0,l+1+\tau} = B_{0,l+1} \\ C_{l+1+\tau} = C_{l+1}, \quad D_{l+1+\tau} = D_{l+1}, \\ D_{0,l+1+\tau} = D_{0,l+1} \end{array}\right\} \tag{3}$$

over $p = 0, 1, \dots, \alpha - 1$, $l = 0, 1, \dots$, where τ is a known positive integer and the boundary conditions are of the form (2).

The matrices which define the state-space models here are said to be τ-periodic and we are considering a discrete linear repetitive process with multiple switched dynamics in the pass-to-pass direction. Also it is clear that there are many more possibilities for switched dynamics in these processes but as shown later in this paper it is this form which is of use in the repetitive process based analysis of metal rolling operations.

For analysis purposes, the process model (3) can be transformed into the non-switched "equivalent lifted process" described by

$$\begin{cases} X_{k+1}(p+1) = \widehat{A}X_{k+1}(p) + \widehat{B}U_{k+1}(p) + \widehat{B}_0 Y_k(p) \\ Y_{k+1}(p) = \widehat{C}X_{k+1}(p) + \widehat{D}U_{k+1}(p) + \widehat{D}_0 Y_k(p), \end{cases} \tag{4}$$

where $p = 0, 1, \dots, \alpha - 1$, $k = 0, 1, \dots$, and

$$X_{k+1}(p) = \begin{bmatrix} x_{\tau k+1}(p) \\ x_{\tau k+2}(p) \\ \vdots \\ x_{\tau k+\tau}(p) \end{bmatrix}, \quad U_{k+1}(p) = \begin{bmatrix} u_{\tau k+1}(p) \\ u_{\tau k+2}(p) \\ \vdots \\ u_{\tau k+\tau}(p) \end{bmatrix}$$

$$Y_k(p) = y_{\tau k}(p), \quad Y_{k+1}(p) = y_{\tau k+\tau}(p)$$

and

$$\widehat{A} = \begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 \\ B_{02}C_1 & A_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{0,\tau-1}D_{0,\tau-2}\dots D_{02}C_1 & B_{0,\tau-1}D_{0,\tau-2}\dots D_{03}C_2 & \cdots & A_{\tau-1} & 0 \\ B_{0,\tau}D_{0,\tau-1}\dots D_{02}C_1 & B_{0,\tau}D_{0,\tau-1}\dots D_{03}C_2 & \cdots & B_{0,\tau}C_{\tau-1} & A_\tau \end{bmatrix}$$

$$\widehat{C} = \begin{bmatrix} D_{0,\tau}D_{0,\tau-1}\dots D_{02}C_1 & D_{0,\tau}D_{0,\tau-1}\dots D_{03}C_2 & \cdots & D_{0,\tau}C_{\tau-1} & C_\tau \end{bmatrix}$$

$$\widehat{B}_0 = \begin{bmatrix} B_{01} \\ B_{02}D_{01} \\ \vdots \\ B_{0,\tau-1}D_{0,\tau-2}\dots D_{01} \\ B_{0,\tau}D_{0,\tau-1}\dots D_{01} \end{bmatrix}, \quad \widehat{D}_0 = D_{0,\tau}D_{0,\tau-1}\dots D_{01}$$

and $\widehat{\Phi}$ and $\widehat{\Pi}$ the "lifted augmented process and input matrices" respectively.

The analysis in this paper will make extensive use of the Schur's complement formula and the following results (Bachelier, Bernussou, de Oliveira and Geromel 1999).

$$\widehat{B} = \begin{bmatrix} B_1 & 0 & \cdots & 0 & 0 \\ B_{02}D_1 & B_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ B_{0,\tau-1}D_{0,\tau-2}\dots D_{02}D_1 & B_{0,\tau-1}D_{0,\tau-2}\dots D_{03}D_2 & \cdots & B_{\tau-1} & 0 \\ B_{0,\tau}D_{0,\tau-1}\dots D_{02}D_1 & B_{0,\tau}D_{0,\tau-1}\dots D_{03}D_2 & \cdots & B_{0,\tau}D_{\tau-1} & B_\tau \end{bmatrix}$$

$$\widehat{D} = \begin{bmatrix} D_{0,\tau}D_{0,\tau-1}\dots D_{02}D_1 & D_{0,\tau}D_{0,\tau-1}\dots D_{03}D_2 & \cdots & D_{0,\tau}D_{\tau-1} & D_\tau \end{bmatrix}$$

with boundary conditions

$$X_{k+1}(0) = \begin{bmatrix} x_{\tau k+1}(0) \\ x_{\tau k+2}(0) \\ \vdots \\ x_{\tau k+\tau}(0) \end{bmatrix} = \begin{bmatrix} d_{\tau k+1} \\ d_{\tau k+2} \\ \vdots \\ d_{\tau k+\tau} \end{bmatrix}, \quad Y_0(p) = f(p).$$

In this paper, we make no further reference to the boundary conditions (except for specifying them in the numerical example given later).

In order to simplify notation, we introduce the following for (3) and (4)

$$\Phi_i = \begin{bmatrix} A_i & B_{0i} \\ C_i & D_{0i} \end{bmatrix}, \quad \Pi_i = \begin{bmatrix} B_i \\ D_i \end{bmatrix}, \text{ for } \quad i = 1, 2, \dots, \tau \tag{5}$$

and

$$\widehat{\Phi} = \begin{bmatrix} \widehat{A} & \widehat{B}_0 \\ \widehat{C} & \widehat{D}_0 \end{bmatrix}, \quad \widehat{\Pi} = \begin{bmatrix} \widehat{B} \\ \widehat{D} \end{bmatrix} \tag{6}$$

respectively. Here the matrices Φ_i and Π_i are termed the "periodic augmented process and input matrices"

Lemma 1: *Let $\mathbb{W}, \mathbb{V}$ be given symmetric matrices with $\mathbb{V} > 0$. Suppose also that $\mathbb{L}$ is a given matrix. Then*

$$\mathbb{W} + \mathbb{L}\mathbb{V}\mathbb{L}^T < 0$$

holds if, and only if,

$$\begin{bmatrix} \mathbb{W} & \mathbb{L}G \\ G^T\mathbb{L}^T & \mathbb{V} - G - G^T \end{bmatrix} < 0, \tag{7}$$

where G is a non-singular matrix.

Lemma 2: *Let $\mathbb{W}, \mathbb{V}$ with $\mathbb{V} > 0$, and $\mathbb{L}_1^0$, $\mathbb{L}_i^j$, for $i = 2, \dots, \tau$, $j = 1, 2$ be given matrices. Then*

$$\mathbb{W} + \mathcal{L}\mathbb{V}\mathcal{L}^T < 0 \tag{8}$$

with

$$\mathcal{L} = \mathbb{L}_1^0 + \big(\mathbb{L}_2^1 + \big(\mathbb{L}_3^1 + \big(\mathbb{L}_4^1 + \cdots + \big(\mathbb{L}_{\tau-1}^1 + \mathbb{L}_\tau^1\mathbb{L}_\tau^2\big)\mathbb{L}_{\tau-1}^2 \cdots\big)\mathbb{L}_4^2\big)\mathbb{L}_3^2\big)\mathbb{L}_2^2$$

holds if

$$\begin{bmatrix} \mathbb{W} & \mathbb{L}_\tau^1 G_\tau & \mathbb{L}_{\tau-1}^1 G_{\tau-1} & \mathbb{L}_{\tau-2}^1 G_{\tau-2} & \cdots & \mathbb{L}_3^1 G_3 & \mathbb{L}_2^1 G_2 & \mathbb{L}_1^0 G_1 \\ G_\tau^T \mathbb{L}_\tau^{1T} & -G_\tau - G_\tau^T & \mathbb{L}_\tau^2 G_{\tau-1} & 0 & \cdots & 0 & 0 & 0 \\ G_{\tau-1}^T \mathbb{L}_{\tau-1}^{1T} & G_{\tau-1}^T \mathbb{L}_\tau^{2T} & -G_{\tau-1} - G_{\tau-1}^T & \mathbb{L}_{\tau-1}^2 G_{\tau-2} & \cdots & 0 & 0 & 0 \\ G_{\tau-2}^T \mathbb{L}_{\tau-2}^{1T} & 0 & G_{\tau-2}^T \mathbb{L}_{\tau-1}^{2T} & -G_{\tau-2} - G_{\tau-2}^T & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ G_3^T \mathbb{L}_3^{1T} & 0 & 0 & 0 & \cdots & -G_3 - G_3^T & \mathbb{L}_3^2 G_2 & 0 \\ G_2^T \mathbb{L}_2^{1T} & 0 & 0 & 0 & \cdots & G_2^T \mathbb{L}_3^{2T} & -G_2 - G_2^T & \mathbb{L}_2^2 G_1 \\ G_1^T \mathbb{L}_1^{0T} & 0 & 0 & 0 & \cdots & 0 & G_1^T \mathbb{L}_2^{2T} & \mathbb{V} - G_1 - G_1^T \end{bmatrix} < 0, \tag{9}$$

where G_i, $i = 1, 2, \dots, \tau$, are non-singular matrices.

Proof: The matrix $\mathcal{L}$ can be rewritten as

$$\begin{aligned}\mathcal{L} &= \mathbb{L}_1^0 + \mathbb{L}_2^1\mathbb{L}_2^2 + \mathbb{L}_3^1\mathbb{L}_3^2\mathbb{L}_2^2 + \mathbb{L}_4^1\mathbb{L}_4^2\mathbb{L}_3^2\mathbb{L}_2^2 + \cdots \\ &\quad + \mathbb{L}_{\tau-1}^1\mathbb{L}_{\tau-1}^2\mathbb{L}_{\tau-2}^2\cdots\mathbb{L}_2^2 + \mathbb{L}_\tau^1\mathbb{L}_\tau^2\mathbb{L}_{\tau-1}^2\cdots\mathbb{L}_2^2 \\ &= \bar{\mathcal{L}} + \mathbb{L}_\tau^1\bar{\Omega}\end{aligned}$$

and hence (8) as

$$\begin{aligned}\mathbb{W} + \mathcal{L}\mathbb{V}\mathcal{L}^T &= \mathbb{W} + \left(\bar{\mathcal{L}} + \mathbb{L}_\tau^1\bar{\Omega}\right)\mathbb{V}\left(\bar{\mathcal{L}} + \mathbb{L}_\tau^1\bar{\Omega}\right)^T \\ &= \begin{bmatrix} I & \mathbb{L}_\tau^1\end{bmatrix}\Gamma\begin{bmatrix} I \\ \mathbb{L}_\tau^{1T}\end{bmatrix} < 0, \end{aligned} \tag{10}$$

where

$$\begin{aligned}\bar{\mathcal{L}} &= \mathbb{L}_1^0 + \mathbb{L}_2^1\mathbb{L}_2^2 + \mathbb{L}_3^1\mathbb{L}_3^2\mathbb{L}_2^2 + \mathbb{L}_4^1\mathbb{L}_4^2\mathbb{L}_3^2\mathbb{L}_2^2 + \cdots \\ &\quad + \mathbb{L}_{\tau-1}^1\mathbb{L}_{\tau-1}^2\mathbb{L}_{\tau-2}^2\cdots\mathbb{L}_2^2 \\ \bar{\Omega} &= \mathbb{L}_\tau^2\mathbb{L}_{\tau-1}^2\cdots\mathbb{L}_2^2 \\ \Gamma &= \begin{bmatrix} \mathbb{W} + \bar{\mathcal{L}}\mathbb{V}\bar{\mathcal{L}}^T & \mathbb{L}_\tau^1 G_\tau + \bar{\mathcal{L}}\mathbb{V}\bar{\Omega}^T \\ G_\tau^T\mathbb{L}_\tau^{1T} + \bar{\Omega}\mathbb{V}\bar{\mathcal{L}}^T & \bar{\Omega}\mathbb{V}\bar{\Omega}^T - G_\tau - G_\tau^T \end{bmatrix}\end{aligned}$$

and the matrix G_τ is non-singular. If $\Gamma < 0$ then (8) holds. Hence we can write

$$\begin{bmatrix} \mathbb{W} & \mathbb{L}_\tau^1 G_\tau \\ G_\tau^T\mathbb{L}_\tau^{1T} & -G_\tau - G_\tau^T \end{bmatrix} + \begin{bmatrix}\bar{\mathcal{L}} \\ \bar{\Omega}\end{bmatrix}\mathbb{V}\begin{bmatrix}\bar{\mathcal{L}}^T & \bar{\Omega}^T\end{bmatrix} < 0, \tag{11}$$

where

$$\begin{aligned}\begin{bmatrix}\bar{\mathcal{L}} \\ \bar{\Omega}\end{bmatrix} &= \begin{bmatrix}\mathbb{L}_1^0 \\ 0\end{bmatrix} + \begin{bmatrix}\mathbb{L}_2^1 \\ 0\end{bmatrix}\mathbb{L}_2^2 + \begin{bmatrix}\mathbb{L}_3^1 \\ 0\end{bmatrix}\mathbb{L}_3^2\mathbb{L}_2^2 + \begin{bmatrix}\mathbb{L}_4^1 \\ 0\end{bmatrix}\mathbb{L}_4^2\mathbb{L}_3^2\mathbb{L}_2^2 \\ &\quad + \cdots + \begin{bmatrix}\mathbb{L}_{\tau-2}^1 \\ 0\end{bmatrix}\mathbb{L}_{\tau-2}^2\mathbb{L}_{\tau-3}^2\cdots\mathbb{L}_2^2 \\ &\quad + \begin{bmatrix}\mathbb{L}_{\tau-1}^1 \\ \mathbb{L}_\tau^2\end{bmatrix}\mathbb{L}_{\tau-1}^2\mathbb{L}_{\tau-2}^2\cdots\mathbb{L}_2^2 \\ &= \bar{\mathcal{L}}_1 + \begin{bmatrix}\mathbb{L}_{\tau-1}^1 \\ \mathbb{L}_\tau^2\end{bmatrix}\bar{\Omega}_1\end{aligned}$$

with

$$\begin{aligned}\bar{\mathcal{L}}_1 &= \begin{bmatrix}\mathbb{L}_1^0 \\ 0\end{bmatrix} + \begin{bmatrix}\mathbb{L}_2^1 \\ 0\end{bmatrix}\mathbb{L}_2^2 + \begin{bmatrix}\mathbb{L}_3^1 \\ 0\end{bmatrix}\mathbb{L}_3^2\mathbb{L}_2^2 + \begin{bmatrix}\mathbb{L}_4^1 \\ 0\end{bmatrix}\mathbb{L}_4^2\mathbb{L}_3^2\mathbb{L}_2^2 \\ &\quad + \cdots + \begin{bmatrix}\mathbb{L}_{\tau-2}^1 \\ 0\end{bmatrix}\mathbb{L}_{\tau-2}^2\mathbb{L}_{\tau-3}^2\cdots\mathbb{L}_2^2 \\ \bar{\Omega}_1 &= \mathbb{L}_{\tau-1}^2\mathbb{L}_{\tau-2}^2\cdots\mathbb{L}_2^2.\end{aligned}$$

Consequently (11) can be rewritten in an analogous form to (10) as

$$\begin{aligned}&\mathbb{W}_1 + \left(\bar{\mathcal{L}}_1 + \begin{bmatrix}\mathbb{L}_{\tau-1}^1 \\ \mathbb{L}_\tau^2\end{bmatrix}\bar{\Omega}_1\right)\mathbb{V}\left(\bar{\mathcal{L}}_1 + \begin{bmatrix}\mathbb{L}_{\tau-1}^1 \\ \mathbb{L}_\tau^2\end{bmatrix}\bar{\Omega}_1\right)^T \\ &\quad = \left[\, I \,\middle|\, \begin{matrix}\mathbb{L}_{\tau-1}^1 \\ \mathbb{L}_\tau^2\end{matrix}\right]\Gamma_1\begin{bmatrix} I \\ \hline \mathbb{L}_{\tau-1}^{1T} \quad \mathbb{L}_\tau^{2T}\end{bmatrix} < 0, \end{aligned} \tag{12}$$

where

$$\begin{aligned}\mathbb{W}_1 &= \begin{bmatrix}\mathbb{W} & \mathbb{L}_\tau^1 G_\tau \\ G_\tau^T\mathbb{L}_\tau^{1T} & -G_\tau - G_\tau^T\end{bmatrix} \\ \Gamma_1 &= \begin{bmatrix} \mathbb{W}_1 + \bar{\mathcal{L}}_1\mathbb{V}\bar{\mathcal{L}}_1^T & \begin{bmatrix}\mathbb{L}_{\tau-1}^1 \\ \mathbb{L}_\tau^2\end{bmatrix}G_{\tau-1} + \bar{\mathcal{L}}_1\mathbb{V}\bar{\Omega}_1^T \\ G_{\tau-1}^T\begin{bmatrix}\mathbb{L}_{\tau-1}^1 \\ \mathbb{L}_\tau^2\end{bmatrix}^T + \bar{\Omega}_1\mathbb{V}\bar{\mathcal{L}}_1^T & \bar{\Omega}_1\mathbb{V}\bar{\Omega}_1^T - G_{\tau-1} - G_{\tau-1}^T \end{bmatrix}\end{aligned}$$

and $G_{\tau-1}$ is non-singular. If $\Gamma_1 < 0$, then (12) holds. Hence we can write

$$\begin{bmatrix} \mathbb{W}_1 & \begin{bmatrix}\mathbb{L}_{\tau-1}^1 \\ \mathbb{L}_\tau^2\end{bmatrix}G_{\tau-1} \\ G_{\tau-1}^T\begin{bmatrix}\mathbb{L}_{\tau-1}^1 \\ \mathbb{L}_\tau^2\end{bmatrix}^T & -G_{\tau-1} - G_{\tau-1}^T \end{bmatrix} + \begin{bmatrix}\bar{\mathcal{L}}_1 \\ \bar{\Omega}_1\end{bmatrix}\mathbb{V}\begin{bmatrix}\bar{\mathcal{L}}_1^T & \bar{\Omega}_1^T\end{bmatrix} < 0, \tag{13}$$

where

$$\begin{aligned}\begin{bmatrix}\bar{\mathcal{L}}_1 \\ \bar{\Omega}_1\end{bmatrix} &= \begin{bmatrix}\mathbb{L}_1^0 \\ 0 \\ 0\end{bmatrix} + \begin{bmatrix}\mathbb{L}_2^1 \\ 0 \\ 0\end{bmatrix}\mathbb{L}_2^2 + \begin{bmatrix}\mathbb{L}_3^1 \\ 0 \\ 0\end{bmatrix}\mathbb{L}_3^2\mathbb{L}_2^2 \\ &\quad + \begin{bmatrix}\mathbb{L}_4^1 \\ 0 \\ 0\end{bmatrix}\mathbb{L}_4^2\mathbb{L}_3^2\mathbb{L}_2^2 + \cdots + \begin{bmatrix}\mathbb{L}_{\tau-3}^1 \\ 0 \\ 0\end{bmatrix}\mathbb{L}_{\tau-3}^2\mathbb{L}_{\tau-4}^2\cdots\mathbb{L}_2^2 \\ &\quad + \begin{bmatrix}\mathbb{L}_{\tau-2}^1 \\ 0 \\ \mathbb{L}_{\tau-1}^2\end{bmatrix}\mathbb{L}_{\tau-2}^2\mathbb{L}_{\tau-3}^2\cdots\mathbb{L}_2^2 \\ &= \bar{\mathcal{L}}_2 + \begin{bmatrix}\mathbb{L}_{\tau-2}^1 \\ 0 \\ \mathbb{L}_{\tau-1}^2\end{bmatrix}\bar{\Omega}_2\end{aligned}$$

with

$$\bar{\mathcal{L}}_2 = \begin{bmatrix} \mathbb{L}_1^0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \mathbb{L}_2^1 \\ 0 \\ 0 \end{bmatrix} \mathbb{L}_2^2 + \begin{bmatrix} \mathbb{L}_3^1 \\ 0 \\ 0 \end{bmatrix} \mathbb{L}_3^2 \mathbb{L}_2^2 + \begin{bmatrix} \mathbb{L}_4^1 \\ 0 \\ 0 \end{bmatrix} \mathbb{L}_4^2 \mathbb{L}_3^2 \mathbb{L}_2^2 + \cdots + \begin{bmatrix} \mathbb{L}_{\tau-3}^1 \\ 0 \\ 0 \end{bmatrix} \mathbb{L}_{\tau-3}^2 \mathbb{L}_{\tau-4}^2 \cdots \mathbb{L}_2^2$$

$$\bar{\Omega}_2 = \mathbb{L}_{\tau-2}^2 \mathbb{L}_{\tau-3}^2 \cdots \mathbb{L}_2^2.$$

We can now rewrite (13) as

$$\mathbb{W}_2 + \left(\bar{\mathcal{L}}_2 + \begin{bmatrix} \mathbb{L}_{\tau-2}^1 \\ 0 \\ \mathbb{L}_{\tau-1}^2 \end{bmatrix} \bar{\Omega}_2 \right) \mathbb{V} \left(\bar{\mathcal{L}}_2 + \begin{bmatrix} \mathbb{L}_{\tau-2}^1 \\ 0 \\ \mathbb{L}_{\tau-1}^2 \end{bmatrix} \bar{\Omega}_2 \right)^T = \left[\begin{array}{c|c} I & \begin{matrix} \mathbb{L}_{\tau-2}^1 \\ 0 \\ \mathbb{L}_{\tau-1}^2 \end{matrix} \end{array} \right] \Gamma_2 \left[\begin{array}{c} I \\ \hline \mathbb{L}_{\tau-2}^{1T} \quad 0 \quad \mathbb{L}_{\tau-1}^{2T} \end{array} \right] < 0, \quad (14)$$

where

$$\mathbb{W}_2 = \begin{bmatrix} \mathbb{W}_1 & \begin{bmatrix} \mathbb{L}_{\tau-1}^1 \\ \mathbb{L}_\tau^2 \end{bmatrix} G_{\tau-1} \\ G_{\tau-1}^T \begin{bmatrix} \mathbb{L}_{\tau-1}^1 \\ \mathbb{L}_\tau^2 \end{bmatrix}^T & -G_{\tau-1} - G_{\tau-1}^T \end{bmatrix}$$

$$\Gamma_2 = \left[\begin{array}{c|c} \mathbb{W}_2 + \bar{\mathcal{L}}_2 \mathbb{V} \bar{\mathcal{L}}_2^T & \begin{bmatrix} \mathbb{L}_{\tau-2}^1 \\ 0 \\ \mathbb{L}_{\tau-1}^2 \end{bmatrix} G_{\tau-2} + \bar{\mathcal{L}}_2 \mathbb{V} \bar{\Omega}_2^T \\ G_{\tau-2}^T \begin{bmatrix} \mathbb{L}_{\tau-2}^1 \\ 0 \\ \mathbb{L}_{\tau-1}^2 \end{bmatrix}^T + \bar{\Omega}_2 \mathbb{V} \bar{\mathcal{L}}_2^T & \bar{\Omega}_2 \mathbb{V} \bar{\Omega}_2^T - G_{\tau-2} - G_{\tau-2}^T \end{array} \right]$$

and the matrix $G_{\tau-2}$ is non-singular. If $\Gamma_2 < 0$ then (14) holds and we can write

$$\begin{bmatrix} \mathbb{W}_2 & \begin{bmatrix} \mathbb{L}_{\tau-2}^1 \\ 0 \\ \mathbb{L}_{\tau-1}^2 \end{bmatrix} G_{\tau-2} \\ G_{\tau-2}^T \begin{bmatrix} \mathbb{L}_{\tau-2}^1 \\ 0 \\ \mathbb{L}_{\tau-1}^2 \end{bmatrix}^T & -G_{\tau-2} - G_{\tau-2}^T \end{bmatrix} + \begin{bmatrix} \bar{\mathcal{L}}_2 \\ \bar{\Omega}_2 \end{bmatrix} \mathbb{V} \begin{bmatrix} \bar{\mathcal{L}}_2^T & \bar{\Omega}_2^T \end{bmatrix} < 0. \quad (15)$$

We can clearly continue this procedure until

$$\mathbb{W}_{\tau-2} + \left(\bar{\mathcal{L}}_{\tau-2} + \begin{bmatrix} \mathbb{L}_2^1 \\ 0 \\ \vdots \\ 0 \\ \mathbb{L}_3^2 \end{bmatrix} \bar{\Omega}_{\tau-2} \right) \mathbb{V} \left(\bar{\mathcal{L}}_{\tau-2} + \begin{bmatrix} \mathbb{L}_2^1 \\ 0 \\ \vdots \\ 0 \\ \mathbb{L}_3^2 \end{bmatrix} \bar{\Omega}_{\tau-2} \right)^T = \left[\begin{array}{c|c} I & \begin{matrix} \mathbb{L}_2^1 \\ 0 \\ \vdots \\ 0 \\ \mathbb{L}_3^2 \end{matrix} \end{array} \right] \Gamma_{\tau-2} \left[\begin{array}{c} I \\ \hline \mathbb{L}_2^{1T} \quad 0 \ \cdots \ 0 \quad \mathbb{L}_3^{2T} \end{array} \right] < 0 \quad (16)$$

where

$$\bar{\mathcal{L}}_{\tau-2} = \begin{bmatrix} \mathbb{L}_1^0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \bar{\Omega}_{\tau-2} = \mathbb{L}_2^2$$

and

$$\mathbb{W}_{\tau-2} = \left[\begin{array}{cc|c|c|c|c} \mathbb{W} & \mathbb{L}_\tau^1 G_\tau & \begin{bmatrix} \mathbb{L}_{\tau-1}^1 \\ \mathbb{L}_\tau^2 \end{bmatrix} G_{\tau-1} & \begin{bmatrix} \mathbb{L}_{\tau-2}^1 \\ 0 \\ \mathbb{L}_{\tau-1}^2 \end{bmatrix} G_{\tau-2} & \cdots & \begin{bmatrix} \mathbb{L}_3^1 \\ 0 \\ \vdots \\ 0 \\ \mathbb{L}_4^2 \end{bmatrix} G_3 \\ G_\tau^T \mathbb{L}_\tau^{1T} & -G_\tau - G_\tau^T & & & & \\ \hline \multicolumn{2}{c|}{G_{\tau-1}^T \begin{bmatrix} \mathbb{L}_{\tau-1}^1 \\ \mathbb{L}_\tau^2 \end{bmatrix}^T} & -G_{\tau-1} - G_{\tau-1}^T & & & \\ \hline \multicolumn{3}{c|}{G_{\tau-2}^T \begin{bmatrix} \mathbb{L}_{\tau-2}^1 \\ 0 \\ \mathbb{L}_{\tau-1}^2 \end{bmatrix}^T} & -G_{\tau-2} - G_{\tau-2}^T & & \\ \hline \multicolumn{4}{c|}{\vdots} & \ddots & \\ \hline \multicolumn{5}{c|}{G_3^T \begin{bmatrix} \mathbb{L}_3^1 \\ 0 \\ \vdots \\ 0 \\ \mathbb{L}_4^2 \end{bmatrix}^T} & -G_3 - G_3^T \end{array} \right]$$

$$\Gamma_{\tau-2} = \begin{bmatrix} \mathbb{W}_{\tau-2} + \bar{\mathcal{L}}_{\tau-2}\mathbb{V}\bar{\mathcal{L}}_{\tau-2}^T & \begin{bmatrix} \mathbb{L}_2^1 \\ 0 \\ \vdots \\ 0 \\ \mathbb{L}_3^2 \end{bmatrix} G_2 + \bar{\mathcal{L}}_{\tau-2}\mathbb{V}\bar{\Omega}_{\tau-2}^T \\ G_2^T \begin{bmatrix} \mathbb{L}_2^1 \\ 0 \\ \vdots \\ 0 \\ \mathbb{L}_3^2 \end{bmatrix}^T + \bar{\Omega}_{\tau-2}\mathbb{V}\bar{\mathcal{L}}_{\tau-2}^T & \bar{\Omega}_{\tau-2}\mathbb{V}\bar{\Omega}_{\tau-2}^T - G_2 - G_2^T \end{bmatrix},$$

where G_2 and G_3 are non-singular. If $\Gamma_{\tau-2} < 0$ then (16) holds and we can write

$$\begin{bmatrix} \mathbb{W}_{\tau-2} & \begin{bmatrix} \mathbb{L}_2^1 \\ 0 \\ \vdots \\ 0 \\ \mathbb{L}_3^2 \end{bmatrix} G_2 \\ G_2^T \begin{bmatrix} \mathbb{L}_2^1 \\ 0 \\ \vdots \\ 0 \\ \mathbb{L}_3^2 \end{bmatrix}^T & -G_2 - G_2^T \end{bmatrix} + \begin{bmatrix} \bar{\mathcal{L}}_{\tau-2} \\ \bar{\Omega}_{\tau-2} \end{bmatrix} \mathbb{V} \begin{bmatrix} \bar{\mathcal{L}}_{\tau-2}^T & \bar{\Omega}_{\tau-2}^T \end{bmatrix} < 0. \qquad (17)$$

Application of Lemma 1 to (17) now yields (9) where G_1 is non-singular and the proof is complete. □

3. Stability

The stability theory for linear repetitive processes consists of two separate concepts, termed asymptotic stability and stability along the pass respectively (Rogers et al. 2007). In effect, both of these are a form of bounded input bounded output stability of the pass profile sequence (recall the unique control problem for these processes) where asymptotic stability demands this property over the finite and constant pass length and as a consequence that the sequence of pass profiles converge to a steady or so-called limit profile as $k \to \infty$. In the case of processes described by (4) (or (1)) the limit profile is described by a standard (or $1D$) discrete linear systems state-space model. The fact that the pass length is finite, however, could mean that this limit profile is unstable, i.e. all eigenvalues of the state matrix do not lie in the open unit circle in the complex plane. (Over a finite duration even a unstable $1D$ linear system can only produce a bounded output.)

Stability along the pass prevents this undesirable situation from arising by demanding the bounded input bounded output property uniformly, i.e. for each possible value of the pass length.

A necessary and sufficient condition for stability along the pass of processes described by (4) is as follows (see, for example, Galkowski, Rogers, Xu, Lam and Owens (2002) for a derivation of this condition and related analysis/ideas).

Theorem 1: *A discrete linear repetitive process described by* (4) (*or* (1)) *is stable along the pass if, and only if,*

$$\det \begin{bmatrix} I - z_1\widehat{A} & -z_1\widehat{B}_0 \\ -z_2\widehat{C} & I - z_2\widehat{D}_0 \end{bmatrix} \neq 0 \quad \text{in } \overline{D}, \qquad (18)$$

where $\overline{D} := \{(z_1, z_2) : \ |z_1| \leq 1, |z_2| \leq 1\}$.

This condition can be reduced to a set which can be tested by applying well known 1D discrete linear systems tests but these have not provided a suitable basis on which to proceed to control law design. A more practical alternative is to use a Lyapunov approach which can be computed using numerically efficient linear matrix inequality (LMI) methods (Boyd, Feron, El Ghaoui and Balakrishnan 1994) but at the possible cost of conservativeness due to the use of a sufficient but not necessary condition for stability along the pass. Next we give two new stability results which can be computed using LMIs and then form the basis for control law design with a formula for computing the matrix in such a law.

The following result is based on the theory of $1D$ periodic systems (Farges, Peaucelle and Arzelier 2005).

Theorem 2: *A discrete linear repetitive process described by* (3) *is stable along the pass if, for* $i = 1, 2, \ldots, \tau$,

- *there exist matrices* $X_i > 0$ *such that*

$$\Phi_i X_i \Phi_i^T - X_{i+1} < 0, \quad X_{\tau+1} = X_1, \qquad (19)$$

where $X_i = \mathrm{diag}(\mathcal{X}_1, \mathcal{X}_{2i})$ *and* Φ_i *is given in* (5),

- *or, there exist matrices* $X_i > 0$ *such that*

$$\begin{bmatrix} -X_{i+1} & \Phi_i X_i \\ X_i \Phi_i^T & -X_i \end{bmatrix} < 0, \quad X_{\tau+1} = X_1. \qquad (20)$$

Proof: Stability along the pass does not depend on the current pass input terms and with these deleted we can write the process state-space model in the form

$$\begin{bmatrix} x_{l+1}(p+1) \\ y_{l+1}(p) \end{bmatrix} = \begin{bmatrix} A_{l+1} & B_{0,l+1} \\ C_{l+1} & D_{0,l+1} \end{bmatrix} \begin{bmatrix} x_{l+1}(p) \\ y_l(p) \end{bmatrix} \qquad (21)$$

over $p=0,1,\ldots,\alpha-1$, $l=0,1,\ldots$, or

$$\xi(x_{l+1}(p+1),y_{l+1}(p)) = \Phi_{l+1}\xi(x_{l+1}(p),y_l(p)), \quad (22)$$

where $\Phi_{l+1+\tau}=\Phi_{l+1}$ and

$$\xi(x_{l+1}(p+1),y_{l+1}(p)) = \begin{bmatrix} x_{l+1}(p+1) \\ y_{l+1}(p) \end{bmatrix},$$

$$\xi(x_{l+1}(p),y_l(p)) = \begin{bmatrix} x_{l+1}(p) \\ y_l(p) \end{bmatrix}.$$

Consider now the following candidate Lyapunov function expressed in terms of (22)

$$\begin{aligned} V(l,p) &= V_1(l,p)+V_2(l,p) \\ &= x_{l+1}^T(p)\mathcal{P}_1 x_{l+1}(p) + y_l^T(p)\mathcal{P}_{2,l+1}y_l(p) \end{aligned}$$

for some $\mathcal{P}_1>0$ and $\mathcal{P}_{2,l+1}>0$, and associated increment

$$\Delta V(l,p) = \Delta V_1(l,p)+\Delta V_2(l,p), \quad (23)$$

where

$$\Delta V_1(l,p) = x_{l+1}^T(p+1)\mathcal{P}_1 x_{l+1}(p+1) - x_{l+1}^T(p)\mathcal{P}_1 x_{l+1}(p)$$
$$\Delta V_2(l,p) = y_{l+1}^T(p)\mathcal{P}_{2,l+2}y_{l+1}(p) - y_l^T(p)\mathcal{P}_{2,l+1}y_l(p).$$

Then by direct application of results in Rogers et al. (2007) we have that stability along the pass holds if

$$\Delta V(l,p) < 0. \quad (24)$$

Routine analysis now shows this last condition can be replaced by

$$\xi^T(x_{l+1}(p),y_l(p))\left(\Phi_{l+1}^T P_{l+2}\Phi_{l+1} - P_{l+1}\right)\xi(x_{l+1}(p),y_l(p)) < 0$$

or, equivalently,

$$\Phi_{l+1}^T P_{l+2}\Phi_{l+1} - P_{l+1} < 0,$$

where $P_{l+1}=\mathrm{diag}(\mathcal{P}_1,\mathcal{P}_{2,l+1})$ is a τ-periodic matrix. Hence a sufficient condition for stability along the pass is that there exist matrices $P_i>0$ such that, for all $i=1,2,\ldots,\tau$,

$$\Phi_i^T P_{i+1}\Phi_i - P_i < 0, \quad P_{\tau+1}=P_1, \quad (25)$$

where $P_i=\mathrm{diag}(\mathcal{P}_1,\mathcal{P}_{2i})$.

Simple algebraic operations applied to this last condition plus setting $X_i=P_i^{-1}$ now establishes (20). □

The following is an alternative result in terms of the lifted process model.

Theorem 3: *A discrete linear repetitive process with dynamics which can be written in the form* (4) *is stable along the pass if*

- *there exists matrix $X>0$ such that*

$$\widehat{\Phi}X\widehat{\Phi}^T - X < 0, \quad (26)$$

where $X=\mathrm{diag}(X_1,X_2)$ and $\widehat{\Phi}$ is given in (6),

- *or, there exist matrices $X>0$, V and Z*

$$\begin{bmatrix} -X & \bar{A}_\tau^1 G_\tau & \bar{A}_{\tau-1}^1 Z & \bar{A}_{\tau-2}^1 Z & \cdots & \bar{A}_2^1 Z & \bar{A}_1^0 V \\ Z^T\bar{A}_\tau^{1T} & -Z-Z^T & \bar{A}_\tau^2 Z & 0 & \cdots & 0 & 0 \\ Z^T\bar{A}_{\tau-1}^{1T} & Z^T\bar{A}_\tau^{2T} & -Z-Z^T & \bar{A}_{\tau-1}^2 Z & \cdots & 0 & 0 \\ Z^T\bar{A}_{\tau-2}^{1T} & 0 & Z^T\bar{A}_{\tau-1}^{2T} & -Z-Z^T & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ Z^T\bar{A}_2^{1T} & 0 & 0 & 0 & \cdots & -Z-Z^T & \bar{A}_2^2 V \\ V^T\bar{A}_1^{0T} & 0 & 0 & 0 & \cdots & V^T\bar{A}_2^{2T} & X-V-V^T \end{bmatrix} < 0 \quad (27)$$

where X is defined in the condition above and $\bar{A}_1^0$, $\bar{A}_i^j$, for $i=2,\ldots,\tau$, $j=1,2$, such that

$$\begin{aligned} \widehat{\Phi} &= \bar{A}_1^0 + \big(\bar{A}_2^1 + \big(\bar{A}_3^1 + \big(\bar{A}_4^1 + \cdots \\ &\quad + \big(\bar{A}_{\tau-1}^1 + \bar{A}_\tau^1\bar{A}_\tau^2\big)\bar{A}_{\tau-1}^2\cdots\big)\bar{A}_4^2\big)\bar{A}_3^2\big)\bar{A}_2^2 \\ &= \bar{A}_1^0 + \bar{A}_2^1\bar{A}_2^2 + \bar{A}_3^1\bar{A}_3^2\bar{A}_2^2 + \bar{A}_4^1\bar{A}_4^2\bar{A}_3^2\bar{A}_2^2 + \cdots \\ &\quad + \bar{A}_\tau^1\bar{A}_\tau^2\bar{A}_{\tau-1}^2\cdots\bar{A}_3^2\bar{A}_2^2 \end{aligned} \quad (28)$$

and are given by

$$\bar{A}_1^0 = \begin{bmatrix} A_1 & 0 & \cdots & 0 & 0 & B_{01} \\ 0 & A_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & A_{\tau-1} & 0 & 0 \\ 0 & 0 & \cdots & 0 & A_\tau & 0 \\ 0 & 0 & \cdots & 0 & C_\tau & 0 \end{bmatrix},$$

$$\bar{A}_2^1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ B_{02} & 0 & \cdots & 0 & 0 & 0 & B_{02} \\ 0 & B_{03} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\ 0 & 0 & \cdots & B_{0,\tau-1} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & B_{0,\tau} & 0 & 0 \\ 0 & 0 & \cdots & 0 & D_{0,\tau} & 0 & 0 \end{bmatrix}$$

$$\bar{A}_2^2 = \begin{bmatrix} C_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & C_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & C_{\tau-1} & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & D_{01} \end{bmatrix},$$

$$\bar{A}_3^1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & B_{03} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & B_{0,\tau-1} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & B_{0,\tau} & 0 & 0 \\ 0 & 0 & \cdots & 0 & D_{0,\tau} & 0 & 0 \end{bmatrix}$$

$$\bar{A}_3^2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ D_{02} & 0 & \cdots & 0 & 0 & 0 & D_{02} \\ 0 & D_{03} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & D_{0,\tau-1} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{A}_4^1 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & B_{04} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & B_{0,\tau-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & B_{0,\tau} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & D_{0,\tau} & 0 & 0 \end{bmatrix}$$

$$\bar{A}_4^2 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & D_{03} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & D_{0,\tau-1} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{A}_5^1 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & B_{05} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & B_{0,\tau} & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & D_{0,\tau} & 0 & 0 \end{bmatrix}$$

$$\bar{A}_5^2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & D_{04} & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & D_{0,\tau-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix}, \quad etc$$

(*Note that if* $\tau \geq 4$ *then the matrices with subscript* $(4+i)$, *for* $i = 0, 1, \ldots,$ *and each superscript* $j = 1, 2$ *are obtained from those with subscript* $(3+i)$ *and superscript* j *by replacing the first non-zero row with a zero row*.)

Proof: With the current pass input terms deleted, write the process dynamics in the form

$$\begin{bmatrix} X_{k+1}(p+1) \\ Y_{k+1}(p) \end{bmatrix} = \begin{bmatrix} \widehat{A} & \widehat{B}_0 \\ \widehat{C} & \widehat{D}_0 \end{bmatrix} \begin{bmatrix} X_{k+1}(p) \\ Y_k(p) \end{bmatrix} \tag{29}$$

over $p = 0, 1, \ldots, \alpha - 1$, $k = 0, 1, \ldots,$ or more compactly

$$\xi(X_{k+1}(p+1), Y_{k+1}(p)) = \widehat{\Phi}\, \xi(X_{k+1}(p), Y_k(p)), \tag{30}$$

where

$$\xi(X_{k+1}(p+1), Y_{k+1}(p)) = \begin{bmatrix} X_{k+1}(p+1) \\ Y_{k+1}(p) \end{bmatrix},$$

$$\xi(X_{k+1}(p), Y_k(p)) = \begin{bmatrix} X_{k+1}(p) \\ Y_k(p) \end{bmatrix}.$$

Proceeding as in the proof of the previous theorem now shows that stability along the pass holds if

$$\widehat{\Phi}^T P \widehat{\Phi} - P < 0,$$

where $P = \text{diag}(P_1, P_2)$. Simple algebraic manipulations on this last condition and setting $X = P^{-1}$ gives (26). Factorising $\widehat{\Phi}$ into the form (28) and applying Lemma 2 leads to (27). □

These last conditions can be computed in a numerically reliable and efficient manner.

4. Stabilisation

In this section we consider the design of control laws to stabilise an example which is unstable along the pass. In particular, we consider a control law of the form

$$\begin{aligned} u_{l+1}(p) &= K_1^{l+1} x_{l+1}(p) + K_2^{l+1} y_l(p) \\ K_1^{l+1+\tau} &= K_1^{l+1}, \\ K_2^{l+1+\tau} &= K_2^{l+1} \end{aligned} \tag{31}$$

with $l = 0, 1, \ldots,$ or, equivalently, in lifted form,

$$U_{k+1}(p) = \widehat{K}_1 X_{k+1}(p) + \widehat{K}_2 Y_k(p) \tag{32}$$

with $k = 0, 1, \ldots,$ where

$$\widehat{K}_1 = \left[\begin{array}{c|c|c|c|c} K_1^1 & 0 & \cdots & 0 & 0 \\ K_2^2(C_1 + D_1 K_1^1) & K_1^2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ K_2^{\tau-1}(D_{0,\tau-2} + D_{\tau-2}K_2^{\tau-2})\ldots(D_{02} + D_2 K_2^2)(C_1 + D_1 K_1^1) & K_2^{\tau-1}(D_{0,\tau-2} + D_{\tau-2}K_2^{\tau-2})\ldots(D_{03} + D_3 K_2^3)(C_2 + D_2 K_1^2) & \cdots & K_1^{\tau-1} & 0 \\ K_2^{\tau}(D_{0,\tau-1} + D_{\tau-1}K_2^{\tau-1})\ldots(D_{02} + D_2 K_2^2)(C_1 + D_1 K_1^1) & K_2^{\tau}(D_{0,\tau-1} + D_{\tau-1}K_2^{\tau-1})\ldots(D_{03} + D_3 K_2^3)(C_2 + D_2 K_1^2) & \cdots & K_2^{\tau}(C_{\tau-1} + D_{\tau-1}K_1^{\tau-1}) & K_1^{\tau} \end{array}\right]$$

$$\widehat{K}_2 = \begin{bmatrix} K_2^1 \\ K_2^2(D_{01} + D_1 K_2^1) \\ \vdots \\ K_2^{\tau-1}(D_{0,\tau-2} + D_{\tau-2}K_2^{\tau-2})\ldots(D_{01} + D_1 K_2^1) \\ K_2^{\tau}(D_{0,\tau-1} + D_{\tau-1}K_2^{\tau-1})\ldots(D_{01} + D_1 K_2^1) \end{bmatrix}.$$

Suppose now that this control law is applied to (3) or (4) respectively. Then the resulting stabilised processes are described by

$$\begin{cases} x_{l+1}(p+1) = \left(A_{l+1} + B_{l+1}K_1^{l+1}\right)x_{l+1}(p) \\ \qquad + \left(B_{0,l+1} + B_{l+1}K_2^{l+1}\right)y_l(p) \\ y_{l+1}(p) = \left(C_{l+1} + D_{l+1}K_1^{l+1}\right)x_{l+1}(p) \\ \qquad + \left(D_{0,l+1} + D_{l+1}K_2^{l+1}\right)y_l(p) \end{cases} \tag{33}$$

and

$$\begin{cases} X_{k+1}(p+1) = (\widehat{A} + \widehat{B}\widehat{K}_1)X_{k+1}(p) \\ \qquad + (\widehat{B}_0 + \widehat{B}\widehat{K}_2)Y_k(p) \\ Y_{k+1}(p) = (\widehat{C} + \widehat{D}\widehat{K}_1)X_{k+1}(p) \\ \qquad + (\widehat{D}_0 + \widehat{D}\widehat{K}_2)Y_k(p) \end{cases} \tag{34}$$

respectively. Also introduce the notation

$$\Phi_{i,new} = \begin{bmatrix} A_{i,new} & B_{0,i,new} \\ C_{i,new} & D_{0,i,new} \end{bmatrix} = \Phi_i + \Pi_i K^i, \quad \text{for} \quad i = 1, 2, \ldots, \tau, \tag{35}$$

where Φ_i and Π_i are again given by (5), $K^i = [\,K_1^i \quad K_2^i\,]$, and

$$\widehat{\Phi}_{new} = \begin{bmatrix} \widehat{A}_{new} & \widehat{B}_{0new} \\ \widehat{C}_{new} & \widehat{D}_{0new} \end{bmatrix} = \widehat{\Phi} + \widehat{\Pi}\widehat{K}, \tag{36}$$

where $\widehat{\Phi}$ and $\widehat{\Pi}$ are again given by (6) and $\widehat{K} = [\,\widehat{K}_1 \quad \widehat{K}_2\,]$

Theorem 4: *Suppose that a control law of the form* (31) *is applied to a discrete linear repetitive process described by* (3). *Then the resulting process is stable along the pass* *if there exist matrices* $X_i > 0$ *and* L_i *such that, for* $i = 1, 2, \ldots, \tau$,

$$\begin{bmatrix} -X_{i+1} & \Phi_i X_i + \Pi_i L_i \\ X_i \Phi_i^T + L_i^T \Pi_i^T & -X_i \end{bmatrix} < 0, \quad X_{\tau+1} = X_1, \tag{37}$$

where $X_i = \text{diag}(\mathcal{X}_1, \mathcal{X}_{2i})$ *and the matrices* Φ_i, Π_i *are again given by* (5).

If this condition holds, then the control law matrices are given by

$$K^i = \begin{bmatrix} K_1^i & K_2^i \end{bmatrix} = L_i X_i^{-1}. \tag{38}$$

Proof: Interpreting Theorem 2 in terms of the controlled process state-space model gives stability along the pass if there exist matrices $X_i > 0$ such that, for all $i = 1, 2, \ldots, \tau$,

$$\begin{bmatrix} -X_{i+1} & \Phi_{i,new} X_i \\ X_i \Phi_{i,new}^T & -X_i \end{bmatrix} < 0, \quad X_{\tau+1} = X_1.$$

where $X_i = \text{diag}(\mathcal{X}_1, \mathcal{X}_{2i})$ and $\Phi_{i,new}$ is given by (35). Setting

$$\Phi_{i,new} X_i = \Phi_i X_i + \Pi_i K^i X_i = \Phi_i X_i + \Pi_i L_i$$

now gives (37) with the control law matrices given by (38) and the proof is complete. □

This last result can introduce conservatism since the matrices X_i have a prescribed block-diagonal structure with a constant matrix in each diagonal entry. Introduction of additional variables may lead to a less conservative condition and we have the following result.

Theorem 5: *Suppose that a control law of the form* (31) *is applied to a discrete linear repetitive process which can be written in the form* (3). *Then the resulting process is stable along the pass if there exist matrices* $X_i > 0$, *non-singular matrices* G_i *and matrices* L_i *such that, for* $i = 1, 2, \ldots, \tau$,

$$\begin{bmatrix} -X_{i+1} & \Phi_i G_i + \Pi_i L_i \\ G_i^T \Phi_i^T + L_i^T \Pi_i^T & X_i - G_i - G_i^T \end{bmatrix} < 0, \quad X_{\tau+1} = X_1, \tag{39}$$

where $X_i = \text{diag}(\mathcal{X}_1, \mathcal{X}_{2i})$ *and the matrices* Φ_i, Π_i *are again given in* (6).

If this condition holds, then the control law matrices are given by

$$K^i = \begin{bmatrix} K_1^i & K_2^i \end{bmatrix} = L_i G_i^{-1} \tag{40}$$

Proof: Interpreting Theorem 2 in terms of the controlled process state-space model gives stability along the pass if there exist matrices $X_i > 0$ and G_i such that, for all $i = 1, 2, \ldots, \tau$,

$$\begin{bmatrix} -X_{i+1} & \Phi_{i,new} G_i \\ G_i^T \Phi_{i,new}^T & X_i - G_i - G_i^T \end{bmatrix} < 0, \quad X_{\tau+1} = X_1,$$

where $X_i = \text{diag}(\mathcal{X}_1, \mathcal{X}_{2i})$ and $\Phi_{i,new}$ is given by (35). Setting

$$\Phi_{i,new} G_i = \Phi_i G_i + \Pi_i K^i G_i = \Phi_i G_i + \Pi_i L_i$$

now gives (39) with the control law matrices given by (35) and the proof is complete. □

In the case of the lifted model, first note that with the control law applied we can write

$$\widehat{\Phi}_{new} = \bar{\Phi} + \bar{\Pi}\big(\bar{\Phi}_1 + \big(\bar{\Phi}_2 + \big(\bar{\Phi}_4 + \cdots + (\Phi_{2\tau-6} + \Phi_{2\tau-4}\Phi_{2\tau-3})\Phi_{2\tau-5}\cdots\big)\bar{\Phi}_5\big)\bar{\Phi}_3\big) \tag{41}$$

where

$$\bar{\Phi} = \widehat{\Phi} + \widehat{\Pi}\bar{\bar{K}}_1, \quad \bar{\Pi} = \widehat{\Pi}\bar{\bar{K}}_2,$$
$$\bar{\Phi}_1 = \bar{A}_1 + \bar{B}_1\bar{\bar{K}}_1, \quad \bar{\Phi}_2 = \bar{A}_2 + \bar{B}_2\bar{\bar{K}}_2, \quad \bar{\Phi}_3 = \bar{A}_3 + \bar{B}_3\bar{\bar{K}}_1,$$
$$\bar{\Phi}_4 = \bar{A}_4 + \bar{B}_4\bar{\bar{K}}_2, \quad \ldots, \quad \bar{\Phi}_{2\tau-3} = \bar{A}_{2\tau-3} + \bar{B}_{2\tau-3}\bar{\bar{K}}_2$$

with $\widehat{\Phi}$ and $\widehat{\Pi}$ given by (6) and

$$\bar{\bar{K}}_1 = \begin{bmatrix} K_1^1 & 0 & \cdots & 0 & 0 & K_2^1 \\ 0 & K_1^2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & K_1^{\tau-1} & 0 & 0 \\ 0 & 0 & \cdots & 0 & K_1^\tau & 0 \end{bmatrix},$$

$$\bar{\bar{K}}_2 = \begin{bmatrix} K_2^2 & 0 & 0 & \cdots & 0 \\ 0 & K_2^2 & 0 & \cdots & 0 \\ 0 & 0 & K_2^3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & K_2^\tau \end{bmatrix}$$

$$\bar{A}_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 \\ C_1 & 0 & \cdots & 0 & 0 & D_{01} \\ 0 & C_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & C_{\tau-1} & 0 & 0 \end{bmatrix},$$

$$\bar{B}_1 = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ D_1 & 0 & \cdots & 0 & 0 \\ 0 & D_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & D_{\tau-1} & 0 \end{bmatrix}$$

$$\bar{A}_2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ D_{02} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & D_{03} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & D_{0,\tau-1} & 0 \end{bmatrix},$$

$$\bar{B}_2 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ D_2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & D_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & D_{\tau-1} & 0 \end{bmatrix}$$

$$\bar{A}_3 = \begin{bmatrix} C_1 & 0 & \cdots & 0 & 0 & 0 & D_{01} \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & C_2 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & C_{\tau-2} & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{B}_3 = \begin{bmatrix} D_1 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & D_2 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & D_{\tau-2} & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{A}_4 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & D_{03} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & D_{0,\tau-1} & 0 \end{bmatrix},$$

$$\bar{B}_4 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & D_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & D_{\tau-1} & 0 \end{bmatrix}$$

$$\bar{A}_5 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ D_{02} & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & D_{03} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & D_{0,\tau-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{B}_5 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ D_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & D_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & D_{\tau-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

$$\bar{A}_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & D_{02} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & D_{0,\tau-1} & 0 \end{bmatrix},$$

$$\bar{B}_6 = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & D_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & D_{\tau-1} & 0 \end{bmatrix}$$

$$\bar{A}_7 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & D_{03} & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & D_{0,\tau-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix},$$

$$\bar{B}_7 = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & D_3 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & D_{\tau-2} & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}$$

etc....

(Note if $\tau \geq 4$ the matrices with indices $(4+2i)$ and $(5+2i)$, for $i = 0, 1, \ldots$ can be obtained from those for $(2+2i)$ by replacing the first non-zero row with a zero row and the last non-zero row with zero row, respectively.)

Now we have the following result.

Theorem 6: *Suppose that a control law of the form* (32) *is applied to a discrete linear repetitive process whose state-space model can be written in the form* (4). *Then the resulting process is stable along the pass if there exists a matrix* $X > 0$, *non-singular matrices* $\bar{\bar{V}}$, $\bar{\bar{Z}}$, *and matrices* $\bar{\bar{L}}$, $\bar{\bar{N}}$, *such that*

$$\begin{bmatrix} -X & \bar{\Pi}\bar{\bar{Z}} & 0 & 0 & \cdots & 0 & 0 & \bar{\Phi}\bar{\bar{V}} \\ \bar{\bar{Z}}^T\bar{\Pi}^T & -\bar{\bar{Z}}-\bar{\bar{Z}}^T & \bar{\Phi}_{2\tau-4}\bar{\bar{Z}} & \bar{\Phi}_{2\tau-6}\bar{\bar{Z}} & \cdots & \bar{\Phi}_4\bar{\bar{Z}} & \bar{\Phi}_2\bar{\bar{Z}} & \bar{\Phi}_1\bar{\bar{V}} \\ 0 & \bar{\bar{Z}}^T\bar{\Phi}^T_{2\tau-4} & -\bar{\bar{Z}}-\bar{\bar{Z}}^T & \bar{\Phi}_{2\tau-3}\bar{\bar{Z}} & \cdots & 0 & 0 & 0 \\ 0 & \bar{\bar{Z}}^T\bar{\Phi}^T_{2\tau-6} & \bar{\bar{Z}}^T\bar{\Phi}^T_{2\tau-3} & -\bar{\bar{Z}}-\bar{\bar{Z}}^T & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & \bar{\bar{Z}}^T\bar{\Phi}^T_4 & 0 & 0 & \cdots & -\bar{\bar{Z}}-\bar{\bar{Z}}^T & \bar{\Phi}_5\bar{\bar{Z}} & 0 \\ 0 & \bar{\bar{Z}}^T\bar{\Phi}^T_2 & 0 & 0 & \cdots & \bar{\bar{Z}}^T\bar{\Phi}^T_5 & -\bar{\bar{Z}}-\bar{\bar{Z}}^T & \bar{\Phi}_3\bar{\bar{V}} \\ \bar{\bar{V}}^T\bar{\Phi}^T & \bar{\bar{V}}^T\bar{\Phi}^T_1 & 0 & 0 & \cdots & 0 & \bar{\bar{V}}^T\bar{\Phi}^T_3 & X-\bar{\bar{V}}-\bar{\bar{V}}^T \end{bmatrix} < 0, \quad (42)$$

where

$$\bar{\Phi}\bar{\bar{V}} = \widehat{\Phi}\bar{\bar{V}} + \widehat{\Pi}\bar{\bar{L}}, \quad \bar{\Pi}\bar{\bar{Z}} = \widehat{\Pi}\bar{\bar{N}}$$

$$\bar{\Phi}_1\bar{\bar{V}} = \bar{A}_1\bar{\bar{V}} + \bar{B}_1\bar{\bar{L}}, \quad \bar{\Phi}_2\bar{\bar{Z}} = \bar{A}_2\bar{\bar{Z}} + \bar{B}_2\bar{\bar{N}},$$

$$\bar{\Phi}_3\bar{\bar{V}} = \bar{A}_3\bar{\bar{V}} + \bar{B}_3\bar{\bar{L}}, \quad \bar{\Phi}_4\bar{\bar{Z}} = \bar{A}_4\bar{\bar{Z}} + \bar{B}_4\bar{\bar{N}}, \quad \ldots,$$

$$\bar{\Phi}_{2\tau-3}\bar{\bar{Z}} = \bar{A}_{2\tau-3}\bar{\bar{Z}} + \bar{B}_{2\tau-3}\bar{\bar{N}}$$

and

$$X = \mathrm{diag}(X_1, X_2), \quad \bar{\bar{V}} = \mathrm{diag}(V_1, V_2, \ldots, V_\tau, V_{\tau+1})$$

$$\bar{\bar{Z}} = \mathrm{diag}(Z_1, Z_1, Z_2, \ldots, Z_{\tau-1}),$$

$$\bar{\bar{N}} = \mathrm{diag}(N_1, N_1, N_2, \ldots, N_{\tau-1})$$

$$\bar{\bar{L}} = \begin{bmatrix} L_1 & 0 & \cdots & 0 & L_{\tau+1} \\ 0 & L_2 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & L_\tau & 0 \end{bmatrix}.$$

If this condition holds, then the control law matrices are given by

$$\bar{\bar{K}}_1 = \bar{\bar{L}}\bar{\bar{V}}^{-1}, \quad \bar{\bar{K}}_2 = \bar{\bar{N}}\bar{\bar{Z}}^{-1}. \quad (43)$$

Proof: Using Theorem 3, stability along the pass of (34) requires the existence of a matrix $X = \mathrm{diag}(X_1, X_2) > 0$ such that

$$\widehat{\Phi}_{new}X\widehat{\Phi}^T_{new} - X < 0. \quad (44)$$

Also factorising $\widehat{\Phi}_{new}$ into the form (41) yields

$$\widehat{\Phi}_{new}X\widehat{\Phi}^T_{new} - X = (\bar{\Phi} + \bar{\Pi}\bar{\Omega})X(\bar{\Phi} + \bar{\Pi}\bar{\Omega})^T - X = \begin{bmatrix} I & \bar{\Pi} \end{bmatrix}\Gamma\begin{bmatrix} I \\ \bar{\Pi}^T \end{bmatrix} < 0,$$

where

$$\bar{\Omega} = \bar{\Phi}_1 + (\bar{\Phi}_2 + (\bar{\Phi}_4 + \cdots + (\Phi_{2\tau-6} + \Phi_{2\tau-4}\Phi_{2\tau-3})\Phi_{2\tau-5}\cdots)\bar{\Phi}_5)\bar{\Phi}_3,$$

$$\Gamma = \begin{bmatrix} -X + \bar{\Phi}X\bar{\Phi}^T & \bar{\Pi}\bar{\bar{Z}} + \bar{\Phi}X\bar{\Omega}^T \\ \bar{\bar{Z}}^T\bar{\Pi}^T + \bar{\Omega}X\bar{\Phi}^T & \bar{\Omega}X\bar{\Omega}^T - \bar{\bar{Z}} - \bar{\bar{Z}}^T \end{bmatrix}$$

and $\bar{\bar{Z}}$ is a non-symmetric non-singular matrix. If $\Gamma < 0$ then (44) holds. Hence, we can write

$$\begin{bmatrix} -X & \bar{\Pi}\bar{\bar{Z}} \\ \bar{\bar{Z}}^T\bar{\Pi}^T & -\bar{\bar{Z}}-\bar{\bar{Z}}^T \end{bmatrix} + \begin{bmatrix} \bar{\Phi} \\ \bar{\Omega} \end{bmatrix} X \begin{bmatrix} \bar{\Phi}^T & \bar{\Omega}^T \end{bmatrix} < 0$$

where

$$\begin{bmatrix} \bar{\Phi} \\ \bar{\Omega} \end{bmatrix} = \begin{bmatrix} \bar{\Phi} \\ \bar{\Phi}_1 \end{bmatrix} + \begin{bmatrix} 0 \\ \bar{\Phi}_2 \end{bmatrix}\bar{\Phi}_3 + \begin{bmatrix} 0 \\ \bar{\Phi}_4 \end{bmatrix}\bar{\Phi}_5\bar{\Phi}_3 + \cdots + \begin{bmatrix} 0 \\ \Phi_{2\tau-4} \end{bmatrix}\Phi_{2\tau-3}\Phi_{2\tau-5}\cdots\bar{\Phi}_5\bar{\Phi}_3$$

Applying Lemma 2 to this last inequality yields (42) with control law matrices given by (43) and the proof is complete. □

Note that, in comparison to the previous results in this section, this last result uses only one LMI but of higher dimensions.

5. Control for tracking and disturbance rejection

Often the design requirement will be to achieve a limit profile which has acceptable along the pass dynamics. Clearly this requires stability along the pass which guarantees the existence of a limit profile in the form of a stable 1D discrete linear system. Any extra transient response characteristics can then be assessed using 1D tools (e.g. damping ratio and undamped natural frequency in the classical second order lag case). Here we consider control law design for this general requirement in the presence of additive disturbances on both the state and pass profile vectors. Related work for the non-switching case can be found in, for example, Sulikowski (2005).

The process state-space model is now assumed to be

$$\left.\begin{array}{l} \left\{\begin{array}{l} x_{l+1}(p+1) = A_{l+1}x_{l+1}(p) + B_{l+1}u_{l+1}(p) \\ \qquad +B_{0,l+1}y_l(p) + B_{1,l+1}w_{l+1}(p) \\ y_{l+1}(p) = C_{l+1}x_{l+1}(p) + D_{l+1}u_{l+1}(p) \\ \qquad +D_{0,l+1}y_l(p) + D_{1,l+1}w_{l+1}(p) \end{array}\right. \\ A_{l+1+\tau} = A_{l+1}, \quad B_{l+1+\tau} = B_{l+1}, \\ B_{0,l+1+\tau} = B_{0,l+1}, \quad B_{1,l+1+\tau} = B_{1,l+1}, \\ C_{l+1+\tau} = C_{l+1}, \quad D_{l+1+\tau} = D_{l+1}, \\ D_{0,l+1+\tau} = D_{0,l+1}, \quad D_{1,l+1+\tau} = D_{1,l+1} \end{array}\right\} \quad (45)$$

over $p=0,1,\dots,\alpha-1$, $l=0,1,\dots$ Here we also assume that the disturbance signals $w_{l+1+\tau}(p)=w_{l+1}(p)$ only depend on the along the pass dynamics. Moreover, we do not require the lifted model representation.

In terms of control law design, the simplest way only involves one variation to the analysis of the previous section. Suppose therefore that stability along the pass holds and hence the limit profile exists. Suppose also that $y_{ref}(p)$ denotes the desired limit profile signal and denote τ successive vectors by the notation $\{x_{1k}(p), x_{2k}(p), \dots, x_{\tau k}(p)\}$ and in the limit as $k\to\infty$ by

$$x_{1\infty}(p), \quad x_{2\infty}(p), \quad \dots, \quad x_{\tau\infty}(p)$$

Next, define the so-called "incremental vectors", for $l=0,1,\dots$ as

$$\hat{x}_{l+1}(p) = x_{l+1}(p) - x_{\varsigma\infty}(p)$$
$$\hat{u}_{l+1}(p) = u_{l+1}(p) - u_{\varsigma\infty}(p)$$
$$\hat{y}_l(p) = y_l(p) - y_{ref}(p)$$

where $\varsigma=1+\text{mod}(l,\tau)$ and $\text{mod}(\cdot,\tau)$ denotes the division modulo τ, or equivalently, for $k=0,1,\dots,$

$$\widehat{X}_{k+1}(p) = \begin{bmatrix} \hat{x}_{\tau k+1}(p) \\ \hat{x}_{\tau k+2}(p) \\ \vdots \\ \hat{x}_{\tau k+\tau}(p) \end{bmatrix} = \begin{bmatrix} x_{\tau k+1}(p) - x_{1\infty}(p) \\ x_{\tau k+2}(p) - x_{2\infty}(p) \\ \vdots \\ x_{\tau k+\tau}(p) - x_{\tau\infty}(p) \end{bmatrix} = X_{k+1}(p) - X_{\infty}(p)$$

$$\widehat{U}_{k+1}(p) = \begin{bmatrix} \hat{u}_{\tau k+1}(p) \\ \hat{u}_{\tau k+2}(p) \\ \vdots \\ \hat{u}_{\tau k+\tau}(p) \end{bmatrix} = \begin{bmatrix} u_{\tau k+1}(p) - u_{1\infty}(p) \\ u_{\tau k+2}(p) - u_{2\infty}(p) \\ \vdots \\ u_{\tau k+\tau}(p) - u_{\tau\infty}(p) \end{bmatrix} = U_{k+1}(p) - U_{\infty}(p)$$

$$\widehat{Y}_{k+1}(p) = \hat{y}_{\tau k+\tau}(p) = y_{\tau k+\tau}(p) - y_{ref}(p) = Y_{k+1}(p) - y_{ref}(p).$$

The vectors $u_{i\infty}(p)$ for $i=1,2,\dots,\tau$ will be defined below.

At this stage we can introduce the so-called "incremental model"

$$\left.\begin{array}{l} \left\{\begin{array}{l} \hat{x}_{l+1}(p+1) = A_{l+1}\hat{x}_{l+1}(p) + B_{l+1}\hat{u}_{l+1}(p) \\ \qquad +B_{0,l+1}\hat{y}_l(p) \\ \hat{y}_{l+1}(p) = C_{l+1}\hat{x}_{l+1}(p) + D_{l+1}\hat{u}_{l+1}(p) \\ \qquad +D_{0,l+1}\hat{y}_l(p) \end{array}\right. \\ A_{l+1+\tau} = A_{l+1}, \quad B_{l+1+\tau} = B_{l+1}, \\ B_{0,l+1+\tau} = B_{0,l+1} \\ C_{l+1+\tau} = C_{l+1}, \quad D_{l+1+\tau} = D_{l+1}, \\ D_{0,l+1+\tau} = D_{0,l+1} \end{array}\right\} \quad (46)$$

over $p=0,1,\ldots,\alpha-1$, $l=0,\ 1,\ldots$ This model has the same structure as (3) but the influence of the disturbance term has been completely decoupled. Moreover, it has the same stability along the pass properties as (3) and (4). Also stability along the pass here means that $\lim_{l\to\infty}\hat{y}_{l+1}(p)=0$ and $\lim_{l\to\infty}\hat{x}_{l+1}(p)=0$. Hence $\lim_{l\to\infty}\ y_{l+1}(p)=y_{ref}(p)$ and the design objectives have been achieved (stability along the pass and the required limit profile) under the control law

$$\left.\begin{aligned}\hat{u}_{l+1}(p)&=K_1^{l+1}\hat{x}_{l+1}(p)+K_2^{l+1}\hat{y}_l(p)\\ K_1^{l+1+\tau}&=K_1^{l+1},\\ K_2^{l+1+\tau}&=K_2^{l+1}\end{aligned}\right\}\tag{47}$$

or, in terms of the original variables,

$$\begin{aligned}u_{l+1}(p)=u_{\varsigma\infty}(p)&+K_1^{\varsigma}\big(x_{l+1}(p)-x_{\varsigma\infty}(p)\big)\\ &+K_2^{\varsigma}\big(y_l(p)-y_{ref}(p)\big).\end{aligned}\tag{48}$$

Equivalently for $k=0,1,\ldots,$

$$\begin{bmatrix}u_{\tau k+1}(p)\\ u_{\tau k+2}(p)\\ \vdots\\ u_{\tau k+\tau}(p)\end{bmatrix}=\begin{bmatrix}u_{1\infty}(p)+K_1^1(x_{\tau k+1}(p)-x_{1\infty}(p))+K_2^1\big(y_{\tau k}(p)-y_{ref}(p)\big)\\ u_{2\infty}(p)+K_1^2(x_{\tau k+2}(p)-x_{2\infty}(p))+K_2^2\big(y_{\tau k+1}(p)-y_{ref}(p)\big)\\ \vdots\\ u_{\tau\infty}(p)+K_1^{\tau}(x_{\tau k+\tau}(p)-x_{\tau\infty}(p))+K_2^{\tau}\big(y_{\tau k+\tau-1}(p)-y_{ref}(p)\big)\end{bmatrix}\tag{49}$$

At this stage, it is a necessity to know sequences $x_{i\infty}(p+1)$ and $u_{i\infty}(p)$, for $i=1,2,\ldots,\tau$, $p=0,1,\ldots,\alpha-1$ (which are not explicitly present in (48)) in order to apply the control. The remaining sequences, i.e. $w_i(p)$, for $i=1,2,\ldots,\tau$, and also $y_{ref}(p)$ are assumed to be known. These can be obtained using

$$\begin{aligned}x_{i\infty}(p+1)=&\big(A_i-B_iD_i^{-1}C_i\big)x_{i\infty}(p)\\ &+\big(B_{0i}+B_iD_i^{-1}(I-D_{0i})\big)y_{ref}(p)\\ &+\big(B_{1i}-B_iD_i^{-1}D_{1i}\big)w_i(p)\\ u_{i\infty}(p)=&D_i^{-1}\big((I-D_{0i})y_{ref}(p)\\ &-C_ix_{i\infty}(p)-D_{1i}w_i(p)\big)\end{aligned}\tag{50}$$

assuming the matrices D_i, for $i=1,2,\ldots,\tau$, are non-singular. If not then replace the inverse by the pseudo-inverse.

Application of the control law (48) (or equivalently (49)) to (45) now yields a process which is stable along the pass and the sequence of pass profiles $y_{l+1}(p)$ produced converge to the required limit profile $y_{ref}(p)$ as $l\to\infty$. The corresponding control law matrices K_1^i, K_2^i for $i=1,2,\ldots,\tau$ are computed by applying Theorem 5 or 6 to (46).

6. Application to multi-roll metal rolling

Previous results on the application of repetitive process theory to metal rolling processes can be found in, for example, Galkowski, Rogers, Paszke and Owens (2003). We consider a multi-roll system consisting of three separate pairs of rolls which are controlled by separate input signals, i.e., different rolling forces. The deformation of the workpiece takes place between these pairs of rolls with parallel axes revolving in opposite directions. The metal strip is to be rolled to a pre-specified thickness (also termed the gauge or shape) by passing it through a series of rolls for successive reductions. The case of more than three sets of rolls follows as a natural generalisation.

In practice, a number of models of this process can be developed depending on the assumptions made about the underlying dynamics and the particular mode of operation under consideration. The particular task is to develop a simplified (but practically feasible) model relating the gauge on the passes through the rolls. The current pass is denoted by $y_{3k+3}(t)$, and the three previous passes by $y_{3k+2}(t)$, $y_{3k+1}(t)$ and $y_{3k}(t)$ respectively. The other process variables and physical constants are defined as follows. $F_{M,3k+1}(t)$, $F_{M,3k+2}(t)$ and $F_{M,3k+3}(t)$ are the forces developed by the motors, $F_{s,3k+1}(t)$, $F_{s,3k+2}(t)$ and $F_{s,3k+3}(t)$ are the forces developed by the springs, M_1, M_2 and M_3 are the lumped masses of the roll-gap adjusting mechanisms, λ_{11}, λ_{12} and λ_{13} are the stiffnesses of the adjustment mechanism springs, λ_2 is the hardness of the metal strip.

Note that this problem is an extension of results in, for example, Bochniak et al. (2006) where only one switch occurred. Here we consider two switches to illustrate the application of the results in this paper with the note that generalisation to more than this number is routine.

With reference to Figure 1, the forces developed by the motors are given by

$$\begin{cases}F_{M,3k+1}(t)=F_{s,3k+1}(t)+M_1\ddot{z}_{3k+1}(t)\\ F_{M,3k+2}(t)=F_{s,3k+2}(t)+M_2\ddot{z}_{3k+2}(t)\\ F_{M,3k+3}(t)=F_{s,3k+3}(t)+M_3\ddot{z}_{3k+3}(t)\end{cases}\tag{51}$$

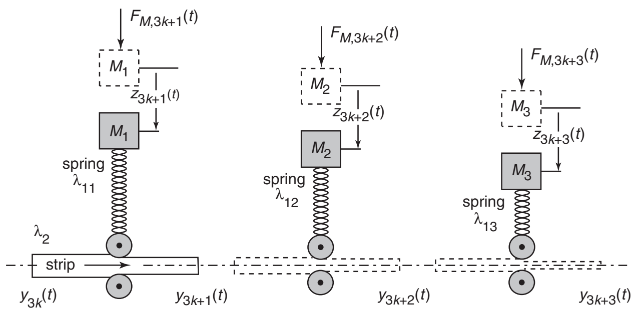

Figure 1. Multi-roll rolling machine operation.

and forces developed by the springs are

$$\begin{cases} F_{s,3k+1}(t) = \lambda_{11}\big(z_{3k+1}(t) + y_{3k+1}(t)\big) \\ F_{s,3k+2}(t) = \lambda_{12}\big(z_{3k+2}(t) + y_{3k+2}(t)\big) \\ F_{s,3k+3}(t) = \lambda_{13}\big(z_{3k+3}(t) + y_{3k+3}(t)\big). \end{cases} \tag{52}$$

Each of these last forces is also applied to the metal strip by the corresponding rolls and hence

$$\begin{cases} F_{s,3k+1}(t) = \lambda_2\big(y_{3k}(t) - y_{3k+1}(t)\big) \\ F_{s,3k+2}(t) = \lambda_2\big(y_{3k+1}(t) - y_{3k+2}(t)\big) \\ F_{s,3k+3}(t) = \lambda_2\big(y_{3k+2}(t) - y_{3k+3}(t)\big). \end{cases} \tag{53}$$

Hence we can write the overall process model in the form of the following three differential-difference equations

$$\begin{cases} \ddot{y}_{3k+1}(t) + a_{01}y_{3k+1}(t) + b_{21}\ddot{y}_{3k}(t) + b_{01}y_{3k}(t) \\ \quad = c_{01}u_{3k+1}(t) \\ \ddot{y}_{3k+2}(t) + a_{02}y_{3k+2}(t) + b_{22}\ddot{y}_{3k+1}(t) + b_{02}y_{3k+1}(t) \\ \quad = c_{02}u_{3k+2}(t) \\ \ddot{y}_{3k+3}(t) + a_{03}y_{3k+3}(t) + b_{23}\ddot{y}_{3k+2}(t) + b_{03}y_{3k+2}(t) \\ \quad = c_{03}u_{3k+3}(t) \end{cases} \tag{54}$$

over $0 \le t \le \alpha$, $k = 0, 1, \ldots$, where, for $i = 1, 2, 3$,

$$a_{0i} = \frac{\lambda_{1i}\lambda_2}{M_i(\lambda_{1i} + \lambda_2)}, \quad b_{2i} = \frac{-\lambda_2}{\lambda_{1i} + \lambda_2},$$
$$b_{0i} = \frac{-\lambda_{1i}\lambda_2}{M_i(\lambda_{1i} + \lambda_2)}, \quad c_{0i} = \frac{-\lambda_{1i}}{M_i(\lambda_{1i} + \lambda_2)}$$

and

$$u_{3k+1}(t) = F_{M,3k+1}(t), \quad u_{3k+2}(t) = F_{M,3k+2}(t),$$
$$u_{3k+3}(t) = F_{M,3k+3}(t).$$

To complete this process description, we specify the boundary conditions, i.e.

$$y_{3k}(0), \quad \dot{y}_{3k}(0), \quad y_{3k+1}(0), \quad \dot{y}_{3k+1}(0), \quad y_{3k+2}(0),$$
$$\dot{y}_{3k+2}(0), \; y_{3k+3}(0), \; \dot{y}_{3k+3}(0)$$

and initial pass profile $y_0(t)$.

The state-space model for the complete operation can now be written as

$$\begin{cases} \dot{x}_{3k+1}(t) = A_{c1}x_{3k+1}(t) + B_{c1}u_{3k+1}(t) + B_{c01}y_{3k}(t) \\ y_{3k+1}(t) = C_{c1}x_{3k+1}(t) + D_{c1}u_{3k+1}(t) + D_{c01}y_{3k}(t) \\ \dot{x}_{3k+2}(t) = A_{c2}x_{3k+2}(t) + B_{c2}u_{3k+2}(t) + B_{c02}y_{3k+1}(t) \\ y_{3k+2}(t) = C_{c2}x_{3k+2}(t) + D_{c2}u_{3k+2}(t) + D_{c02}y_{3k+1}(t) \\ \dot{x}_{3k+3}(t) = A_{c3}x_{3k+3}(t) + B_{c3}u_{3k+3}(t) + B_{c03}y_{3k+2}(t) \\ y_{3k+3}(t) = C_{c3}x_{3k+3}(t) + D_{c3}u_{3k+3}(t) + D_{c03}y_{3k+2}(t) \end{cases} \tag{55}$$

over $0 \le t \le \alpha$, $k = 0, 1, \ldots$, where for $i = 1, 2, 3$,

$$A_{ci} = \begin{bmatrix} 0 & 1 \\ -a_{0i} & 0 \end{bmatrix}, \quad B_{ci} = \begin{bmatrix} 0 \\ c_{0i} \end{bmatrix},$$
$$B_{c0i} = \begin{bmatrix} 0 \\ -b_{0i} + a_{0i}b_{2i} \end{bmatrix}, \quad C_{ci} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad D_{ci} = 0,$$
$$D_{c0i} = -b_{2i} \tag{56}$$

with

$$x_{3k+1}(t) = \begin{bmatrix} y_{3k+1}(t) + b_{21}y_{3k}(t) \\ \dot{y}_{3k+1}(t) + b_{21}\dot{y}_{3k}(t) \end{bmatrix},$$
$$x_{3k+2}(t) = \begin{bmatrix} y_{3k+2}(t) + b_{22}y_{3k+1}(t) \\ \dot{y}_{3k+2}(t) + b_{22}\dot{y}_{3k+1}(t) \end{bmatrix}$$
$$x_{3k+3}(t) = \begin{bmatrix} y_{3k+3}(t) + b_{23}y_{3k+2}(t) \\ \dot{y}_{3k+3}(t) + b_{23}\dot{y}_{3k+2}(t) \end{bmatrix}$$

and boundary conditions

$$x_{3k+1}(0) = d_{3k+1}, \quad x_{3k+2}(0) = d_{3k+2},$$
$$x_{3k+3}(0) = d_{3k+3} \quad \text{and} \quad y_0(t) = f(t)$$

where the vectors d_{3k+1}, d_{3k+2} and d_{3k+3} have known constant entries and $f(t)$ is a known function of t.

Applying the backward Euler discretisation method with sampling period T in the along the pass direction converts (55) to the following model of the form (3) with $\tau = 3$

$$\begin{cases} x_{3k+1}(p+1) = A_1x_{3k+1}(p) + B_1u_{3k+1}(p) + B_{01}y_{3k}(p) \\ y_{3k+1}(p) = C_1x_{3k+1}(p) + D_1u_{3k+1}(p) + D_{01}y_{3k}(p) \\ x_{3k+2}(p+1) = A_2x_{3k+2}(p) + B_2u_{3k+2}(p) + B_{02}y_{3k+1}(p) \\ y_{3k+2}(p) = C_2x_{3k+2}(p) + D_2u_{3k+2}(p) + D_{02}y_{3k+1}(p) \\ x_{3k+3}(p+1) = A_3x_{3k+3}(p) + B_3u_{3k+3}(p) + B_{03}y_{3k+2}(p) \\ y_{3k+3}(p) = C_3x_{3k+3}(p) + D_3u_{3k+3}(p) + D_{03}y_{3k+2}(p) \end{cases} \tag{57}$$

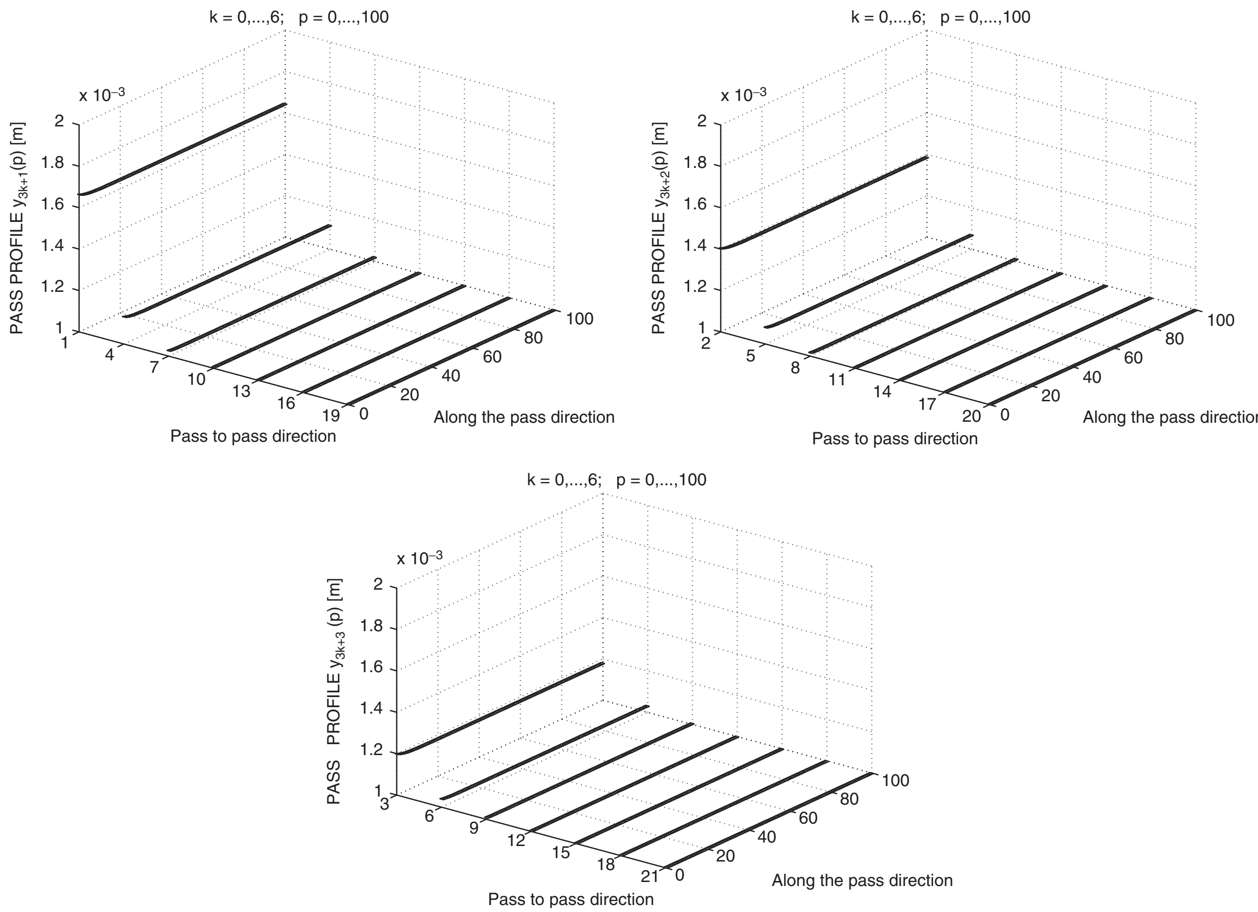

Figure 2. Reductions in the pass profile after each of the three pairs of rolls.

over $p = 0, 1, \ldots, \alpha - 1$, $k = 0, 1, \ldots$ and for $i = 1, 2, 3$

$$A_i = (I - A_{ci}T)^{-1} = \frac{1}{1 + a_{0i}T^2}\begin{bmatrix} 1 & T \\ -a_{0i}T & 1 \end{bmatrix}$$

$$B_i = A_i B_{ci} T = \frac{c_{0i}T}{1 + a_{0i}T^2}\begin{bmatrix} T \\ 1 \end{bmatrix}$$

$$B_{0i} = A_i B_{c0i} T = \frac{(-b_{0i} + a_{0i}b_{2i})T}{1 + a_{0i}T^2}\begin{bmatrix} T \\ 1 \end{bmatrix}$$

$$C_i = C_{ci}A_i = \frac{1}{1 + a_{0i}T^2}\begin{bmatrix} 1 & T \end{bmatrix}$$

$$D_i = C_{ci}A_iB_{ci}T + D_{ci} = \frac{c_{0i}T^2}{1 + a_{0i}T^2}$$

$$D_{0i} = C_{ci}A_iB_{c0i}T + D_{c0i} = \frac{-b_{2i} - b_{0i}T^2}{1 + a_{0i}T^2}$$

and state vectors

$$x_{3k+1}(p) = \begin{bmatrix} \upsilon_{3k+1}(p-1) \\ T^{-1}\{\upsilon_{3k+1}(p-1) - \upsilon_{3k+1}(p-2)\} \end{bmatrix}$$

$$x_{3k+2}(p) = \begin{bmatrix} \upsilon_{3k+2}(p-1) \\ T^{-1}\{\upsilon_{3k+2}(p-1) - \upsilon_{3k+2}(p-2)\} \end{bmatrix}$$

$$x_{3k+3}(p) = \begin{bmatrix} \upsilon_{3k+3}(p-1) \\ T^{-1}\{\upsilon_{3k+3}(p-1) - \upsilon_{3k+3}(p-2)\} \end{bmatrix}$$

with

$$\upsilon_{3k+1}(p) = y_{3k+1}(p) + b_{21}y_{3k}(p),$$
$$\upsilon_{3k+2}(p) = y_{3k+2}(p) + b_{22}y_{3k+1}(p),$$
$$\upsilon_{3k+3}(p) = y_{3k+3}(p) + b_{23}y_{3k+2}(p)$$

with partial boundary conditions

$$x_{3k+1}(0) = d_{3k+1}, \quad x_{3k+2}(0) = d_{3k+2},$$
$$x_{3k+3}(0) = d_{3k+3} \quad \text{and} \quad y_0(p) = f(p)$$

To completely specify the boundary conditions we must obtain values for $y_l(-1)$ and $y_l(-2)$, where these terms arise from the discretisation method used.

Here, it is required that the initial state vectors

$$x_{l+1}(0) = \begin{bmatrix} \upsilon_{l+1}(-1) \\ T^{-1}\{\upsilon_{l+1}(-1) - \upsilon_{l+1}(-2)\} \end{bmatrix},$$
$$\text{over} \quad l = 0, 1, \ldots,$$

with

$$\upsilon_{l+1}(-1) = y_{l+1}(-1) + b_{2\varsigma} y_l(-1),$$
$$\upsilon_{l+1}(-2) = y_{l+1}(-2) + b_{2\varsigma} y_l(-2),$$

where $\varsigma = 1 + \mathrm{mod}(l, \tau)$, must be consistent with the dynamics of process. They can not be taken to be arbitrary (or zero) because they are dependent on $y_l(-1)$ and $y_l(-2)$ and $y_{l+1}(-1)$ and $y_{l+1}(-2)$. Ill-chosen values of the initial state vectors could result in highly undesirable behaviour, such as the control vectors $u_{l+1}(p)$, having negative entries at the beginning of each pass. This is inadmissible for the case of the multi-roll metal rolling process considered here. To prevent this, the following procedure can be used to obtain $x_{l+1}(0)$, for $l = 0, 1, \ldots$

Step 1: Assume that we know all the matrices of state-space model (3) or, in general, (45). together with the disturbance signals $w_1(p), w_2(p), \ldots, w_\tau(p)$. Now, assume that the initial state vectors $x_{l+1}(0)$ are arbitrary and calculate the control law matrices using (48) or equivalently (49), and simulate process with these arbitrary initial conditions. Choose the appropriate number of passes $\beta = \varpi\tau$, where ϖ is a positive integer, necessary to achieve the control objectives independent of what form of control law is used.

Step 2: Determine the vectors $x_{i\infty}(0)$ and $u_{i\infty}(0)$, for $i = 1, 2, \ldots, \tau$, where

$$x_{i\infty}(0) = x_{\beta-\tau+i}(0)$$

and $u_{i\infty}(0)$ is then computed using (50).

Step 3: Compute the control vectors $u_{l+1}(0)$, using (48) or equivalently (49), for $p = 0$ on the each pass, over $l = 0, 1, \ldots, \beta - 1$, i.e.

$$\mathcal{U}(0) = \begin{bmatrix} u_1(0) & u_2(0) & u_3(0) & \cdots & u_\beta(0) \end{bmatrix}$$

and recover from simulating the process response, or direct calculations, the following:

$$\mathcal{X}(0) = \left[\begin{array}{ccccc} x_1(0) & x_2(0) & x_3(0) & \cdots & x_\beta(0) \\ \hline x_1(1) & x_2(1) & x_3(1) & \cdots & x_\beta(1) \end{array}\right]$$
$$\mathcal{Y}(0) = \left[\begin{array}{c|cccc} y_0(0) & y_1(0) & y_2(0) & \cdots & y_\beta(0) \end{array}\right].$$

Step 4: Check if the entries in $\mathcal{U}(0)$ are acceptable, i.e., none of them are negative. If yes, the initial state vectors are acceptable, if no take the second vector row from $\mathcal{X}(0)$ as the new update of the state initial vectors and repeat the procedure.

As a numerical example, we consider the following parameter values

$$\lambda_{11} = 40\,\mathrm{N/m}, \quad \lambda_{12} = 60\,\mathrm{N/m}, \quad \lambda_{13} = 80\,\mathrm{N/m},$$
$$\lambda_2 = 100\,\mathrm{N/m}, \quad M_1 = 10\,\mathrm{kg}, \quad M_2 = 20\,\mathrm{kg},$$
$$M_3 = 30\,\mathrm{kg}.$$

Sampling with sampling period $T = 0.1$ seconds gives the following discrete model state-space matrices

$$\begin{bmatrix} A_1 & B_{01} & B_1 \\ C_1 & D_{01} & D_1 \\ A_2 & B_{02} & B_2 \\ C_2 & D_{02} & D_2 \\ A_3 & B_{03} & B_3 \\ C_3 & D_{03} & D_3 \end{bmatrix}$$
$$= 10^{-3} \left[\begin{array}{cc|c|c} 972.2222 & 97.2222 & 7.9365 & -0.2778 \\ -277.7778 & 972.2222 & 79.3651 & -2.7778 \\ \hline 972.2222 & 97.2222 & 722.2222 & -0.2778 \\ \hline 981.5951 & 98.1595 & 6.9018 & -0.1840 \\ -184.0491 & 981.5951 & 69.0184 & -1.8405 \\ \hline 981.5951 & 98.1595 & 631.9018 & -0.1840 \\ \hline 985.4015 & 98.5401 & 6.4882 & -0.1460 \\ -145.9854 & 985.4015 & 64.8824 & -1.4599 \\ \hline 985.4015 & 98.5401 & 562.0438 & -0.1460 \end{array}\right] \tag{58}$$

The stabilisation condition of Theorem 5 holds and the control law matrices are

$$K_1^1 = \begin{bmatrix} 442.0971 & 350.0000 \end{bmatrix}, \quad K_2^1 = 53.9684$$
$$K_1^2 = \begin{bmatrix} 725.1304 & 533.3333 \end{bmatrix}, \quad K_2^2 = 71.0827$$
$$K_1^3 = \begin{bmatrix} 952.0547 & 675.0000 \end{bmatrix}, \quad K_2^3 = 89.6064.$$

The stabilisation condition of Theorem 6 also holds and the control law matrices are

$$K_1^1 = \begin{bmatrix} 735.3100 & 346.3186 \end{bmatrix}, \quad K_2^1 = 87.1643$$
$$K_1^2 = \begin{bmatrix} 983.5134 & 530.7128 \end{bmatrix}, \quad K_2^2 = 64.4260$$
$$K_1^3 = \begin{bmatrix} 1338.3637 & 670.3836 \end{bmatrix}, \quad K_2^3 = 174.4187.$$

The overall controlled process is stable along the pass without oscillations and disturbances, as shown in Figure 2 with initial pass profile $y_0(p) = 2\,\mathrm{mm}$ over $p = 0, 1, \ldots, 100$ and with the estimated (by applying

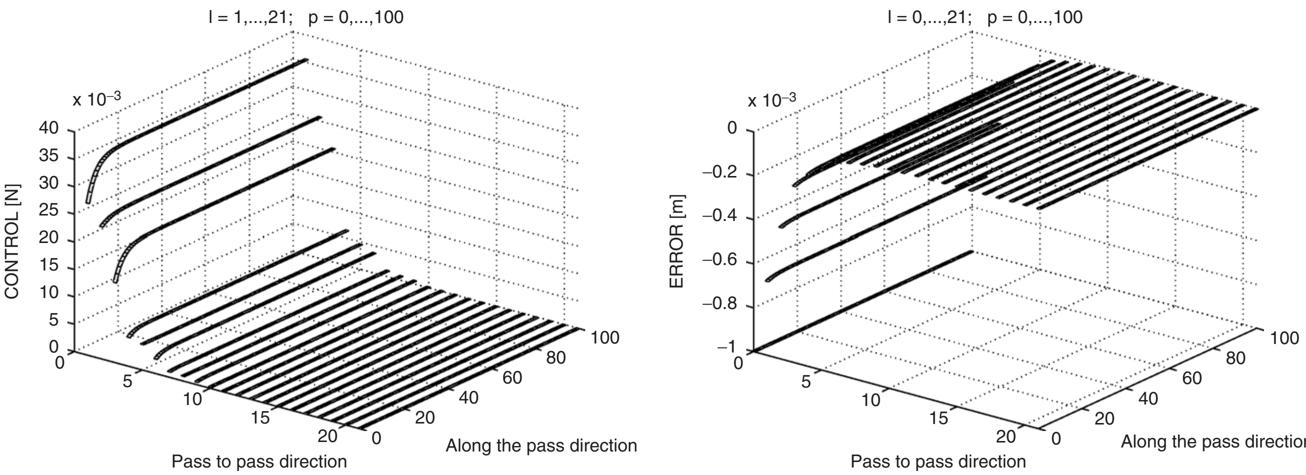

Figure 3. The control input $u_l(p)$ and the error $\Delta y_l(p) = y_{ref}(p) - y_l(p)$.

the procedure given above, which for this example requires 4 steps to be completed) state initial vectors $x_{l+1}(0)$ for $l = 0, 1, \ldots, 20$:

$$
\begin{aligned}
&\begin{bmatrix} x_1(0) & x_2(0) & \cdots & x_{21}(0) \end{bmatrix} \\
&= 10^{-3}\left[\begin{matrix} 0.240 & 0.365 & 0.421 & 0.276 & 0.373 & 0.440 & 0.284 & 0.375 & 0.443 \\ -0.074 & -0.017 & -0.037 & -0.014 & -0.003 & -0.007 & -0.002 & 0 & -0.001 \end{matrix}\right. \\
&\qquad \left.\begin{matrix} 0.285 & 0.375 & 0.444 & 0.286 & 0.375 & 0.444 & 0.286 & 0.375 & 0.444 & 0.286 & 0.375 & 0.444 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{matrix}\right]
\end{aligned}
$$

The required pass profile is $y_{ref}(p) = 1$ mm and the disturbances are randomly selected.

Finally, the control $u_l(p)$ and error $\Delta y_l(p) = y_{ref}(p) - y_l(p)$ signals are shown in Figure 3.

7. Conclusions

This paper has developed new results on control law design for discrete linear repetitive process in the presence of switching in the pass-to-pass direction, which can be computed in a numerically reliable and efficient manner using linear matrix inequality methods. Such models naturally arise in modelling multi-machine operations and an example of this has been given here. The control law design results developed consider two aspects: stability and tracking/performance where the sequence of pass profiles produced are required to converge to a specified limit profile despite the presence of disturbances.

The switching problems addressed in this paper have many possible areas for further development. One of these is to model and control so-called bi-directional processes. For example, suppose metal rolling is taking place with only one set of rolls. Then the mode of operation where the workpiece involved is returned to the input side before the next pass is not the most efficient. The alternative is to successively process the workpiece by passing it through on one pass from left-to-right and on the next from right-to-left (or vice versa). (It is also possible to envisage cases where it is required to complete a number of passes in one direction and then a different number in the opposite direction.) A switching setting is definitely one way this could be treated (whereas alternative currently available repetitive process theory only allows uni-directional operations). This idea is the subject of on-going research (together with in depth development/extension of the results given here) and outcome from this work will be reported in due course.

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