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Higher-order linear lossless systems

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We define a lossless autonomous system as one having a quadratic differential form associated with it, called the total energy, which obeys the property of positivity and which is conserved. In this paper, we show that an autonomous system is lossless if and only if it is oscillatory. Next we discuss a suitable way of splitting the total energy function into its kinetic and potential energy components. We also extend the investigation to the case of open systems.

Keywords: linear oscillatory systems; quadratic differential forms; conserved quantities; positivity; inconsequential inputs; energy function

1. Introduction

What is a lossless system? This problem has occupied theoretical physicists and applied mathematicians alike for quite some time. In theoretical physics (Young, Freedman, Sandin, and Ford 1999), a system is called lossless if the work done by a force is path-independent and equal to the difference between the final and initial values of an energy function that remains positive for non-zero trajectories of the system.

In most of the work done so far in the area of lossless systems, characterisation of losslessness is done assuming a given supply rate or the rate at which external work is done. An example for such a characterisation is the one by Willems (1972), in which losslessness was defined with respect to a given scalar function associated with the system, known as the supply rate. The system is called lossless if the given scalar function is the derivative of another scalar function, known as the storage function, along the trajectories of the system. Pillai and Willems (2002) have extended the concept of losslessness introduced by Willems (1972) to the case of distributed systems.

A lot of research has been carried out in the area of characterisation of lossless systems in the state-space. For example, Hill and Moylan (1976, 1980) have characterised lossless non-linear systems in the state space in a way similar to Willems (1972), and have proved that under certain conditions, there exists a positive definite storage function for the system. Weiss, Staffans, and Tucsnak (2001) and Weiss and Tucsnak (2003), have given algebraic characterisations of energy preserving and of conservative linear systems based on a state space description of the system. Here, a system is called energy preserving if the rate of change of a scalar positive definite function defined on its state space called energy, is equal to the difference between an incoming power and an outgoing power, which are respectively assumed to be the square of the norms of the input signal u and the output signal y. Note that in the sense of Willems (1972), if a system is energy preserving, then it is lossless with respect to the difference between the incoming and outgoing power. For a given energy preserving system, a related system known as its dual has been defined by Weiss et al. (2001). Here, a system is called conservative if both the system and its dual are energy preserving. In addition, Weiss et al. (2001) have also given results about the stability, controllability and observability of conservative systems and have illustrated these with the help of a model of a controlled beam. Malinen, Staffans, and Weiss (2006), have extended the characterisation of Weiss and Tucsnak (2003) for the case of infinite dimensional linear systems.

In many cases, the term "conservative" has been used instead of "lossless". In the following papers, special assumptions have been made in order to characterise conservative systems. Jacyno (1984) has constructed a class of non-linear autonomous conservative systems, starting from the general class of non-linear systems given by $\dot{x} = F(x)$, by deriving a certain condition on $F(x)$ and the total energy function $Q(x)$ for the system. Here it is assumed that the total

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energy function $Q(x)$ is a positive definite function of the state variables x. Van der Schaft (2000, 2004) has studied the properties of Hamiltonian and port-Hamiltonian conservative systems starting from sets of equations namely Hamiltonian, respectively port-Hamiltonian equations of motion. Here, it is assumed that the Hamiltonian (total energy) for the system is given *a priori*, i.e., one does not begin with the basic equations of motion.

The purpose of this paper is to give a definition of linear lossless systems which agrees with the basic intuition, derived from physics, that the external work done on such a system is equal to the difference between the final and initial values of the total energy for the system. We also make use of the fact that the total energy of such a system is a quadratic functional in the system variables and their derivatives that is positive for all infinitely differentiable non-zero trajectories. In a sense, our approach is similar to the one by Jacyno (1984), as we assume positivity of energy function. The main differences are that unlike Jacyno (1984), we restrict our characterisation to only linear systems and we do not restrict our analysis to systems described by the equation $\dot{x} = F(x)$. Unlike popular literature on lossless systems, we do not assume a given supply rate or derivative of energy function. We first characterise losslessness for the case of autonomous systems, which is defined below.

An autonomous system is a system with no inputs or free variables, i.e., a system which evolves on its own. For such a system, the future of every trajectory is completely determined by its past. We first characterise autonomous lossless systems based on the observation that the total energy of a physical system of this type is conserved. For this characterisation and for that of open lossless systems, we make use of the concept of quadratic differential forms (QDF), which we now define.

Consider the set of bilinear functionals acting on infinitely differentiable trajectories w_1 and w_2 of the form

$$L_\Phi(w_1, w_2) = \sum_{h,k=0}^{N} \left(\frac{\mathrm{d}^h w_1}{\mathrm{d}t^h}\right)^T \Phi_{h,k} \left(\frac{\mathrm{d}^k w_2}{\mathrm{d}t^k}\right),$$

where $\Phi_{h,k}$ are $w_1 \times w_2$-dimensional real matrices, w_1 and w_2 respectively stand for the dimensions of the trajectories w_1 and w_2, and N is a non-negative integer. Such a functional is called a bilinear differential form (BDF). We denote $L_\Phi(w, w)$ by $Q_\Phi(w)$, and we call such a quadratic functional acting on an infinitely differentiable trajectory w, a quadratic differential form (QDF).

A conserved quantity associated with a behaviour is a QDF, whose time-derivative along the trajectories of the behavior is zero. We define a lossless autonomous system as one for which there exists a conserved quantity that remains positive for all infinitely differentiable non-zero trajectories. We show the equivalence between linear autonomous lossless systems and oscillatory systems, i.e., systems whose trajectories are linear combinations of vector sinusoidal functions. Physical examples of oscillatory systems are mechanical systems consisting of frictionless springs and masses having as external variables the displacements or the velocities of the masses from the equilibrium positions; and electrical systems consisting of the interconnection of inductors and capacitors, having as external variables the voltages across the capacitors or the currents in the inductance components. We show that a linear autonomous system is lossless if and only if it is oscillatory.

We extend the characterisation of losslessness to open systems by making use of two properties. The first property is that the total energy of such a system is always positive for all infinitely differentiable non-zero trajectories. The second property is that the rate of change of total energy is zero if the inputs of the system are made equal to zero.

We assume that the reader is familiar with the calculus of B/QDF's, and with the behavioral framework, and we refer to Polderman and Willems (1997) and Willems and Trentelman (1998) respectively, for a thorough exposition of the concepts and mathematical techniques.

The structure of the paper is as follows. In § 2, we discuss properties of oscillatory systems and the notions of conserved quantities for oscillatory systems and of $\mathfrak{B}$-canonicity and positivity of QDFs. In § 3, we prove the equivalence of autonomous lossless and oscillatory systems and in § 4, we do the same for the state space case. In § 5, we extend the characterisation of losslessness to the case of open systems.

The notation used in this paper is standard: we denote the space of n dimensional real, respectively complex vectors by $\mathbb{R}^{\mathtt{n}}$, respectively $\mathbb{C}^{\mathtt{n}}$, the space of $\mathtt{m} \times \mathtt{n}$ real, respectively complex matrices, by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$, respectively $\mathbb{C}^{\mathtt{m}\times\mathtt{n}}$ and the space of $\mathtt{m} \times \mathtt{n}$ symmetric real matrices, by $\mathbb{R}_s^{\mathtt{m}\times\mathtt{n}}$. Whenever one of the two dimensions is not specified, a bullet $\bullet$ is used; so that for example, $\mathbb{R}^{\bullet\times\mathtt{n}}$ denotes the set of real matrices with n columns and an unspecified number of rows. In order to enhance readability, when dealing with a vector space $\mathbb{R}^\bullet$ whose elements are commonly denoted with w, we use the notation $\mathbb{R}^{\mathtt{w}}$ (note the typewriter font type!); similar considerations hold for matrices representing linear operators on such spaces. The ring of polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$; the set of two-variable polynomials with real coefficients in the indeterminates ζ and η is denoted by $\mathbb{R}[\zeta,\eta]$. The space of all $\mathrm{n}\times\mathrm{m}$ polynomial matrices in the indeterminate ξ is denoted by $\mathbb{R}^{\mathrm{n}\times\mathrm{m}}[\xi]$, and that consisting of all $\mathrm{n}\times\mathrm{m}$ polynomial matrices in the indeterminates ζ and η by $\mathbb{R}^{\mathrm{n}\times\mathrm{m}}[\zeta,\eta]$. We denote with $\mathbb{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathrm{q}})$ the set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathrm{q}}$. $\mathbb{R}^{+}$ denotes the set of positive real numbers. I_{p} stands for identity matrix of size p. $0_{\mathrm{p}\times\mathrm{q}}$ denotes a matrix of size $\mathrm{p}\times\mathrm{q}$ consisiting of zeroes. $\mathrm{col}(L_1,L_2)$ denotes the matrix obtained by stacking the matrix L_1 over L_2. $\mathrm{Re}(s)$ and $\mathrm{Im}(s)$ denote the real and imaginary parts of a complex quantity s. A^* denotes the matrix obtained by transposing the complex conjugate of the matrix A. The class of linear differential behaviours with w external variables is denoted by $\mathcal{L}^{\mathrm{w}}$. $\mathrm{diag}(a_1,\ldots,a_n)$ denotes the diagonal matrix whose diagonal entries are $a_1,\ldots,a_n$ in the given order. $\mathrm{colrank}(P)$ denotes the column rank of the polynomial matrix P, as defined by Kailath (1980, p. 652).

2. Preliminaries

In this section, we illustrate the basic definitions and concepts of Kailath (1980, p. 188) necessary to understand the results illustrated in this article.

2.1 *Oscillatory systems*

Definition 1: A behaviour $\mathfrak{B}$ defines a linear oscillatory system if

- $\mathfrak{B}$ is the set of solutions of a system of linear constant-coefficient differential equations

$$R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w=0,\quad R\in\mathbb{R}^{\bullet\times\mathrm{w}}[\xi];$$

  equivalently, $\mathfrak{B}$ belongs to the class of linear differential behaviors with $\mathtt{w}$ external variables;
- every solution $w:\mathbb{R}\to\mathbb{R}^{\mathrm{w}}$ is bounded on $(-\infty,\infty)$.

From the definition, it follows that an oscillatory system is necessarily autonomous: if there were any input variables in w, then those components of w could be chosen to be unbounded.

In the following, the case of multivariable ($\mathtt{w}>1$) oscillatory systems will be often reduced to the scalar case by using the Smith form of a polynomial matrix. Consequently, we now examine in more detail the properties of scalar oscillatory systems and of their representation.

It was proved in proposition 2 of Rapisarda and Willems (2005) that any behaviour $\mathfrak{B}$ is oscillatory if and only if every non-zero invariant polynomial of $\mathfrak{B}$ has distinct and purely imaginary roots. Consequently, if $r\in\mathbb{R}[\xi]$ then $\mathfrak{B}=\ker(r(\mathrm{d}/\mathrm{d}t))$ defines an oscillatory system if and only if all the roots of r are distinct and on the imaginary axis. From this it follows that r has one of the following two forms

$$r(\xi)=(\xi^2+\omega_0^2)(\xi^2+\omega_1^2)\ldots(\xi^2+\omega_{n-1}^2)$$

or

$$r(\xi)=\xi(\xi^2+\omega_0^2)(\xi^2+\omega_1^2)\ldots(\xi^2+\omega_{n-1}^2)$$

where $\omega_0,\ldots,\omega_{n-1}\in\mathbb{R}^{+}$. Recall from Polderman and Willems (1997, p. 69) that the dimension of $\ker(r(\mathrm{d}/\mathrm{d}t))$ as a linear subspace of $\mathbb{C}^{\infty}(\mathbb{R},\mathbb{R})$ equals the degree of the polynomial r and that the roots of r are called the characteristic frequencies of $\ker(r(\mathrm{d}/\mathrm{d}t))\cdot$

In the following, a polynomial matrix will be called oscillatory if all its invariant polynomials have distinct and purely imaginary roots.

In the following proposition, we give a condition on the state space equation of an autonomous system under which it is oscillatory.

Lemma 1: *A linear system described by the state space equation* $\mathrm{d}x/\mathrm{d}t=Ax$, *where* $x\in\mathbb{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathrm{x}})$ *and* $A\in\mathbb{R}^{\mathrm{x}\times\mathrm{x}}$ *is oscillatory if and only if A has purely imaginary eigenvalues that occur in conjugate pairs, and A is diagonalisable, i.e.,* $\exists$ *an invertible matrix* $V\in\mathbb{C}^{\mathrm{x}\times\mathrm{x}}$ *such that* $VAV^{-1}=A_{\mathrm{d}}$, *where* A_{d} *is a diagonal matrix whose diagonal entries are purely imaginary and occur in conjugate pairs.*

Proof: (*If*) Let $Vx=z\in\mathbb{C}^{\mathrm{x}\times 1}$. Consider the system $\mathrm{d}z/\mathrm{d}t=A_dz$. Each component of z is bounded because the diagonal entries of A_d are purely imaginary and occur in conjugate pairs. Since V is invertible, this implies that each component of x is also bounded. Hence, the system is oscillatory.

(*Only if*) By contradiction, if A is not diagonalisable, it implies that A has at least one eigenvalue with geometric multiplicity less than its algebraic multiplicity, which in turn implies that the system is not bounded on $(-\infty,\infty)$. Hence A is diagonalisable. Again by contradiction, if any of the eigenvalues of A is not purely imaginary, then one of the components of $z=Vx$, is unbounded on $(-\infty,\infty)$, which implies that one or more components of x are unbounded. Hence A has purely imaginary eigenvalues. Since the characteristic polynomial of A has real coefficients, the eigenvalues of A occur in conjugate pairs. □

2.2 $\mathfrak{B}$-*canonicity of QDFs*

Equip the set of QDFs associated with a behaviour $\mathfrak{B}$, with the equivalence relation defined by

$$Q_\Phi \overset{\mathfrak{B}}{\sim} Q_\Psi \Leftrightarrow Q_\Phi(w) = Q_\Psi(w) \quad \forall w \in \mathfrak{B}.$$

It is easy to see that the set of equivalence classes under $\overset{\mathfrak{B}}{\sim}$ is a linear vector space over $\mathbb{R}$. With every equivalence class of QDFs associated with an autonomous behaviour $\mathfrak{B}$, we associate a certain representative known as the $\mathfrak{B}$-canonical representative. Below, we define the notion of $\mathfrak{B}$-canonicity of QDFs.

Definition 2: Let $\mathfrak{B}$ be an autonomous behaviour given by $\mathfrak{B} = \ker(R(\mathrm{d}/\mathrm{d}t))$, where $R \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\xi]$. Then a QDF Q_Φ is $\mathfrak{B}$-canonical if $R(\zeta)^{-T}\Phi(\zeta, \eta)R(\eta)^{-1}$ is strictly proper.

If $R \in \mathbb{R}[\xi]$ and has degree n, then from the definition, it follows that the two-variable polynomials associated with $\mathfrak{B}$-canonical QDFs are spanned by monomials $\zeta^k\eta^j$, with $k, j \leq n-1$. It is easy to see that every QDF has a $\mathfrak{B}$-canonical representative.

2.3 *Conserved quantities associated with an oscillatory behaviour*

Definition 3: Let $\mathfrak{B} \in \mathcal{L}^{\mathrm{w}}$ be a linear behaviour. A QDF Q_Φ is a conserved quantity if

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_\Phi(w) = 0 \quad \forall w \in \mathfrak{B}.$$

Note that the trivial QDF $Q_\Phi = 0$ is always conserved. Any conserved QDF which is identically not equal to zero will be called "non-trivial conserved quantity" in the following. Consider an oscillatory behaviour $\mathfrak{B} = \ker(r(\mathrm{d/dt}))$, where $r \in \mathbb{R}[\xi]$. If r is an even polynomial of degree $2n$, then it can be shown (see Rapisarda and Willems 2005) that the two-variable polynomials $\gamma_i(\zeta, \eta)$ given by

$$\gamma_i(\zeta, \eta) = \frac{r(\zeta)\eta^{2i+1} + r(\eta)\zeta^{2i+1}}{\zeta + \eta}$$

$i = 0, 1, \ldots, n-1$, induce a basis for the space of $\mathfrak{B}$-canonical conserved quantities over $\mathfrak{B}$. If r is an odd polynomial of degree $2n+1$, then it can be shown that a basis of conserved quantities over $\mathfrak{B}$ is induced by the set $\{\gamma_i'(\zeta, \eta)\}_{i=0,1,\ldots,n}$, where

$$\gamma_i'(\zeta, \eta) = \frac{r(\zeta)\eta^{2i} + r(\eta)\zeta^{2i}}{\zeta + \eta}.$$

2.4 *Positivity of QDFs*

Definition 4: Let $\Phi \in \mathbb{R}_s^{\mathrm{w}\times\mathrm{w}}[\zeta, \eta]$. Q_Φ is said to be positive denoted by $Q_\Phi > 0$, if $Q_\Phi \geq 0$ for all $w \in \mathbb{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{w}})$, and $Q_\Phi(w) = 0$ implies $w = 0$.

It can shown (see Willems and Trentelman (1998, p. 1712) that a QDF Q_Φ, where $\Phi \in \mathbb{R}_s^{\mathrm{w}\times\mathrm{w}}[\zeta, \eta]$ is positive if $\exists\, D \in \mathbb{R}^{\bullet\times\mathrm{w}}[\xi]$ such that $\Phi(\zeta, \eta) = D(\zeta)^\top D(\eta)$, and $D(\lambda)$ has full column rank w for all $\lambda \in \mathbb{C}$.

3. Autonomous conservative systems

In this section, we define an autonomous lossless system as an autonomous system for which there exists a positive conserved quantity. We then prove the equivalence between autonomous lossless and oscillatory systems. This is first done for the case of scalar systems and then extended to the case of multivariable systems. We also discuss a few properties of energy functions of scalar lossless systems.

The main advantage of the higher-order approach over the state space method lies in its ability to deal with higher-order differential equations obtained directly from the modelling of the system, instead of having to set up a system of first order differential equations.

We begin with the following definition for autonomous lossless systems.

Definition 5: A linear autonomous behaviour $\mathfrak{B} \in \mathcal{L}^{\mathrm{w}}$ is lossless if there exists a conserved quantity Q_E associated with $\mathfrak{B}$, such that $Q_E > 0$. Such a Q_E is called an energy function for the system.

Remark 1: The total energy of any physical system does not have an absolute measure as such. It is always defined with respect to an arbitrary choice of a reference level, which is hence indeterminate. However this indeterminacy is not important as in any physical application, it is always the difference between the initial and final values of energy that matters, and this difference is independent of the reference level. Hence it is convenient to define the reference level for the total energy of a system as its lower bound. This point has been elaborated upon in Sears (1946, pp. 128–129). While defining lossless systems, we fix the reference level or lower bound of the energy functions for the system at zero, which leads to positivity of energy functions. We implicitly assume that an energy function of a lossless system is bounded from below.

For proving that lossless autonomous systems are necessarily oscillatory, we examine all the linear autonomous scalar systems, for which conserved QDFs exist. To this end, we first determine the conditions under which a linear system has conserved

QDFs associated with it, and the dimension of the space of conserved QDFs for such systems. We begin with the following definition.

Definition 6: Let $r \in \mathbb{R}[\xi]$. The maximal even polynomial factor of r is its monic even factor polynomial of maximal degree.

For any given polynomial $r \in \mathbb{R}[\xi]$, it is easy to see that there exists a unique maximal even polynomial factor. In the next proposition, we examine the conditions under which a linear behaviour $\mathfrak{B}$ has conserved QDFs associated with it.

Proposition 1: *Consider a linear behaviour* $\mathfrak{B} = \ker(r(\mathrm{d}/\mathrm{d}t))$, *where* $r \in \mathbb{R}[\xi]$. *There exists a non-trivial conserved quantity for* $\mathfrak{B}$ *if and only if either r has a non-unity maximal even polynomial factor* r_e *or* $r(\xi) = \xi$. *Moreover if* $p := r/r_e$ *is such that* $p(0) \neq 0$, *then the dimension of the space of conserved QDFs is* $\deg(r_e)/2$, *otherwise it is equal to* $(\deg(r_e)/2) + 1$.

Proof: Let the degree of r be equal to n. Let $r = r_e p$. Assume that $\mathfrak{B}$ has a conserved QDF whose two-variable polynomial representation is $\phi(\zeta, \eta)$. Then

$$\phi(\zeta, \eta) = \frac{r(\zeta) f_1(\zeta, \eta) + r(\eta) f_1(\eta, \zeta)}{\zeta + \eta}$$

for some $f_1 \in \mathbb{R}[\zeta, \eta]$. It is easy to see that since ϕ is $\mathfrak{B}$-canonical, f_1 is independent of ζ and is of degree less than or equal to $n-1$ in η. Let $f(\eta) = f_1(\zeta, \eta)$. Since ϕ exists, the numerator is divisible by $\zeta + \eta$. Consequently $r(-\xi)f(\xi) + r(\xi)f(-\xi) = 0$. This implies that $g(\xi) = r(\xi)f(-\xi) = r_e(\xi)p(\xi)f(-\xi)$ is an odd function. Hence

$$p(\xi)f(-\xi) = -p(-\xi)f(\xi). \tag{1}$$

Two cases arise.

- **Case 1:** $p(\xi)$ is not divisible by ξ. In this case, for Equation (1) to hold, it is easy to see that f should be of the form

  $$f(\xi) = p(\xi) f_o(\xi),$$

  where $f_o(\xi)$ is an odd function such that $\deg(f) \leq n-1$. Hence, we obtain

  $$\deg(f_o) \leq \deg(r_e) - 1. \tag{2}$$

From property (2), it follows that the dimension of the space of all possible polynomials $f_o(\xi)$ for a given even polynomial $r_e(\xi)$ and hence that of the space of conserved QDFs in this case is $\deg(r_e(\xi))/2$.

- **Case 2:** $p(\xi)$ is divisible by ξ. Let $p(\xi) = \xi p_1(\xi)$. In this case, since $p_1(\xi)$ does not have a root at zero, for Equation (1) to hold, it is easy to see that f should be of the form

  $$f(\xi) = p_1(\xi) f_e(\xi),$$

  where $f_e(\xi)$ is an even function such that $\deg(f) \leq n-1$. Hence, we obtain

  $$\deg(f_e) \leq \deg(r_e). \tag{3}$$

From property (3), it follows that the dimension of the space of all possible polynomials $f_e(\xi)$ for a given even polynomial $r_e(\xi)$ and hence that of the space of conserved QDFs in this case is equal to $(\deg(r_e(\xi))/2) + 1$. □

Remark 2: We now associate a concept called reversibility with the existence of conserved quantities for the case of scalar behaviours. An autonomous behaviour $\mathfrak{B}$ is said to be reversible if $w \in \mathfrak{B} \Rightarrow \mathrm{rev}(w) \in \mathfrak{B}$ where $\mathrm{rev}(w)$ is defined as $\mathrm{rev}(w)(t) := w(-t)$. For further details and physical insight into the concept of reversibility, the reader is referred to Fagnani and Willems (1991) and Lamb and Roberts (1998). We now examine what kind of scalar behaviours are reversible.

Let $r \in \mathbb{R}[\xi]$ have distinct roots $\lambda_i \in \mathbb{C}$, $i = 1, \ldots, m + 2N$ of multiplicity n_i, i.e., $r(\xi) = \prod_{k=1}^{m+2N} (\xi - \lambda_k)^{n_k}$. Assume that the first m distinct roots are real numbers and the remaining distinct roots are the conjugate pairs $\lambda_{m+1}, \bar{\lambda}_{m+1}, \lambda_{m+2}, \bar{\lambda}_{m+2}, \ldots, \lambda_{m+N}, \bar{\lambda}_{m+N}$. Let $\mathfrak{B} = \ker(r(\mathrm{d}/\mathrm{d}t))$. Then from Polderman and Willems (1997), Corollary 3.2.13, p. 75), it follows that $w \in \mathfrak{B}$ iff it is of the form

$$w(t) = \sum_{k=1}^{m} \sum_{l=0}^{n_k - 1} r_{kl} t^l e^{\lambda_k t} + \sum_{k=m+1}^{m+N} \sum_{l=0}^{n_k - 1} t^l \left(r_{kl} e^{\lambda_k t} + \bar{r}_{kl} e^{\bar{\lambda}_k t} \right)$$

with r_{kl}, arbitrary real numbers for $k = 1, 2, \ldots, m$ and arbitrary complex numbers with non-zero imaginary parts for $k = m+1, m+2, \ldots, m+N$. We have

$$w(-t) = \sum_{k=1}^{m} \sum_{l=0}^{n_k - 1} (-1)^l r_{kl} t^l e^{-\lambda_k t} + \sum_{k=m+1}^{m+N} \sum_{l=0}^{n_k - 1} (-1)^l t^l \left(r_{kl} e^{-\lambda_k t} + \bar{r}_{kl} e^{-\bar{\lambda}_k t} \right).$$

Thus if the system is reversible, it follows that every non-zero root λ_k of r is accompanied by another root $-\lambda_k$ of r. This implies that r is either even or odd. Hence from Proposition 1, it can be inferred that every reversible scalar behaviour has conserved quantities associated with it.

In order to prove the equivalence between oscillatory systems and autonomous lossless systems, we first consider the case of scalar behaviours.

Theorem 1: *A behaviour $\mathfrak{B} \in \mathcal{L}^1$ is lossless if and only if it is oscillatory.*

Proof: (*If*) We consider the two forms of scalar oscillatory behaviours mentioned in § 2.1. For each of these forms of oscillatory behaviour, we construct an energy function that is positive.

- **Case 1:** The oscillatory behaviour is of the form $\mathfrak{B} = \ker(r(\mathrm{d}/\mathrm{d}t))$, where $r(\xi) = (\xi^2 + \omega_0^2)(\xi^2 + \omega_1^2)\ldots(\xi^2 + \omega_{n-1}^2)$ and $\omega_0, \ldots, \omega_{n-1} \in \mathbb{R}^+$. From the discussion of § 2.3, it can be said that the two-variable polynomial associated with a general $\mathfrak{B}$-canonical conserved quantity for this case has the form

$$\phi(\zeta, \eta) = \frac{\eta r(\zeta) f_e(\eta) + \zeta r(\eta) f_e(\zeta)}{\zeta + \eta}, \tag{4}$$

where f_e is an even function of degree less than or equal to $2n-2$. Define $v_p(\xi) := r(\xi)/(\xi^2 + \omega_p^2)$. It can be seen that the set $\{v_p(\xi)\}_{p=0,\ldots,n-1}$ is a basis of even polynomials of degree less than or equal to $2n-2$. It follows that there exist, $b_p \in \mathbb{R}$, $p = 0, \ldots, n-1$ such that $f_e(\xi) = \sum_{p=0}^{n-1} b_p v_p(\xi)$. Now

$$\begin{aligned}\phi(\zeta, \eta) &= \sum_{p=0}^{n-1} b_p \left[\frac{\eta r(\zeta) v_p(\eta) + \zeta r(\eta) v_p(\zeta)}{\zeta + \eta}\right] \\ &= \sum_{p=0}^{n-1} b_p v_p(\zeta) v_p(\eta)(\zeta\eta + \omega_p^2).\end{aligned}$$

Define $\phi_p(\zeta, \eta) := v_p(\zeta) v_p(\eta)(\zeta\eta + \omega_p^2)$. From Equation (4), it can be seen that linearly independent f_e' s produce linearly independent ϕ's. Hence $\{\phi_p(\zeta, \eta)\}_{p=0,\ldots,n-1}$ is a basis of the space of two-variable polynomials that induce $\mathfrak{B}$-canonical conserved quantities. Now consider $E(\zeta, \eta) = \sum_{p=0}^{n-1} a_p^2 \phi_p(\zeta, \eta) = D(\zeta)^\top D(\eta)$, where $a_p \in \mathbb{R}\backslash\{0\}$ for $p = 0, \ldots, n-1$ and

$$D(\xi) = \begin{bmatrix} a_0 \xi v_0(\xi) \\ a_0 \omega_0 v_0(\xi) \\ a_1 \xi v_1(\xi) \\ a_1 \omega_1 v_1(\xi) \\ \vdots \\ a_{n-1} \xi v_{n-1}(\xi) \\ a_{n-1} \omega_{n-1} v_{n-1}(\xi)) \end{bmatrix}.$$

It can be verified that $D(\lambda) \neq 0_{2n\times 1}$ for any $\lambda \in \mathbb{C}$. This proves that E induces an energy function for Case 1 oscillatory systems.

- **Case 2:** The oscillatory behaviour is of the form $\mathfrak{B} = \ker(r(\mathrm{d}/\mathrm{d}t))$ where $r(\xi) = \xi(\xi^2 + \omega_0^2)(\xi^2 + \omega_1^2)\ldots(\xi^2 + \omega_{n-1}^2) = \xi r_e(\xi)$ and $\omega_0, \ldots, \omega_{n-1} \in \mathbb{R}^+$. From the discussion of § 2.3, we conclude that the two-variable polynomial associated with a general $\mathfrak{B}$-canonical conserved quantity for this case has the form

$$\phi(\zeta, \eta) = \frac{r(\zeta) f_e(\eta) + r(\eta) f_e(\zeta)}{\zeta + \eta}, \tag{5}$$

where f_e is an even function of degree less than or equal to $2n$. Define $v_p(\xi) := r_e(\xi)/(\xi^2 + \omega_p^2)$. It can be seen that the set $\{r_e(\xi)\} \cup \{\xi^2 v_p(\xi)\}_{p=0,\ldots,n-1}$ is a basis of even polynomials of degree less than or equal to $2n$. It follows that there exist $b_p \in \mathbb{R}, p = 0, \ldots, n$, such that $f_e(\xi) = \sum_{p=0}^{n-1} b_p \xi^2 v_p(\xi) + b_n r_e(\xi)$. Now

$$\begin{aligned}\phi(\zeta, \eta) &= \sum_{p=0}^{n-1} b_p \left[\frac{\eta^2 r(\zeta) v_p(\eta) + \zeta^2 r(\eta) v_p(\zeta)}{\zeta + \eta}\right] \\ &\quad + b_n \left[\frac{r(\zeta) r_e(\eta) + r(\eta) r_e(\zeta)}{\zeta + \eta}\right] \\ &= \sum_{p=0}^{n-1} b_p \zeta\eta v_p(\zeta) v_p(\eta)(\zeta\eta + \omega_p^2) + b_n r_e(\zeta) r_e(\eta).\end{aligned}$$

Define $\phi_p(\zeta, \eta) := v_p(\zeta) v_p(\eta)(\zeta\eta + \omega_p^2)$. From Equation (5), it can be seen that linearly independent f_e's produce linearly independent ϕ's. Hence $\{r_e(\zeta) r_e(\eta)\} \cup \{\zeta\eta\phi_p(\zeta, \eta)\}_{p=0,\ldots,n-1}$ is a basis of two-variable polynomials that induce conserved quantities associated with $\mathfrak{B}$. Now consider $\sum_{p=0}^{n-1} a_p^2 \zeta\eta\phi_p(\zeta, \eta) + a_n^2 r_e(\zeta) r_e(\eta) = D(\zeta)^\top D(\eta)$, where $a_p \in \mathbb{R}^+$ for $p = 0, \ldots, n$ and

$$D(\xi) = \begin{bmatrix} a_0 \xi^2 v_0(\xi) \\ a_0 \omega_0 \xi v_0(\xi) \\ a_1 \xi^2 v_1(\xi) \\ a_1 \omega_1 \xi v_1(\xi) \\ \vdots \\ a_{n-1} \xi^2 v_{n-1}(\xi) \\ a_{n-1} \omega_{n-1} \xi v_{n-1}(\xi) \\ a_n r_e(\xi) \end{bmatrix}.$$

It can be verified that $D(\lambda) \neq 0_{(2n+1)\times 1}$ for any $\lambda \in \mathbb{C}$. This proves that E induces an energy function for Case 2 oscillatory systems.

(*Only if*) We consider all scalar systems for which conserved quantities exist and prove that a conserved quantity cannot be positive unless the system is oscillatory. Let $\mathfrak{B}$ be a behaviour whose kernel representation is $r(\mathrm{d}/\mathrm{d}t)w = 0$. Let $r(\xi) = r_e(\xi) p(\xi)$ where r_e is the maximal even polynomial factor of r. If $p(\xi)$ is not a constant and $p(\xi) \neq a\xi$, where $a \in \mathbb{R}$, then it has at least one root, say $\lambda \in \mathbb{R}\backslash\{0\}$ or two roots, say $\lambda, \bar{\lambda} \in \mathbb{C}\backslash\mathbb{R}$. From the proof of Proposition 1, depending on whether $p(\xi)$ is divisible by ξ or not, any two-variable polynomial inducing conserved QDF over $\mathfrak{B}$ can either have the form

$$\phi_1(\zeta,\eta) = \frac{r(\zeta)p_1(\eta)f_e(\eta) + r(\eta)p_1(\zeta)f_e(\zeta)}{\zeta+\eta},$$

where $p_1(\xi) = p(\xi)/\xi$ and f_e is an even function, or the form

$$\phi_2(\zeta,\eta) = \frac{r(\zeta)p(\eta)f_o(\eta) + r(\eta)p(\zeta)f_o(\zeta)}{\zeta+\eta},$$

where $f_o(\xi)$ is an odd function. It can be seen that both ϕ_1 and ϕ_2 are divisible by $(\zeta-\lambda)(\eta-\lambda)$ if $\lambda \in \mathbb{R}$ and divisible by $(\zeta-\lambda)(\zeta-\bar{\lambda})(\eta-\lambda)(\eta-\bar{\lambda})$ if $\lambda \in \mathbb{C}\backslash\mathbb{R}$. Hence along the trajectory $w(t) = e^{\lambda t} + e^{\bar{\lambda} t} \in \mathfrak{B}$, the QDFs induced by ϕ_1 and ϕ_2 are equal to zero. This implies that $\mathfrak{B}$ does not have a positive conserved QDF. This eliminates all scalar systems except those for which the kernel representation is $r(\mathrm{d}/\mathrm{d}t)w = 0$, such that either $r(\xi)$ is even, or $r(\xi) = \xi r_e(\xi)$, where $r_e(\xi)$ is an even function.

We now consider all those remaining cases except the oscillatory one, for which $\mathfrak{B} = \ker(r(\mathrm{d}/\mathrm{d}t))$ is such that r does not have repeated roots. For each of these, we will construct a general conserved quantity which is zero on at least one of the trajectories of $\mathfrak{B}$, thus proving the claim by contradiction.

- **Case 1:** r is even and has roots at λ_0 and $-\lambda_0$, where $\lambda_0 \in \mathbb{R}$. Let $r(\xi) = r'(\xi^2) = (\xi^2 - \lambda_0^2)r_1(\xi)$, where $r_1(\xi) = \prod_{p=1}^{n-1}(\xi^2 - \lambda_p^2)$ and $\lambda_p \in \mathbb{C}$ for $p = 1,\ldots,n-1$. Any two-variable polynomial that induces a conserved QDF over $\mathfrak{B}$ has the form

$$\phi(\zeta,\eta) = \frac{\eta r(\zeta)f(\eta^2) + \zeta r(\eta)f(\zeta^2)}{\zeta+\eta}.$$

We can write $f(\xi^2)$ in terms of the basis $\{u'_p(\xi^2)\}_{p=0,\ldots,n-1}$, where $u'_p(\xi^2) = r'(\xi^2)/(\xi^2 - \lambda_p^2)$. Hence

$$\phi(\zeta,\eta) = b_0 r_1(\zeta)r_1(\eta)(\zeta\eta - \lambda_0^2) + (\zeta^2 - \lambda_0^2)(\eta^2 - \lambda_0^2)\phi_1(\zeta,\eta).$$

Along the trajectory $w(t) = e^{\lambda_0 t} \in \mathfrak{B}$, the QDF induced by the above polynomial is zero. Hence in this case, a positive conserved QDF does not exist.

- **Case 2:** r is odd and has roots at λ, $-\lambda$, $\bar{\lambda}$ and $-\bar{\lambda}$, where λ is a point in the complex plane that is not on any of the co-ordinate axes. Let $r(\xi) = \xi r'(\xi^2) = \xi(\xi^2 - \lambda^2)(\xi^2 - \bar{\lambda}^2)r_1(\xi)$, where $r_1(\xi) = \prod_{p=1}^{n-2}(\xi^2 - \lambda_p^2)$ and $\lambda_p \in \mathbb{C}$ for $p = 1,\ldots,n-2$. Any two-variable polynomial that induces a conserved QDF over $\mathfrak{B}$ has the form

$$\phi(\zeta,\eta) = \frac{r(\zeta)f(\eta^2) + r(\eta)f(\zeta^2)}{\zeta+\eta}.$$

We can write $f(\xi^2)$ in terms of a new basis as follows:

$$f(\xi^2) = b_0\frac{\xi^2 r'(\xi^2)}{\xi^2 - \lambda^2} + \bar{b}_0\frac{\xi^2 r'(\xi^2)}{\xi^2 - \bar{\lambda}^2} + \sum_{p=1}^{n-2} b_p\frac{\xi^2 r'(\xi^2)}{\xi^2 - \lambda_p^2} + b_{n-1}r'(\xi^2).$$

Hence

$$\begin{aligned}\phi(\zeta,\eta) &= b_0 r_1(\zeta)r_1(\eta)\zeta\eta(\zeta^2 - \bar{\lambda}^2)(\eta^2 - \bar{\lambda}^2)(\zeta\eta - \lambda^2)\\ &\quad + \bar{b}_0 r_1(\zeta)r_1(\eta)\zeta\eta(\zeta^2 - \lambda^2)(\eta^2 - \lambda^2)(\zeta\eta - \bar{\lambda}^2)\\ &\quad + (\zeta^2 - \lambda^2)(\zeta^2 - \bar{\lambda}^2)(\eta^2 - \lambda^2)(\eta^2 - \bar{\lambda}^2)\phi_1(\zeta,\eta).\end{aligned}$$

Along the trajectory $w(t) = e^{\lambda t} + e^{\bar{\lambda} t} \in \mathfrak{B}$, the QDF induced by the above polynomial is zero. Hence in this case, a positive conserved QDF does not exist.

There are two other cases, namely:

- **Case 3:** r is odd and has roots at λ_0 and $-\lambda_0$, where $\lambda_0 \in \mathbb{R}$.
- **Case 4:** r is even and has roots at λ, $-\lambda$, $\bar{\lambda}$ and $-\bar{\lambda}$, where λ is a point in the complex plane that is not on any of the co-ordinate axes.

The proofs for the last two cases are very similar to the ones for the first two cases and will not be given explicitly. Finally, we consider those remaining cases, for which $\mathfrak{B} = \ker(r(\mathrm{d}/\mathrm{d}t))$ is such that r has repeated roots.

- **Case 1:** r is even and has at least twice repeated roots at $\pm\lambda$, where λ is either purely real or purely imaginary. Let $r(\xi) = (\xi^2 - \lambda^2)^2(\sum_{p=0}^{n-2} a_p\xi^{2p})$. In this case, any two-variable polynomial that induces a conserved QDF over $\mathfrak{B}$ has the form

$$\begin{aligned}\phi(\zeta,\eta) &= \frac{\eta r(\zeta)\left(\sum_{p=0}^{n-1} b_p\eta^{2p}\right) + \zeta r(\eta)\left(\sum_{p=0}^{n-1} b_p\zeta^{2p}\right)}{\zeta+\eta}\\ &= \sum_{i=0}^{n-2}\sum_{j=0}^{n-1} a_{ij}\phi_{ij}(\zeta,\eta),\end{aligned}$$

where

$$\phi_{ij}(\zeta,\eta)=\frac{\zeta^{2i}(\zeta^2-\lambda^2)^2\eta^{2j+1}+\eta^{2i}(\eta^2-\lambda^2)^2\zeta^{2j+1}}{\zeta+\eta}. \quad (6)$$

We show that ϕ_{ij} can be written as

$$\phi_{ij}(\zeta,\eta)=(\zeta^2-\lambda^2)g_{ij}(\zeta,\eta)+g_{ij}(\eta,\zeta)(\eta^2-\lambda^2). \quad (7)$$

Assuming that this is true, comparing Equations (6) and (7), we obtain

$$g_{ij}(\eta,\zeta)(\zeta+\eta)=(\eta^2-\lambda^2)\eta^{2i}\zeta^{2j+1}$$
$$g_{ij}(\zeta,\eta)(\zeta+\eta)=(\zeta^2-\lambda^2)\zeta^{2i}\eta^{2j+1}.$$

Adding the above equations, we get

$$g_{ij}(\zeta,\eta)+g_{ij}(\eta,\zeta)=\frac{(\eta^2-\lambda^2)\eta^{2i}\zeta^{2j+1}+(\zeta^2-\lambda^2)\zeta^{2i}\eta^{2j+1}}{\zeta+\eta}.$$

It can be seen that the numerator in the right hand side of the above equation is divisible by the denominator. Let $g_{ij}(\zeta,\eta)+g_{ij}(\eta,\zeta)=\pi_{ij}(\zeta,\eta)$. Since π_{ij} is a symmetric polynomial, we can take $g_{ij}(\zeta,\eta)=\pi_{ij}(\zeta,\ \eta)/2$. This shows that $g_{ij}(\zeta,\eta)$ is a symmetric polynomial.
Hence from Equation (7), along the trajectory $w(t)=e^{\lambda t}+e^{-\lambda t}\in \ker(r(\mathrm{d}/\mathrm{d}t))$, the QDF induced by $\phi(\zeta,\ \eta)$ is zero. Hence no conserved QDF is positive in this case.

- **Case 2:** r is odd and has at least twice repeated roots at $\pm\lambda$ and $\pm\bar{\lambda}$ where λ is a point in the complex plane that does not lie on any of the co-ordinate axes. Let $r(\xi)=\xi(\xi^2-\lambda^2)^2\ \ (\xi^2-\bar{\lambda}^2)^2(\sum_{p=0}^{n-4}a_p\xi^{2p})$. As in the previous case, any two-variable polynomial that induces a conserved QDF over $\mathfrak{B}$ has the form

$$\phi(\zeta,\eta)=\sum_{i=0}^{n-4}\sum_{j=0}^{n}a_{ij}\phi_{ij}(\zeta,\eta),$$

where

$$\phi_{ij}(\zeta,\eta)=\frac{\zeta^{2i+1}(\zeta^2-\lambda^2)^2(\zeta^2-\bar{\lambda}^2)^2\eta^{2j}+\eta^{2i+1}(\eta^2-\lambda^2)^2(\eta^2-\bar{\lambda}^2)^2\zeta^{2j}}{\zeta+\eta}.$$

Following the argument used in the proof of Case 1, it can be shown that ϕ_{ij} can be written as

$$\phi_{ij}(\zeta,\eta)=(\zeta^2-\lambda^2)(\zeta^2-\bar{\lambda}^2)g_{ij}(\zeta,\eta)+g_{ij}(\eta,\zeta)(\eta^2-\lambda^2)(\eta^2-\bar{\lambda}^2).$$

Thus along the trajectory $w(t)=e^{\lambda t}+e^{\bar{\lambda}t}\in \ker(r(\mathrm{d}/\mathrm{d}t))$, the QDF induced by $\phi(\zeta,\ \eta)$ is zero. Hence no conserved QDF is positive in this case.

There are two other cases, namely:

- **Case 3:** r is even and has at least twice repeated roots at $\pm\lambda$ and $\pm\bar{\lambda}$ where λ is a point in the complex plane that does not lie on any of the co-ordinate axes.
- **Case 4:** r is odd and has at least twice repeated roots at $\pm\lambda$, where λ is either purely real or purely imaginary.

The proofs for these cases are very similar to the ones for Cases 1 and 2 and will not be given explicitly.

We have considered all linear scalar systems apart from oscillatory ones for which conserved QDFs exist and we have shown that a positive conserved QDF does not exist for any of these cases. Since we have already proved the existence of a positive conserved QDF for oscillatory systems, this concludes the proof. □

An alternate proof for the (*Only if*) part of Theorem 1 has been given in the Appendix. This proof makes use of concepts from Lyapunov theory of stability and the following proposition, whose proof is also given in the Appendix.

Proposition 2: *If $r\in\mathbb{R}[\xi]$ given by $r(\xi)=r'(\xi^2)+\xi r''(\xi^2)$, where $r', r''\in\mathbb{R}[\xi]$, is Hurwitz then r' and r'' have distinct roots on the negative real axis.*

We now discuss a few properties of energy functions for scalar oscillatory behaviours. We first present an analysis of the conditions under which a conserved quantity for a scalar oscillatory behaviour is positive. The following lemma can be used to construct an energy function for a scalar oscillatory behaviour.

Lemma 2: *Let $r_1\in\mathbb{R}[\xi]$ be given by $r_1(\xi)=(\xi^2+\omega_0^2)(\xi^2+\omega_1^2)\ldots(\xi^2+\omega_{n-1}^2)$, where $\omega_0,\ldots,\omega_{n-1}\in\mathbb{R}^+$ and n is a positive integer. Define $v_p(\xi):=r_1(\xi)/(\xi^2+\omega_p^2)$ $p=0,\ldots,n-1$. Define $r_2(\xi):=\xi r_1(\xi)$. Then the following hold.*

(1) *Let $\mathfrak{B}_1=\ker(r_1(\mathrm{d}/\mathrm{d}t))$. If the conserved quantity for $\mathfrak{B}_1$ induced by $\phi_1(\zeta,\eta)=\sum_{p=0}^{n-1}b_p v_p(\zeta)v_p(\eta)(\zeta\eta+\omega_p^2)$ is positive, then $b_p>0$ for $p=0,\ldots,n-1$.*
(2) *Let $\mathfrak{B}_2=\ker(r_2(\mathrm{d}/\mathrm{d}t))$. If the conserved quantity for $\mathfrak{B}_2$ induced by $\phi_2(\zeta,\eta)=\sum_{p=0}^{n-1}b_p\zeta\eta v_p(\zeta)v_p(\eta)(\zeta\eta+\omega_p^2)+b_n r_1(\zeta)r_1(\eta)$ is positive, then $b_p>0$ for $p=0,\ldots,n$.*

Proof: Assume that $b_i\leq 0$ for some $i\in\{0,\ldots,n-1\}$. Consider a trajectory $w(t)=ke^{j\omega_i t}+\bar{k}e^{-j\omega_i t}\in\mathfrak{B}_1,\mathfrak{B}_2$. Along this trajectory, $v_p(\mathrm{d}/\mathrm{d}t)w=0$ for

$p \in \{0, \ldots, n-1\} \backslash \{i\}$. Since $\phi_p(\zeta,\eta) = v_p(\zeta)v_p(\eta)(\zeta\eta + \omega_p^2)$ and $\zeta\eta\phi_p(\zeta,\eta)$ are non-negative, the QDF induced by $\phi_1(\zeta,\eta)$ and $\phi_2(\zeta,\eta)$ over $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively along this trajectory turns out to be non-positive. Hence by contradiction, $b_p > 0$ for $p = 0, \ldots, n-1$ in both cases.

In order to complete the proof consider now statement 2 of the Lemma and assume by contradiction that $b_n \leq 0$. Consider a trajectory $w(t) = k \in \mathfrak{B}_2$. Along this trajectory $v_p(\mathrm{d}/\mathrm{d}t)w = 0$ for $p \in \{0, \ldots, n-1\}$. Since $r_1(\zeta)r_1(\eta)$ is non-negative, the QDF induced by $\phi_2(\zeta,\eta)$ over $\mathfrak{B}_2$ turns out to be non-positive. Hence, $b_n > 0$. This concludes the proof. □

The next Theorem relates the positivity of a conserved quantity to an important property known as interlacing property, which also arises in applications like electrical network theory.

Theorem 2: *Let $r_1 \in \mathbb{R}[\xi]$ be given by $r_1(\xi) = (\xi^2 + \omega_0^2)(\xi^2 + \omega_1^2)\ldots(\xi^2 + \omega_{n-1}^2)$, where $\omega_0 < \omega_1 \cdots < \omega_{n-1} \in \mathbb{R}^+$ and n is a positive integer. Define $r'(\xi^2) := r_1(\xi)$; $r_2(\xi) := \xi r_1(\xi)$ and $\breve{r}(\xi) := \xi r'(\xi)$. Then the following hold.*

(1) *Let $\mathfrak{B}_2 = \ker(r_1(\mathrm{d}/\mathrm{d}t))$. Let $f_1(\xi)$ be a polynomial of degree less than or equal to $n-1$. A conserved quantity for $\mathfrak{B}_1$ induced by*

$$\phi_1(\zeta,\eta) = \frac{\eta r'(\zeta^2)f_1(\eta^2) + \zeta r'(\eta^2)f_1(\zeta^2)}{\zeta + \eta} \tag{8}$$

is positive if and only if $f_1(-\omega_0^2) > 0$ and the roots of f_1 are interlaced between those of r', i.e., along the real axis, exactly one root of f_1 occurs between any two consecutive roots of r'.

(2) *Let $\mathfrak{B}_2 = \ker(r_2(\mathrm{d}/\mathrm{d}t))$. Let $f_2(\xi)$ be a polynomial of degree less than or equal to n. A conserved quantity associated with $\mathfrak{B}_2$ induced by*

$$\phi_2(\zeta,\eta) = \frac{\zeta r'(\zeta^2)f_2(\eta^2) + \eta r'(\eta^2)f_2(\zeta^2)}{\zeta + \eta}$$

is positive if and only if $f_2(0) > 0$ and the roots of f_2 are interlaced between those of $\breve{r}$.

Proof: (*Only if*) From Lemma 2 and Theorem 1, we know that f_1 and f_2 are of the form

$$f_1(\xi^2) = \sum_{p=0}^{n-1} b_p v_p'(\xi^2) \tag{9}$$

$$f_2(\xi^2) = \sum_{p=0}^{n-1} b_p \xi^2 v_p'(\xi^2) + b_n r'(\xi^2), \tag{10}$$

where $b_p \in \mathbb{R}^+$ and $v_p'(\xi^2) = r'(\xi^2)/(\xi^2 + \omega_p^2)$. The roots of r' are $-\omega_0^2, -\omega_1^2, \ldots, -\omega_{n-1}^2$ and the roots of $\breve{r}$ are $0, -\omega_0^2, -\omega_1^2, \ldots, -\omega_{n-1}^2$. From Equation (9), it can be seen that

$$f_1(-\omega_0^2) = b_0\left(\prod_{p=1}^{n-1}(\omega_p^2 - \omega_0^2)\right) > 0$$

$$f_1(-\omega_1^2) = b_1\left(\prod_{p\neq 1}(\omega_p^2 - \omega_1^2)\right) < 0$$

$$f_1(-\omega_2^2) = b_2\left(\prod_{p\neq 2}(\omega_p^2 - \omega_2^2)\right) > 0$$

$$\vdots$$

Since f_1 is a continuous function and can have a maximum of $n-1$ real roots, it follows that the roots of f are interlaced between those of r'.

In a similar way, it can be verified that $f_2(0) > 0$ and the roots of f_2 are interlaced between those of $\breve{r}$.

(*If*) Assuming that the roots of f_1 are interlaced between those of r' and $f_1(-\omega_0^2) > 0$, since f_1 is continuous and can have a maximum of $n-1$ roots, we have

$$f_1(-\omega_0^2) = b_0\left(\prod_{p=1}^{n-1}(\omega_p^2 - \omega_0^2)\right) > 0$$

$$f_1(-\omega_1^2) = b_1\left(\prod_{p\neq 1}(\omega_p^2 - \omega_1^2)\right) < 0$$

$$f_1(-\omega_2^2) = b_2\left(\prod_{p\neq 2}(\omega_p^2 - \omega_2^2)\right) > 0$$

$$\vdots$$

This implies that $b_p > 0$ for $p = 0, \ldots, n-1$, which in turn implies that $\phi_1(\zeta,\eta)$ is positive.

Assuming that the roots of f_2 are interlaced between those of $\breve{r}$ and $f_2(0) > 0$, from the continuity of f_2, using the method used above, it can be proved that $\phi_2(\zeta,\eta)$ is positive. □

Remark 3: The above property known as interlacing property can be deduced from Fuhrmann (1996, Theorem 9.1.8, p. 258). This property also arises in the case of positive real transfer functions of lossless electrical networks (see Baher (1984, p. 50)), wherein the transfer function is of the form

$$Z(\xi) = \left[\frac{H(\xi^2 + \omega_1^2)(\xi^2 + \omega_3^2)\ldots(\xi^2 + \omega_{2n-1}^2)}{\xi(\xi^2 + \omega_2^2)(\xi^2 + \omega_4^2)\ldots(\xi^2 + \omega_{2n-2}^2)}\right]^{\pm 1}$$

where $H \in \mathbb{R}^+$, and

$$0 < \omega_1 < \omega_2 < \omega_3 < \omega_4 < \cdots.$$

We show the link between interlacing property of positive real transfer functions of lossless electrical networks and the interlacing property of Theorem 2. Assume that the input voltage of a lossless electrical network is set to zero. Define

$$V(\xi) = r(\xi) = r'(\xi^2) = H(\xi^2+\omega_1^2)(\xi^2+\omega_3^2)\ldots(\xi^2+\omega_{2n-1}^2)$$
$$\frac{I(\xi)}{\xi} = f_1(\xi^2) = (\xi^2+\omega_2^2)(\xi^2+\omega_4^2)\ldots(\xi^2+\omega_{2n-2}^2).$$

Observe that the behaviour $\mathfrak{B} = \ker(V(\mathrm{d}/\mathrm{d}t)) = \ker(r(\mathrm{d}/\mathrm{d}t))$ corresponds to an autonomous lossless electrical network and also that r' and f_1 obey the interlacing property mentioned in Theorem 2. The two-variable polynomial corresponding to the power delivered to the network is given by

$$P(\zeta,\eta) = V(\zeta)I(\eta) + I(\zeta)V(\eta) = \eta r'(\zeta^2)f_1(\eta^2) + \zeta r'(\eta^2)f_1(\zeta^2).$$

If $\phi_1(\zeta,\eta)$ represents the two-variable polynomial corresponding to the energy function for the lossless network, then $(\zeta+\eta)\phi_1(\zeta,\eta) = P(\zeta,\eta)$ and this corresponds with Equation (8).

In the next corollary, we give the general expression for an energy function of a scalar conservative behaviour that has no characteristic frequency at zero.

Corollary 1: *Let $\mathfrak{B} = ker\ (r(\mathrm{d}/\mathrm{d}t))$ be an oscillatory behaviour, where $r(\xi) = (\xi^2+\omega_0^2)(\xi^2+\omega_1^2)\ldots(\xi^2+\omega_{n-1}^2)$. Define $v_p(\xi) := r(\xi)/(\xi^2+\omega_p^2)$, $V(\xi) := \mathrm{col}(v_0(\xi), v_1(\xi),\ldots,v_{n-1}(\xi))$ and $\Omega := \mathrm{diag}(\omega_0, \omega_1,\ldots,\omega_{n-1})$. A two-variable polynomial E induces an energy function for $\mathfrak{B}$, if and only if there exists a diagonal matrix $C\in\mathbb{R}^{n\times n}$ with positive diagonal entries, such that*

$$E(\zeta,\eta) = \zeta\eta V(\zeta)^T C^2 V(\eta) + V(\zeta)^T C^2\Omega^2 V(\eta). \qquad (11)$$

Proof: (*If*) From the (*if*) part of the proof of Theorem 1, it follows that

$$E(\zeta,\eta) = \sum_{p=0}^{n-1} a_p^2 v_p(\zeta)v_p(\eta)(\zeta\eta+\omega_p^2)$$

induces an energy function for $\mathfrak{B}$ if $a_p\in\mathbb{R}^+$ for $p=0,\ldots,n-1$. If C is defined as

$$C := \mathrm{diag}(a_0, a_1,\ldots,a_{n-1})$$

then, it is easy to see that Equation (11) holds.

(*Only if*) From the proof of Lemma 2, it follows that every energy function for $\mathfrak{B}$ has an associated two-variable polynomial of the form

$$E(\zeta,\eta) = \sum_{p=0}^{n-1} b_p v_p(\zeta)v_p(\eta)(\zeta\eta+\omega_p^2),$$

where $b_p\in\mathbb{R}^+$ for $p=0,\ldots,n-1$. Let $a_p\in\mathbb{R}^+$ be such that $a_p^2 = b_p$ for $p=0,\ldots,n-1$. Define

$$C := \mathrm{diag}(a_0, a_1,\ldots,a_{n-1}).$$

Then, it is easy to see that Equation (11) holds. □

With reference to the previous Corollary, if we interpret $q = V(\mathrm{d}/\mathrm{d}t)w$ as a generalised position, then $\mathrm{d}q/\mathrm{d}t = (\mathrm{d}/\mathrm{d}t)V(\mathrm{d}/\mathrm{d}t)w$ is a generalised velocity. Define $M := 2C^2$ and $K := 2C^2\Omega^2$. Using these expressions the system equations can be written in a way similar to the equations describing a second order mechanical system as

$$M\frac{\mathrm{d}^2q}{\mathrm{d}t^2} + Kq = 0$$
$$C^2\left(I\frac{\mathrm{d}^2}{\mathrm{d}t^2}+\Omega^2\right)V\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w = 0$$

which reduces to $\mathrm{col}(r(\mathrm{d}/\mathrm{d}t), r(\mathrm{d}/\mathrm{d}t),\ldots)w(t) = 0$. Thus M and K can be interpreted as the mass and the stiffness matrix respectively. This leads to the two-variable polynomials K and P corresponding to the kinetic energy $(\frac{1}{2}M(\mathrm{d}q/\mathrm{d}t)^2)$ and potential energy $(\frac{1}{2}Kq^2)$ respectively being given by

$$K(\zeta,\eta) = \zeta\eta V(\zeta)^T C^2 V(\eta) \qquad (12)$$

$$P(\zeta,\eta) = V(\zeta)^T C^2\Omega^2 V(\eta). \qquad (13)$$

We now illustrate the concepts discussed so far in this section using the example of a mechanical system.

Example 1: Consider two masses m_1 and m_2 attached to springs with constants k_1 and k_2. The first mass is connected to the second one via the first spring, and the second mass is connected to the wall with the second spring as shown in Figure 1. Denote by w_1 the position of the first mass. The differential equation governing w_1 is

$$r\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)w_1 = \frac{\mathrm{d}^4}{\mathrm{d}t^4}w_1 + \left(\frac{k_1+k_2}{m_2}+\frac{k_1}{m_1}\right)\frac{\mathrm{d}^2}{\mathrm{d}t^2}w_1 + \left(\frac{k_1k_2}{m_1m_2}\right)w_1 = 0.$$

Let $m_1 = m_2 = 1$, $k_1 = 2$ and $k_2 = 3$. Then $r(\xi) = \xi^4 + 7\xi^2 + 6 = (\xi^2+6)(\xi^2+1)$. The natural

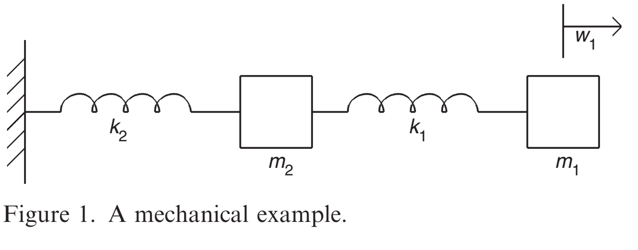

Figure 1. A mechanical example.

frequencies of the system are given by $\omega_0 = \sqrt{6}$ and $\omega_1 = 1$. The total kinetic energy and the total potential energy for the system can be expressed as QDFs in terms of only w_1. The two variable polynomials corresponding to these are

$$K(\zeta,\eta) = \frac{1}{8}[\zeta^3\eta^3 + 2(\zeta\eta^3 + \zeta^3\eta) + 8\zeta\eta] \qquad (14)$$

$$P(\zeta,\eta) = \frac{1}{8}[5\zeta^2\eta^2 + 6(\zeta^2+\eta^2) + 12]. \qquad (15)$$

The total energy of the system is a positive conserved quantity and hence from Lemma 2 will correspond to the two-variable polynomial of the form

$$E(\zeta,\eta) = a_0^2(\zeta\eta+6)(\zeta^2+1)(\eta^2+1) + a_1^2(\zeta\eta+1)(\zeta^2+6)(\eta^2+6)$$

Indeed by comparison with Equations (14) and (15), we obtain real values for a_0 and a_1 as

$$a_0 = \sqrt{0.1} \quad a_1 = \sqrt{0.025}$$

In this case, with $C = \mathrm{diag}(a_0, a_1)$ and $\Omega = \mathrm{diag}(\omega_0, \omega_1)$, it can be verified that Equations (12) and (13) reduce to Equations (14) and (15), respectively.

We now build upon the result of Theorem 1 and extend it to the multivariable case.

Theorem 3: *A linear autonomous system* $\mathfrak{B} \in \mathcal{L}^{\mathrm{w}}$ *is lossless if and only if it is oscillatory.*

Proof: We proceed by reduction of the multivariable case to the scalar case by use of the Smith form. Consider a kernel representation of $\mathfrak{B}$ given by $\mathfrak{B} = \ker(R(\mathrm{d}/\mathrm{d}t))$, where $R \in \mathbb{R}^{\mathrm{w}\times\mathrm{w}}[\xi]$ and $\det(R(\xi)) \neq 0$. Let $R = U\Delta V$ be the Smith form decomposition of R. Let the behaviour $\mathfrak{B}'$ be given by $\mathfrak{B}' = \ker(\Delta(\mathrm{d}/\mathrm{d}t))$. Denote the number of invariant polynomials of R equal to one with w_1 and let $\{r_i(\xi)\}_{i=\mathrm{w}_1+1,\ldots,\mathrm{w}}$ be the set consisting of the remaining invariant polynomials of R. Let $\mathfrak{B}'_i = \ker(r_i(\mathrm{d}/\mathrm{d}t))$.

(*Only If*) We assume that $\mathfrak{B}$ and hence $\mathfrak{B}'$ are lossless. Consider a trajectory $w' \in \mathfrak{B}'$. Let $\{w'_i\}_{i=1,\ldots,\mathrm{w}}$ be the components of w'. Consider an energy function Q_ϕ of $\mathfrak{B}$ acting on w. Let $\phi'(\zeta,\eta) = (V(\zeta))^{-\mathrm{T}}\phi(\zeta,\eta)(V(\eta))^{-1}$. Let $\phi'_i(\zeta,\eta)$ be the i^{th} diagonal entry and $\phi'_{ik}(\zeta,\eta)$ be the entry corresponding to the i^{th} row and k^{th} column of the polynomial matrix $\phi'(\zeta,\eta)$. Then

$$Q_\phi(w) = Q_{\phi'}(w') = \sum_{i=1}^{\mathrm{w}} Q_{\phi'_i}(w'_i) + \sum_{i\neq k} L_{\phi'_{ik}}(w'_i, w'_k) \quad (16)$$

Since $Q_\phi > 0$, also $Q_{\phi'} > 0$. Since each component of w' can be chosen independently of each other, it follows that $Q_{\phi'_i} > 0$ and is conserved over $\mathfrak{B}'_i$ for $i = 1, 2, \ldots, \mathrm{w}$. This is possible only if each of $\mathfrak{B}'_i$ is oscillatory for $i = 1, 2, \ldots, \mathrm{w}$, which implies that $\mathfrak{B}'$ and hence $\mathfrak{B}$ is oscillatory.

(*If*) Assume that $\mathfrak{B}$ and hence $\mathfrak{B}'$ is oscillatory. We construct a QDF that is positive and conserved along $\mathfrak{B}$ and hence prove that the system is lossless. For $i = \mathrm{w}_1+1, \ldots, \mathrm{w}$, let r_i have non-zero roots at $\pm j\omega_{0i}$, $\pm j\omega_{1i}$, $\pm j\omega_{2i}, \ldots$ and maximal even polynomial factor equal to s_i. Define $v_{pq}(\xi) := s_q(\xi)/(\xi^2 + \omega_{pq}^2)$. Consider

$$D(\xi) = \begin{bmatrix} 0_{\mathrm{w}_1\times\mathrm{w}_1} & 0_{\mathrm{w}_1\times 1} & \cdots & \cdots & \cdots & \\ 0_{\bullet\times\mathrm{w}_1} & D_{\mathrm{w}_1+1} & 0_{\bullet\times 1} & \cdots & \cdots & \cdots \\ 0_{\bullet\times\mathrm{w}_1} & 0_{\bullet\times 1} & D_{\mathrm{w}_1+2} & 0_{\bullet\times 1} & \cdots & \cdots \\ \vdots & \vdots & 0_{\bullet\times 1} & \ddots & \ddots & \vdots \\ 0_{\bullet\times\mathrm{w}_1} & 0_{\bullet\times 1} & \vdots & \ddots & \ddots & 0_{\bullet\times 1} \\ 0_{\bullet\times\mathrm{w}_1} & 0_{\bullet\times 1} & \cdots & \cdots & \cdots & D_{\mathrm{w}}(\xi) \end{bmatrix}, \quad (17)$$

where $D_i = \mathrm{col}(a_{0i}\xi v_{0i}(\xi),\ a_{0i}\omega_{0i}v_{0i}(\xi),\ a_{1i}\xi v_{1i}(\xi),\ a_{1i}\omega_{1i}v_{1i}(\xi),\ \ldots)$ if r_i is even and $D_i = \mathrm{col}(a_{0i}\xi^2 v_{0i}(\xi),\ a_{0i}\omega_{0i}\xi v_{0i}(\xi),\ a_{1i}\xi^2 v_{1i}(\xi),\ a_{1i}\omega_{1i}\xi v_{1i}(\xi), \ldots)$ if r_i is odd, $a_{ik} \in \mathbb{R}^+$ as in the proof of the sufficiency part of Theorem 1. From the argument used in order to prove the scalar case, it is easy to see that $\phi'(\zeta,\eta) = D(\zeta)^T D(\eta)$ is positive and conserved along $\mathfrak{B}'$, and hence $\phi(\zeta,\eta) = V(\zeta)^\top D(\zeta)^\top D(\eta)V(\eta)$ is positive and conserved along $\mathfrak{B}$. This concludes the proof. □

4. State space case

In this section, we define an autonomous lossless system for the state space case in a way analogous to the one in higher-order approach. We then prove that an autonomous system is lossless if and only if it is oscillatory. We begin with the following definition.

Definition 7: A linear system given by the state space equation $\mathrm{d}x/\mathrm{d}t = Ax$, where $x \in \mathbb{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{x}})$ and $A \in \mathbb{R}^{\mathrm{x}\times\mathrm{x}}$ is lossless if there exists $S \in \mathbb{R}_s^{\mathrm{x}\times\mathrm{x}}$, such that $S > 0$ and the functional Q_E defined by $Q_E(x)(t) := x(t)^T S x(t)$ is conserved along the trajectories of the system, i.e., $(\mathrm{d}/\mathrm{d}t)Q_E(x) = 0\ \forall x$ that satisfy $\mathrm{d}x/\mathrm{d}t = Ax$. We call E an energy function for the system.

We now prove the main result of this section, namely equivalence of autonomous lossless and oscillatory systems using the state space method.

Theorem 4: *Consider a linear autonomous system that obeys the state space equation* $\mathrm{d}x/\mathrm{d}t = Ax$ *where* $x \in \mathbb{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{x}})$ *and* $A \in \mathbb{R}^{\mathrm{x}\times\mathrm{x}}$. $\mathfrak{B}$ *is oscillatory if and only if it is lossless.*

Proof: (*If*) We consider two cases, namely A diagonalisable and A not diagonalisable. We prove that in the diagonalisable case, the existence of an energy function implies that the system is oscillatory. We then prove that an energy function cannot exist for the non-diagonalisable case of linear systems.

- **Case 1:** A is diagonalisable. Let $Vx = z \in \mathbb{C}^{\infty}(\mathbb{R}, \mathbb{C}^{x})$, where V is invertible and $V A V^{-1} = A_d$ is a diagonal matrix whose i^{th} diagonal element is $\delta_i \in \mathbb{C}$. We have

$$\frac{dz}{dt} = V\frac{dx}{dt} = V\,Ax = V\,AV^{-1}z = A_d z$$

Assume that there exists a matrix $K \in \mathbb{C}^{x \times x}$ such that $K = K^*$ and $z^*Kz > 0$ for all non-zero $z \in \mathbb{C}^{x \times 1}$. Since x is a real vector and V is invertible,

$$z^*Kz = x^T V^* K V x > 0 \quad \forall x \neq 0.$$

This implies that $S = V^*KV$ is positive definite and real. Let $E = z^*Kz = x^T Sx$. This implies that

$$\frac{dE}{dt} = z^*(KA_d + A_d^* K)z.$$

Let K_{ik} be the element of the matrix K corresponding to the i^{th} row and k^{th} column. Then Equation (18) holds

$$K\,A_d + A_d^* K = \begin{bmatrix} 2\mathrm{Re}(\delta_1)K_{11} & (\bar{\delta}_1 + \delta_2)\bar{K}_{12} & (\bar{\delta}_1 + \delta_3)\bar{K}_{13} & \dots \\ (\delta_1 + \bar{\delta}_2)K_{12} & 2\mathrm{Re}(\delta_2)K_{22} & (\bar{\delta}_2 + \delta_3)\bar{K}_{23} & \dots \\ (\delta_1 + \bar{\delta}_3)K_{13} & (\delta_2 + \bar{\delta}_3)K_{23} & 2\mathrm{Re}(\delta_3)K_{33} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (18)$$

Now $dE/dt = 0 \;\forall\, y \in \mathbb{C}^{x \times 1} \Rightarrow KA_d + A_d^*K = 0$. Also since K is positive definite, $K_{pp} > 0$, for $p = 1, \dots, x$. This implies that each of the diagonal entries of A_d is purely imaginary. Hence the system whose governing equation is $dz/dt = A_d z$, is oscillatory. Since oscillatory nature is invariant under similarity transformation, the given system is also oscillatory.

- **Case 2:** A is not diagonalisable. Let $Vx = z \in \mathbb{C}^{x \times 1}$, where V is invertible and $VAV^{-1} = A_J$ is the Jordan form of A. Let $A_J = \mathrm{diag}(\Delta_1, \dots, \Delta_n)$, where

$$\Delta_i = \begin{bmatrix} \delta_i & 1 & 0 & \dots & 0 \\ 0 & \delta_i & 1 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & \delta_i & 1 \\ 0 & \dots & 0 & 0 & \delta_i \end{bmatrix}.$$

As in the earlier case, let $E = z^*Kz = x^\top Sx$, where K is positive definite. This implies that

$$\frac{dE}{dt} = z^*(KA_J + A_J^* K)z.$$

Let K_{ik} be the element of the matrix K corresponding to the i^{th} row and k^{th} column. Then Equation (19) holds.

$$KA_J + A_J^*K = \begin{bmatrix} \ddots & \vdots & \vdots & \vdots \\ \dots & 2\mathrm{Re}(\delta_g)K_{ii} & K_{ii} + 2\mathrm{Re}(\delta_g)\bar{K}_{i,i+1} & \dots \\ \dots & K_{ii} + 2\mathrm{Re}(\delta_g)K_{i,i+1} & 2\mathrm{Re}(\delta_g)K_{i+1,i+1} + 2\mathrm{Re}(K_{i,i+1}) & \dots \\ \dots & K_{i,i+1} + 2\mathrm{Re}(\delta_g)K_{i,i+2} & K_{i,i+2} + K_{i+1,i+1} + 2\mathrm{Re}(\delta_g)K_{i+1,i+2} & \dots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} \quad (19)$$

$$\frac{dE}{dt} = 0$$

implies that the right hand side of Equation (19) is zero. Since $K_{ii} \neq 0$, we obtain $\mathrm{Re}(\delta_g) = 0$ for $g = 1, \dots, n$, which in turn implies that $K_{ii} + 2\mathrm{Re}(\delta_g)\bar{K}_{i,i+1} = K_{ii} = 0$, which is not possible since K is positive definite. Hence a non-diagonalisable system cannot be lossless.

Thus an autonomous lossless system is necessarily oscillatory.

(*Only if*) We prove that for an oscillatory system, there exists a quadratic functional which is positive definite and is conserved. Let $Vx = z \in \mathbb{C}^{\infty}(\mathbb{R}, \mathbb{C}^{x})$, where V is invertible and $VAV^{-1} = A_d$ is a diagonal matrix whose i^{th} diagonal element is δ_i, which is purely imaginary. We have

$$\frac{dz}{dt} = V\frac{dx}{dt} = V\,Ax = V\,AV^{-1}z = A_d z.$$

Now consider a diagonal matrix K of size $x \times x$, each of whose diagonal entries is real and positive. It is easy to see that $z^*Kz > 0$ for all non-zero $z \in \mathbb{C}^{x \times 1}$. Since x is a real vector and V is invertible,

$$z^*Kz = x^T V^* K V x > 0 \quad \forall x \neq 0.$$

This implies that $S = V^*KV$ is positive definite and real. Let $E = z^*Kz = x^{\mathrm{T}}Sx$. This implies that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = z^*(KA_d + A_d^*K)z = 0.$$

Hence an oscillatory system is necessarily lossless. □

5. Open lossless systems

In this section, we consider systems that are not autonomous, i.e., systems which have inputs. Here we define open lossless systems based on the observation that the total energy of a physical system of this type is positive definite and that the rate of change of total energy of such a system is zero if the inputs of the system are made equal to zero.

Definition 8: A controllable behaviour $\mathfrak{B}$ is called lossless with respect to an input/output partition $\mathrm{col}(u, y)$ of $\mathfrak{B}$, if $\exists$ *a QDF* $Q_E > 0$ ($E \in \mathbb{R}^{(\mathrm{u}+\mathrm{y})\times(\mathrm{u}+\mathrm{y})}[\zeta, \eta]$), such that

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_E(w) = 0 \quad \forall w = \mathrm{col}(0, y) \in \mathfrak{B}$$

Any QDF Q_E that satisfies the properties of the above definition is called an energy function for the system.

Lemma 3: *Consider a controllable behaviour* $\mathfrak{B} \in \mathcal{L}^{\mathrm{u}+\mathrm{y}}$. $\mathfrak{B}$ *is lossless with respect to an input/output partition* $w = \mathrm{col}(u, y)$ *of* $\mathfrak{B}$, *if and only if* $\mathfrak{B}_y := \{y \in \mathbb{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathrm{y}}) | \mathrm{col}(0, y) \in \mathfrak{B}\}$ *is oscillatory*.

Proof: (*If*) Assume that $\exists$ an input/output partition $w = \mathrm{col}(u, y)$ of $\mathfrak{B}$ such that $\mathfrak{B}_y$ is oscillatory and hence lossless. Let Q_{E_1} be an energy function for $\mathfrak{B}_y$ in the sense of Definition 5. Define

$$E(\zeta, \eta) := \begin{bmatrix} I_u & 0 \\ 0 & E_1(\zeta, \eta) \end{bmatrix}.$$

Since $Q_{E_1} > 0$, it follows that $Q_E > 0$. Also $(\mathrm{d}/\mathrm{d}t)Q_E(w) = 0 \ \forall \ w = \mathrm{col}(0, y) \in \mathfrak{B}$. Thus Q_E is an energy function for $\mathfrak{B}$ in the sense of Definition 8. Hence $\mathfrak{B}$ is lossless with respect to the input/output partition $w = \mathrm{col}(u, y)$.

(*Only If*) Assume that $\mathfrak{B}$ is lossless with respect to an input/output partition $w = \mathrm{col}(u, y)$ of $\mathfrak{B}$. Hence there exists an energy function Q_E, such that $(\mathrm{d}/\mathrm{d}t)Q_E(w) = 0 \ \ \forall \ w = \mathrm{col}(0, y) \in \mathfrak{B}$. Partition the two-variable polynomial matrix $E(\zeta, \eta)$ inducing Q_E as

$$E(\zeta, \eta) = \begin{bmatrix} E_{11}(\zeta, \eta) & E_{12}(\zeta, \eta) \\ E_{12}(\eta, \zeta)^T & E_{22}(\zeta, \eta) \end{bmatrix},$$

where $E_{11} \in \mathbb{R}^{\mathrm{u}\times\mathrm{u}}[\zeta, \eta]$, $E_{12} \in R^{\mathrm{u}\times\mathrm{y}}[\zeta, \eta]$ and $E_{22} \in \mathbb{R}^{\mathrm{y}\times\mathrm{y}}[\zeta, \eta]$. Since $Q_E > 0$, it follows that $Q_{E_{22}} > 0$. Also since Q_E is an energy function for $\mathfrak{B}$ in the sense of Definition 8, $(\mathrm{d}/\mathrm{d}t)Q_{E_{22}}(w) = 0 \ \ \forall \ w \in \mathfrak{B}_y$. Hence $\mathfrak{B}_y$ is lossless, which in turn implies that it is oscillatory. □

Remark 4: With reference to the above lemma, it is easy to see that if $P(\mathrm{d}/\mathrm{d}t)y = Q(\mathrm{d}/\mathrm{d}t)u$ and $u = D(\mathrm{d}/\mathrm{d}t)\ell$, $y = N(\mathrm{d}/\mathrm{d}t)\ell$ are respectively a minimal kernel representation and an observable image representation for $\mathfrak{B}$ then P, respectively D are oscillatory.

In the next algorithm, we show the computation of an energy function of a controllable lossless behaviour, starting from an observable image representation of the behaviour.

Algorithm 1: Input: An observable image representation of $\mathfrak{B} \in \mathcal{L}^{\mathrm{u}+\mathrm{y}}$ of the form $u = D(\mathrm{d}/\mathrm{d}t)\ell$, $y = N(\mathrm{d}/\mathrm{d}t)\ell$, where $N \in \mathbb{R}^{\mathrm{y}\times\mathrm{u}}[\xi]$ and $D \in \mathbb{R}^{\mathrm{u}\times\mathrm{u}}[\xi]$ is oscillatory.

Output: A two-variable polynomial matrix $E \in \mathbb{R}^{(\mathrm{u}+\mathrm{y})\times(\mathrm{u}+\mathrm{y})}[\zeta, \eta]$ that induces an energy function for $\mathfrak{B}$ in the sense of Definition 8.

Step 1: Compute a Smith form decomposition of D given by $D = U\Delta V$.

Step 2: Let w_1 = number of invariant polynomials of D equal to one.

Step 3: Let $\{r_i(\xi)\}_{i=\mathrm{w}_1+1,\ldots,\mathrm{u}}$ be the set consisting of the non-unity invariant polynomials of D.

Step 4: For $i = \mathrm{w}_1 + 1, \ldots, \mathrm{u}$, let $\pm j\omega_{0i}$, $\pm j\omega_{1i}$, $\pm j\omega_{2i}, \ldots$ be the non-zero roots of r_i and let s_i be the maximal even polynomial factor of r_i.

Step 5: Define $v_{pq}(\xi) := s_q(\xi)/(\xi^2 + \omega_{pq}^2)$.

Step 6: Construct the matrix

$$D'(\xi) = \begin{bmatrix} 0_{\mathrm{w}_1\times\mathrm{w}_1} & 0_{\mathrm{w}_1\times 1} & \ldots & \ldots & \ldots & \\ 0_{\bullet\times\mathrm{w}_1} & D_{\mathrm{w}_1+1} & 0_{\bullet\times 1} & \ldots & \ldots & \ldots \\ 0_{\bullet\times\mathrm{w}_1} & 0_{\bullet\times 1} & D_{\mathrm{w}_1+2} & 0_{\bullet\times 1} & \ldots & \ldots \\ \vdots & \vdots & 0_{\bullet\times 1} & \ddots & \ddots & \vdots \\ 0_{\bullet\times\mathrm{w}_1} & 0_{\bullet\times 1} & \vdots & \ddots & \ddots & 0_{\bullet\times 1} \\ 0_{\bullet\times w_1} & 0_{\bullet\times 1} & \ldots & \ldots & \ldots & D_{\mathrm{u}}(\xi) \end{bmatrix},$$

where $D_i = \mathrm{col}(a_{0i}\xi v_{0i}(\xi), \ a_{0i}\omega_{0i}v_{0i}(\xi), \ a_{1i}\xi v_{1i}(\xi), \ a_{1i}\omega_{1i}v_{1i}(\xi), \ldots)$ if r_i is even, $D_i = \mathrm{col}(a_{0i}\xi^2 v_{0i}(\xi), \ a_{0i}\omega_{0i}\xi v_{0i}(\xi), \ a_{1i}\xi^2 v_{1i}(\xi), \ a_{1i}\omega_{1i}\xi v_{1i}(\xi), \ldots)$ if r_i is odd and $a_{ik} \in \mathbb{R}^+$.

Step 7: Define $M := \mathrm{col}(D, N)$. Compute a left inverse C_0 of M.

Step 8: Compute

$$E(\zeta,\eta) = C_0(\zeta)^T V(\zeta)^T D'(\zeta)^T D'(\eta) V(\eta) C_0(\eta). \qquad (18)$$

The next lemma gives an explanation for the steps in the previous algorithm.

Lemma 4: *With reference to Algorithm* 1, *the two-variable polynomial matrix E given by Equation* (18) *induces an energy function for* $\mathfrak{B}$.

Proof: Let $Q_{E'}$ be a QDF, such that for any trajectory $w \in \mathfrak{B}$, $Q_E(w) = Q_{E'}(\ell)$, where $w = M(\mathrm{d}/\mathrm{d}t)\ell$ is an observable image representation of $\mathfrak{B}$. Then

$$E'(\zeta,\eta) = M(\zeta)^T E(\zeta,\eta) M(\eta) = V(\zeta)^T D'(\zeta)^T D'(\eta) V(\eta).$$

Consider the behaviour $\mathfrak{B}_{\mathrm{aut}} \in \mathcal{L}^{\mathrm{u}}$, defined as

$$\mathfrak{B}_{\mathrm{aut}} := \left\{ \ell \in \mathbb{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathrm{u}}) \middle| D\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\ell = 0 \right\}.$$

From the method of construction of an energy function for an autonomous lossless system given in the proof of Theorem 3, it follows that $Q_{E'}$ is an energy function for the lossless autonomous behaviour $\mathfrak{B}_{\mathrm{aut}}$ in the sense of Definition 5. Now $w = M(\mathrm{d}/\mathrm{d}t)\ell \Rightarrow \ell = C_0(\mathrm{d}/\mathrm{d}t)w$. Hence from Definition 8, it follows that

$$E(\zeta,\eta) = C_0(\zeta)^T V(\zeta)^T D'(\zeta)^T D'(\eta) V(\eta) C_0(\eta)$$

induces an energy function for $\mathfrak{B}$. □

If we consider a mechanical system, the total power delivered to the system is equal to the the summation of scalar products of various forces (inputs) acting on the system and the velocities (outputs) at the points of application of the respective forces. Similarly, the total power delivered to an electrical system is equal to the summation of the products of input voltages across various branches and the currents (outputs) through them. Hence the total power delivered to such systems can be written as a quadratic functional, each term of which involves a certain derivative of an input variable and a certain derivative of an output variable. We now investigate whether a similar property holds for the derivative of an energy function of a controllable lossless behaviour which we may call a "power function". We begin with the following definition.

Definition 9: Consider a behaviour $\mathfrak{B}$ with an input/output partition $\mathrm{col}(u,y)$ ($u \in \mathbb{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathrm{u}})$, $y \in \mathbb{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathrm{y}})$). u is said to have inconsequential components if the behaviour $\mathfrak{B}_u = \{u \in \mathbb{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathrm{u}}) | \mathrm{col}(u,0) \in \mathfrak{B}\}$ is not autonomous.

The next lemma gives the condition on the kernel representation of a behaviour under which its input does not have inconsequential components.

Lemma 5: *Consider a behaviour* $\mathfrak{B}$ *with an input/output partition* $\mathrm{col}(u, y)$ ($u \in \mathbb{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathrm{u}}), y \in \mathbb{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathrm{y}})$). *Let* $P(\frac{\mathrm{d}}{\mathrm{d}t})\, y = Q(\frac{\mathrm{d}}{\mathrm{d}t})u$ $(P \in \mathbb{R}^{\mathrm{y}\times\mathrm{y}}[\xi], Q \in \mathbb{R}^{\mathrm{y}\times\mathrm{u}}[\xi])$ *be a minimal kernel representation of* $\mathfrak{B}$. *u does not have inconsequential components iff* $\mathrm{colrank}(Q) = \mathrm{u}$.

Proof: Consider the behaviour $\mathfrak{B}_u = \ker(Q(\mathrm{d}/\mathrm{d}t))$. Since $\mathfrak{B}_u$ is autonomous iff Q has full column rank, hence u does not have inconsequential inputs iff $\mathrm{colrank}(Q) = \mathrm{u}$. □

The next lemma gives the condition on the image representation of a controllable behaviour under which its input does not have inconsequential components.

Lemma 6: *Consider a controllable behaviour* $\mathfrak{B}$ *with an input/output partition* $\mathrm{col}(u,y)$ ($u \in \mathbb{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathrm{u}})$, $y \in \mathbb{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathrm{y}})$). *Let* $y = N(\mathrm{d}/\mathrm{d}t)\ell$, $u = D(\mathrm{d}/\mathrm{d}t)\ell$, ($N \in \mathbb{R}^{\mathrm{y}\times\mathrm{u}}[\xi]$, $D \in \mathbb{R}^{\mathrm{u}\times\mathrm{u}}[\xi]$) *be an observable image representation of* $\mathfrak{B}$. *u does not have inconsequential components iff* $\mathrm{colrank}(N) = \mathrm{u}$.

Proof: Observe that $\mathfrak{B}_u$ is autonomous iff the behaviour $\mathfrak{B}_l := \ker(N(\mathrm{d}/\mathrm{d}t))$ is autonomous, which in turn holds iff N has full column rank. Conclude from this that u does not have inconsequential inputs iff $\mathrm{colrank}(N) = \mathrm{u}$. □

In the next theorem, using the concept of *inconsequential components*, we give the condition on the kernel representation of a controllable lossless system under which its power function can be written as a QDF, each term of which involves a certain derivative of an input variable and a certain derivative of an output variable.

Theorem 5: *Consider a controllable behaviour* $\mathfrak{B}$ *which is lossless with respect to an input/output partition* $\mathrm{col}(u,y)$, ($u \in \mathbb{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathrm{y}})$, $y \in \mathbb{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathrm{y}})$) *of* $\mathfrak{B}$. *There exists an energy function* Q_E, *such that*

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_E(\mathrm{col}(u,y)) = \left(R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)u\right)^T \left(S\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)y\right) \qquad (20)$$

where $R \in \mathbb{R}^{\bullet\times\mathrm{u}}[\xi]$ *and* $S \in \mathbb{R}^{\bullet\times\mathrm{y}}[\xi]$, *iff the following two conditions hold*:

(1) *u does not have inconsequential components*;
(2) *all the invariant polynomials of Q in any minimal kernel representation of* $\mathfrak{B}$ *given by* $P(\mathrm{d}/\mathrm{d}t)y = Q(\mathrm{d}/\mathrm{d}t)u$ *are oscillatory*.

Proof: (*If*) Assume that Q has full column rank and has all its invariant polynomials oscillatory. Let $\mathfrak{B}_y = \{y \in \mathbb{C}^{\infty}(\mathbb{R},\mathbb{R}^{\mathrm{y}}) | \mathrm{col}(0,y) \in \mathfrak{B}\}$. Let Q_{E_1} be a $\mathfrak{B}_y$-canonical energy function for $\mathfrak{B}_y$ in the sense of Definition 5. Since Q has full column rank, $y \geq u$. Hence, there exists a unimodular matrix $U \in \mathbb{R}^{\mathrm{y}\times\mathrm{y}}[\xi]$, such that

$$U(\xi)Q(\xi) = \begin{bmatrix} Q_1(\xi) \\ 0_{(\mathrm{y}-\mathrm{u})\times\mathrm{u}} \end{bmatrix},$$

where $Q_1 \in \mathbb{R}^{\mathrm{u}\times\mathrm{u}}[\xi]$ is such that $\det(Q_1(\xi)) \neq 0$. Now consider the behaviour $\mathfrak{B}_1 := \ker(Q_1(\mathrm{d}/\mathrm{d}t))$. Since all the invariant polynomials of Q are oscillatory, $\mathfrak{B}_1$ is an autonomous lossless behaviour. Let Q_{E_2} be a $\mathfrak{B}_1$-canonical energy function for $\mathfrak{B}_1$ in the sense of Definition 5. Consider the energy function Q_E for $\mathfrak{B}$ given by

$$E(\zeta,\eta) = \begin{bmatrix} E_2(\zeta,\eta) & 0_{\mathrm{u}\times\mathrm{y}} \\ 0_{\mathrm{y}\times\mathrm{u}} & E_1(\zeta,\eta) \end{bmatrix}.$$

Define

$$E_1'(\zeta,\eta) := \begin{bmatrix} 0_{\mathrm{u}\times\mathrm{u}} & 0_{\mathrm{u}\times\mathrm{y}} \\ 0_{\mathrm{y}\times\mathrm{u}} & E_1(\zeta,\eta) \end{bmatrix}.$$

Let the QDF Q_{P_1} be such that $Q_{P_1} \stackrel{\mathfrak{B}}{=} (\mathrm{d}/\mathrm{d}t)Q_{E_1'}$. Let

$$P_1(\zeta,\eta) = \begin{bmatrix} P_{11}(\zeta,\eta) & P_{12}(\zeta,\eta) \\ P_{12}(\eta,\zeta)^T & P_{13}(\zeta,\eta) \end{bmatrix},$$

where $P_{11} \in \mathbb{R}^{\mathrm{u}\times\mathrm{u}}[\zeta,\eta]$, $P_{12} \in \mathbb{R}^{\mathrm{u}\times\mathrm{y}}[\zeta,\eta]$ and $P_{13} \in \mathbb{R}^{\mathrm{y}\times\mathrm{y}}[\zeta,\eta]$. Since Q_{E_1} is an energy function for $\mathfrak{B}_y$, $\forall\, w = \mathrm{col}(0,y) \in \mathfrak{B}$, $Q_{P_1}(w) = 0$. Hence $P_{13}(\zeta,\eta) = 0_{\mathrm{y}\times\mathrm{y}}$. Now $\forall\, w = \mathrm{col}(u,0) \in \mathfrak{B}$, $Q_{E_1'}(w) = 0$ and consequently $Q_{P_1}(w) = 0$. Hence $P_{11}(\zeta,\eta) = 0_{\mathrm{u}\times\mathrm{u}}$ Define

$$E_2'(\zeta,\eta) := \begin{bmatrix} E_2(\zeta,\eta) & 0_{\mathrm{u}\times\mathrm{y}} \\ 0_{\mathrm{y}\times\mathrm{u}} & 0_{\mathrm{y}\times\mathrm{y}} \end{bmatrix}.$$

Consider a QDF Q_{P_2}, which is equivalent to $(\mathrm{d}/\mathrm{d}t)Q_{E_2'}$ along $\mathfrak{B}$. Let

$$P_2(\zeta,\eta) = \begin{bmatrix} P_{21}(\zeta,\eta) & P_{22}(\zeta,\eta) \\ P_{22}(\eta,\zeta)^T & P_{23}(\zeta,\eta) \end{bmatrix},$$

where $P_{21} \in \mathbb{R}^{\mathrm{u}\times\mathrm{u}}[\zeta,\eta]$, $P_{22} \in \mathbb{R}^{\mathrm{u}\times\mathrm{y}}[\zeta,\eta]$ and $P_{23} \in \mathbb{R}^{\mathrm{y}\times\mathrm{y}}[\zeta,\eta]$. Since Q_{E_2} is an energy function for $\mathfrak{B}_1$, $\forall\, w = \mathrm{col}(0,y) \in \mathfrak{B}$, $Q_{E_2'}(w) = 0$, and consequently $Q_{P_2}(w) = 0$. Hence $P_{23}(\zeta,\eta) = 0_{\mathrm{y}\times\mathrm{y}}$.

Since Q_{E_2} is an energy function for $\mathfrak{B}_u$, $\forall\, w = \mathrm{col}(u,0) \in \mathfrak{B}$, $Q_{P_2}(w) = 0$. Hence $P_{21}(\zeta,\eta) = 0_{\mathrm{u}\times\mathrm{u}}$. It is easy to see that

$$P_1(\zeta,\eta) + P_2(\zeta,\eta) = \begin{bmatrix} 0_{\mathrm{u}\times\mathrm{u}} & Z(\zeta,\eta) \\ Z(\eta,\zeta)^T & 0_{\mathrm{y}\times\mathrm{y}} \end{bmatrix} \stackrel{\mathfrak{B}}{=} (\zeta+\eta)E(\zeta,\eta),$$

where $Z(\zeta,\ \eta) = P_{12}(\zeta,\ \eta) + P_{22}(\zeta,\ \eta)$. This implies that

$$\frac{\mathrm{d}}{\mathrm{d}t} Q_E(\mathrm{col}(u,y)) = \left(R\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)u\right)^T \left(S\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)y\right),$$

where $R(\zeta)^{\mathrm{T}}S(\eta)$ is a canonical factorisation of $2Z(\zeta,\eta)$.

(*Only If*) Assume that u has inconsequential components. Let $Q = U\Delta V$ be a Smith form decomposition of Q. Define $u' := V(\mathrm{d}/\mathrm{d}t)u$. Let $\mathfrak{B}' = \{\mathrm{col}(u',y) \in \mathbb{C}^\infty(\mathbb{R},\mathbb{R}^{\mathrm{u}+\mathrm{y}}) \,|\, (V_1(\mathrm{d}/\mathrm{d}t)u', y) \in \mathfrak{B}\}$, where $V_1(\xi) = V(\xi)^{-1}$. Since $\mathfrak{B}$ is lossless with respect to its input/output partition $w = \mathrm{col}(u,y)$, $\mathfrak{B}'$ is also lossless with respect to its input/output partition $w' = \mathrm{col}(u',\ y)$. Define

$$R(\xi) := [\,\Delta(\xi) \quad -U(\xi)^{-1}P(\xi)\,].$$

Then $\mathfrak{B}' = \ker(R(\mathrm{d}/\mathrm{d}t))$. Since u has inconsequential components, we can write

$$\Delta(\xi) = [\,D(\xi) \quad 0_{\mathrm{y}\times\mathrm{u}_1}\,],$$

where $D \in \mathbb{R}^{\mathrm{y}\times(\mathrm{u}-\mathrm{u}_1)}[\xi]$. Consider an energy function $Q_{E'}$ for $\mathfrak{B}'$ given by

$$E'(\zeta,\eta) = \begin{bmatrix} E_{11}(\zeta,\eta) & E_{12}(\zeta,\eta) & E_{13}(\zeta,\eta) \\ E_{12}(\eta,\zeta)^T & E_{22}(\zeta,\eta) & E_{23}(\zeta,\eta) \\ E_{13}(\eta,\zeta)^T & E_{23}(\eta,\zeta)^\top & E_{33}(\zeta,\eta) \end{bmatrix},$$

where $E_{33} \in \mathbb{R}^{\mathrm{y}\times\mathrm{y}}[\zeta,\eta]$, $E_{12} \in \mathbb{R}^{(\mathrm{u}-\mathrm{u}_1)\times\mathrm{u}_1}[\zeta,\eta]$, $E_{13} \in \mathbb{R}^{(\mathrm{u}-\mathrm{u}_1)\times\mathrm{y}}[\zeta,\eta]$ and $E_{23} \in \mathbb{R}^{\mathrm{u}_1\times\mathrm{y}}[\zeta,\eta]$, $E_{22} \in \mathbb{R}^{\mathrm{u}_1\times\mathrm{u}_1}[\zeta,\eta]$ and $E_{11} \in \mathbb{R}^{(\mathrm{u}-\mathrm{u}_1)\times(\mathrm{u}-\mathrm{u}_1)}[\zeta,\eta]$. It is easy to see that $Q_{E_{22}} > 0$. Property (20) implies that the derivative of the energy function Q_E is equivalent to another QDF along the behaviour whose associated two-variable polynomial P_0 has the form

$$P_0(\zeta,\eta) = \begin{bmatrix} 0_{\mathrm{u}\times\mathrm{u}} & Z(\zeta,\eta) \\ Z(\eta,\zeta)^T & 0_{\mathrm{y}\times\mathrm{y}} \end{bmatrix}, \tag{21}$$

where $2Z(\zeta,\ \eta) = R(\zeta)^T S(\eta)$. Let the QDF $Q_{P'}$ be such that $Q_{P'} \stackrel{\mathfrak{B}'}{=} (\mathrm{d}/\mathrm{d}t)Q_{E'}$, where

$$P'(\zeta,\eta) = \begin{bmatrix} 0_{\mathrm{u}\times\mathrm{u}} & P_2(\zeta,\eta) \\ P_2(\eta,\zeta)^T & 0_{\mathrm{u}\times\mathrm{u}} \end{bmatrix},$$

where $P_2 \in \mathbb{R}^{\mathrm{u}\times\mathrm{y}}[\zeta,\eta]$. Since $(\zeta+\eta)E'(\zeta,\eta) \stackrel{\mathfrak{B}'}{=} P'(\zeta,\eta)$,

$$(\zeta+\eta)E'(\zeta,\eta) = R(\zeta)^T F(\zeta,\eta) + F(\eta,\zeta)^T R(\eta) + P'(\zeta,\eta),$$

where $F \in \mathbb{R}^{\mathrm{y}\times(\mathrm{y}+\mathrm{u})}[\zeta,\eta]$. Let the right hand side of the above equation be denoted by $P''(\zeta,\ \eta)$. Let

$$P''(\zeta,\eta) = \begin{bmatrix} P_{11}(\zeta,\eta) & P_{12}(\zeta,\eta) & P_{13}(\zeta,\eta) \\ P_{12}(\eta,\zeta)^T & P_{22}(\zeta,\eta) & P_{23}(\zeta,\eta) \\ P_{13}(\eta,\zeta)^T & P_{23}(\eta,\zeta)^T & P_{33}(\zeta,\eta) \end{bmatrix},$$

where $P_{33} \in \mathbb{R}^{\mathrm{y}\times\mathrm{y}}[\zeta,\eta]$, $P_{12} \in \mathbb{R}^{(\mathrm{u}-\mathrm{u}_1)\times\mathrm{u}_1}[\zeta,\eta]$, $P_{13} \in \mathbb{R}^{(\mathrm{u}-\mathrm{u}_1)\times\mathrm{y}}[\zeta,\eta]$ and $P_{23} \in \mathbb{R}^{\mathrm{u}_1\times\mathrm{y}}[\zeta,\eta]$, $P_{22} \in \mathbb{R}^{\mathrm{u}_1\times\mathrm{u}_1}[\zeta,\eta]$ and $P_{11} \in \mathbb{R}^{(\mathrm{u}-\mathrm{u}_1)\times(\mathrm{u}-\mathrm{u}_1)}[\zeta,\eta]$. Then it can be verified that $P_{22}(\zeta,\eta) = 0_{\mathrm{u}_1\times\mathrm{u}_1}$, which is a contradiction as $E_{22}(\zeta,\ \eta) > 0$. Hence there does not exist a QDF $Q_{P'}$ which is equivalent to the derivative of an energy function along the behaviour $\mathfrak{B}'$, with $P_1(\zeta,\ \eta) = 0_{\mathrm{u}\times\mathrm{u}}$. This implies that there does not exist a QDF Q_{P_0} which is equivalent to the derivative of an energy function along the behaviour $\mathfrak{B}$ such that Equation (21) holds.

Now assume that colrank$(Q) = \mathrm{u}$, but at least one of the invariant polynomials of Q is not oscillatory. Hence, there exists a unimodular matrix $U \in \mathbb{R}^{\mathrm{y}\times\mathrm{y}}[\xi]$, such that

$$U(\xi)Q(\xi) = \begin{bmatrix} Q_1(\xi) \\ 0_{(\mathrm{y}-\mathrm{u})\times\mathrm{u}} \end{bmatrix},$$

where $Q_1 \in \mathbb{R}^{\mathrm{u}\times\mathrm{u}}[\xi]$ is such that $\det(Q_1(\xi)) \neq 0$. Now consider the behaviour $\mathfrak{B}_1 := \ker(Q_1(\mathrm{d}/\mathrm{d}t))$. Since all invariant polynomials of Q are not oscillatory, $\mathfrak{B}_1$ is not lossless. Consider an energy function Q_E for $\mathfrak{B}$ given by

$$E(\zeta,\eta) = \begin{bmatrix} E_1(\zeta,\eta) & E_2(\zeta,\eta) \\ E_2(\eta,\zeta)^T & E_3(\zeta,\eta) \end{bmatrix}$$

where $E_1 \in \mathbb{R}^{\mathrm{u}\times\mathrm{u}}[\zeta,\eta]$, $E_2 \in \mathbb{R}^{\mathrm{u}\times\mathrm{y}}[\zeta,\eta]$ and $E_3 \in \mathbb{R}^{\mathrm{y}\times\mathrm{y}}[\zeta,\eta]$. Let the QDF Q_{P_0} be such that $Q_{P_0} \stackrel{\mathfrak{B}}{=} (\mathrm{d}/\mathrm{d}t)Q_E$. Let

$$P_0(\zeta,\eta) = \begin{bmatrix} P_1(\zeta,\eta) & P_2(\zeta,\eta) \\ P_2(\eta,\zeta)^T & P_3(\zeta,\eta) \end{bmatrix},$$

where $P_1 \in \mathbb{R}^{\mathrm{u}\times\mathrm{u}}[\zeta,\eta]$, $P_2 \in \mathbb{R}^{\mathrm{u}\times\mathrm{y}}[\zeta,\eta]$ and $P_3 \in \mathbb{R}^{\mathrm{y}\times\mathrm{y}}[\zeta,\eta]$. Since $\mathfrak{B}_1$ is not lossless, there does not exist a positive QDF Q_{E_1} such that $(\mathrm{d}/\mathrm{d}t)Q_{E_1} \stackrel{\mathfrak{B}_1}{=} 0$. Hence $Q_{P_0}(w) \neq 0 \ \forall\ w = \mathrm{col}(u,0) \in \mathfrak{B}$. Thus $P_1(\zeta,\eta) \neq 0_{\mathrm{u}\times\mathrm{u}}$. It follows that there does not exist a QDF Q_{P_0} which is equivalent to the derivative of an energy function along the behaviour $\mathfrak{B}$ such that Equation (21) holds. □

Remark 5: We now state a result that is equivalent to the one stated in Theorem 5 in terms of an observable image representation of the behaviour. With reference to the previous Theorem, if instead of a kernel representation of $\mathfrak{B}$, an observable image representation of the form $y = N(\mathrm{d}/\mathrm{d}t)\ell$, $u = D(\mathrm{d}/\mathrm{d}t)\ell$ ($N \in \mathbb{R}^{\mathrm{y}\times\mathrm{u}}[\xi]$, $D \in \mathbb{R}^{\mathrm{u}\times\mathrm{u}}[\xi]$) is given, then for Equation (20) to hold, a necessary condition is that N should have all its invariant polynomials oscillatory.

Example 2: Consider a behaviour $\mathfrak{B}_m$ whose external variables are the generalised position vector $q \in \mathbb{R}^{\mathrm{q}}$ and generalised force vector $F \in \mathbb{R}^{\mathrm{q}}$ of a second order undamped mechanical system given by the equation

$$M\frac{\mathrm{d}^2 q}{\mathrm{d}t^2} + Kq = F \tag{22}$$

where M, $K \in \mathbb{R}^{\mathrm{q}\times\mathrm{q}}$ denote generalised mass and stiffness matrices respectively. $\mathfrak{B}_m$ can be represented in kernel form as $\mathfrak{B}_m = \ker(R_m(\mathrm{d}/\mathrm{d}t))$, where

$$R_m(\xi) = [\,(M\xi^2 + K) \quad -I_{\mathrm{q}}\,].$$

If q and F are considered as output and input respectively, then it easy to see that the behaviour is controllable and lossless. Also from Theorem 5, it follows that there exists an energy function Q_E, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_E(\mathrm{col}(q,F)) = \left(M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)F\right)^T\left(S\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)q\right),$$

where M, $S \in \mathbb{R}^{\bullet\times\mathrm{q}}[\xi]$. Indeed the power delivered to the second order mechanical system given by Equation (22) is $F^T(\mathrm{d}q/\mathrm{d}t)$, which implies that $M(\xi) = I_{\mathrm{q}}$ and $S(\xi) = \xi I_{\mathrm{q}}$.

Example 3: Consider a behaviour $\mathfrak{B}_e$ whose external variables are the voltage V across and the current I through a one-port lossless electrical network given by the equation

$$d\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)V = n\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)I$$

where n, $d \in \mathbb{R}[\xi]$. It is well known that Z defined by $Z(\xi) := n(\xi)/d(\xi)$ is lossless positive real and hence both n and d are oscillatory. Hence by Theorem 5, there exists an energy function Q_E, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}Q_E(\mathrm{col}(V,I)) = \left(M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)V\right)^T\left(S\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)I\right)$$

where M, $S \in \mathbb{R}^{\bullet}[\xi]$. Indeed the power delivered to the one-port electrical network is equal to VI which implies that $M(\xi) = S(\xi) = 1$.

6. Conclusion

In this article, the main focus has been to give a characterisation for higher order linear lossless systems as opposed to the characterisation for first order systems using state space method (see Weiss et al. (2001), Weiss and Tucsnak (2003) and Malinen et al. (2006)). Using the material covered in this paper, one can easily implement a computer program wherein the input is a higher order description of a scalar oscillatory system and the outputs are its energy functions and the kinetic and potential energy components of a given energy function for the system. Given a multivariable oscillatory system, using the material in this paper, one can implement a program to compute an energy function for the system. Similarly one can also implement a computer program for open lossless systems wherein the inputs are either a kernel or an image description of a controllable system and a given input/output partition of the system and the outputs of the program are the following:

- whether the system is lossless with respect to the given input/output partition or not;
- if the answer to the previous question is yes, then an energy function for the system;

- whether the power delivered to the system is of the form mentioned in Equation (20).

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Appendix

Proof of Proposition 2: Define $n(\xi) := r'(\xi^2)$ and $d(\xi) := \xi r''(\xi^2)$ and

$$S(\xi) := \frac{n(\xi) - d(\xi)}{n(\xi) + d(\xi)}.$$

We have

$$\begin{aligned} 1 - S(\xi)S(-\xi) &= 2\left\{\frac{n(\xi)d(-\xi) + n(-\xi)d(\xi)}{(n(\xi) + d(\xi))(n(-\xi) + d(-\xi))}\right\} \\ 1 - \left|S(j\omega)\right|^2 &= 0 \quad \forall \omega \in \mathbb{R}. \end{aligned}$$

Now since r is Hurwitz, S is analytic in the right half plane. Since $|S(j\omega)| = 1\ \forall \omega \in \mathbb{R}$, by maximum modulus theorem, it follows that $|S(\lambda)| < 1$ for $\mathrm{Re}(\lambda) > 0$, i.e.,

$$|n(\lambda) - d(\lambda)| < |n(\lambda) + d(\lambda)| \quad \text{for } \mathrm{Re}(\lambda) > 0. \tag{23}$$

Assume that n has a zero λ_1 in the open right half plane. Then it follows from inequality (23), that $|d(\lambda_1)| < |d(\lambda_1)|$, which is not true. Hence by contradiction, it follows that both n and d have no roots in the open right half plane. Since both n and d are even, this implies that they have no roots in the open left half plane as well. From inequality (23), it follows that

$$|n(\lambda) - d(\lambda)|^2 < |n(\lambda) + d(\lambda)|^2 \quad \text{for } \mathrm{Re}(\lambda) > 0.$$

Consequently, for $\mathrm{Re}(\lambda) > 0$, we have

$$n(\lambda)d(\bar{\lambda}) + n(\bar{\lambda})d(\lambda) > 0. \tag{24}$$

Dividing the inequality (24) by $|d(\lambda)|^2$, we get

$$\mathrm{Re}\left(\frac{n(\lambda)}{d(\lambda)}\right) > 0 \quad \text{for } \mathrm{Re}(\lambda) > 0.$$

Define $Z(\xi) := n(\xi)/d(\xi)$. We assume that Z has a pole of order n at a point λ_0 on the imaginary axis. Close to λ_0, Z has a Laurent series expansion of the form

$$Z(\xi) = \sum_{k=0}^{n} \frac{a_k}{(\xi - \lambda_0)^k} + \sum_{k=0}^{\infty} b_k(\xi - \lambda_0)^k. \tag{25}$$

In polar form, we may write $a_n = K_1 e^{j\theta_1}$ and $\xi - \lambda_0 = K_2 e^{j\theta_2}$. Thus the real part of the dominant term of the expansion (25) is

$$\mathrm{Re}\left\{\frac{a_n}{(\xi - \lambda_0)^n}\right\} = \frac{K_1}{K_2^n}\cos(\theta_1 - n\theta_2)$$

This is required to be positive for $-\pi/2 \le \theta_2 \le \pi/2$. This is possible only if $n = 1$ and $\theta_1 = 0$, i.e., a_n is real and positive and multiplicity of the pole is one. This implies that d is oscillatory. A similar argument for $Y(\xi)=(Z(\xi))^{-1}$ shows that n is also oscillatory.

Alternate proof for the *Only if* part of Theorem 1: Assume that $\mathfrak{B}$ has the kernel representation $r(\mathrm{d}/\mathrm{d}t)w=0$. Let $r(\xi)=r_e(\xi)p(\xi)$ where r_e is the maximal even polynomial factor of r. If $p(\xi)$ is not a constant and $p(\xi)\neq a\xi$, where $a\in\mathbb{R}$, then it has at least one root, say $\lambda\in\mathbb{R}\backslash\{0\}$ or two roots, say λ, $\bar{\lambda}\in\mathbb{C}\backslash\mathbb{R}$. From the proof of Proposition 1, depending on whether $p(\xi)$ is divisible by ξ or not, any two-variable polynomial inducing conserved QDF over $\mathfrak{B}$ can either have the form

$$\phi_1(\zeta,\eta)=\frac{r(\zeta)p_1(\eta)f_e(\eta)+r(\eta)p_1(\zeta)f_e(\zeta)}{\zeta+\eta},$$

where $p_1(\xi)=p(\xi)/\xi$ and f_e is an even function, or the form

$$\phi_2(\zeta,\eta)=\frac{r(\zeta)p(\eta)f_o(\eta)+r(\eta)p(\zeta)f_o(\zeta)}{\zeta+\eta},$$

where $f_o(\xi)$ is an odd function. It can be seen that both ϕ_1 and ϕ_2 are divisible by $(\zeta-\lambda)(\eta-\lambda)$ if $\lambda\in\mathbb{R}$ and divisible by $(\zeta-\lambda)(\zeta-\bar{\lambda})(\eta-\lambda)(\eta-\bar{\lambda})$ if $\lambda\in\mathbb{C}\backslash\mathbb{R}$. Hence along the trajectory $w(t)=e^{\lambda t}+e^{\bar{\lambda}t}\in\mathfrak{B}$, the QDFs induced by ϕ_1 and ϕ_2 are equal to zero. This implies that $\mathfrak{B}$ does not have a positive conserved QDF. This eliminates all scalar systems except those for which the kernel representation is $r(\mathrm{d}/\mathrm{d}t)w=0$, such that either $r(\xi)$ is even, or $r(\xi)=\xi r_e(\xi)$, where $r_e(\xi)$ is an even function. We now consider two cases.

- **Case 1:** r is even. Define $r'(\xi^2):=r(\xi)$. In this case, from the proof of proposition 1, any conserved quantity for $\mathfrak{B}$ has its associated two-variable polynomial of the form

$$\Phi(\zeta,\eta)=\frac{\eta r'(\zeta^2)r''(\eta^2)+\zeta r'(\eta^2)r''(\zeta^2)}{\zeta+\eta},$$

where r'' has degree less than that of r'. Assume that $Q_\Phi>0$. Define $r_1(\xi):=r'(\xi^2)+\xi r''(\xi^2)$ and $\mathfrak{B}':=\ker(r_1(\mathrm{d}/\mathrm{d}t))$. Let $\dot{\Phi}(\zeta,\eta)$ denote the two-variable polynomial that induces the derivative of Q_Φ. Then

$$\dot{\Phi}(\zeta,\eta)=(\zeta+\eta)\Phi(\zeta,\eta)=r_1(\zeta)r''(\eta^2)+r_1(\eta)r''(\zeta^2)-2\zeta\eta r''(\zeta^2)r''(\eta^2)$$

Hence $Q_{\dot{\Phi}}\overset{\mathfrak{B}'}{=}Q_{\Phi_1}$, where

$$\Phi_1(\zeta,\eta)=-2\zeta\eta r''(\zeta^2)r''(\eta^2).$$

This implies that $Q_{\dot{\Phi}}\overset{\mathfrak{B}'}{<}0$. Hence Q_Φ is a Lyapunov function for $\mathfrak{B}'$, which implies that $\mathfrak{B}'$ is asymptotically stable or that r_1 is Hurwitz. From Proposition 2, it follows that r is oscillatory.

- **Case 2:** r is odd. The proof for this case is very similar to the one for Case 1.