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New Algorithms for Polynomial J-Spectral Factorization*

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Abstract. In this paper new algorithms are developed for J-spectral factorization of polynomial matrices. These algorithms are based on the calculus of two-variable polynomial matrices and associated quadratic differential forms, and share the common feature that the problem is lifted from the original one-variable polynomial context to a two-variable polynomial context. The problem of polynomial J-spectral factorization is thus reduced to a problem of factoring a constant matrix obtained from the coefficient matrices of the polynomial matrix to be factored. In the second part of the paper, we specifically address the problem of computing polynomial J-spectral factors in the context of H_∞ control. For this, we propose an algorithm that uses the notion of a Pick matrix associated with a given two-variable polynomial matrix.

Key words. Polynomial J-spectral factorization, Two-variable polynomial matrix, Quadratic differential form, Dissipativity, Pick matrix.

1. Introduction

In this paper new algorithms are developed for J-spectral factorization of polynomial matrices, using the notion of quadratic differential form, and the calculus of two-variable polynomial matrices as recently developed in [19]. The problem of polynomial J-spectral factorization arises in different areas of systems and control. Perhaps the best known is the special case in which the polynomial matrix to be factored is positive semidefinite on the imaginary axis (equivalently, the signature matrix J is equal to the identity matrix). This problem arises in Wiener filtering, and in the polynomial and behavioral approaches to LQG theory (see, for instance, [4], [7], [1], and [3]). The general case in which the polynomial matrix to be factored is indefinite on the imaginary axis arises most notably in the polynomial and behavioral approaches to the H_∞ control problem (see, for instance, [8], [10], [11], [16], and [2]).

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As also noted in [9], only very few practical algorithms for the J-spectral factorization of polynomial matrices can be found in the literature. An overview of existing algorithms was given in [9]. In the present paper some fundamentally new algorithms are developed that are based on the calculus of two-variable polynomial matrices and associated quadratic differential forms that was developed in [19]. Our algorithms share the common feature that the problem is lifted from the original one-variable polynomial context to a two-variable polynomial context. In the two-variable polynomial context, factorizations can essentially be performed by doing factorizations of *constant* real symmetric matrices.

The problem of polynomial J-spectral factorization is stated as follows. Given a para-hermitian real $q \times q$ polynomial matrix $Z(\xi)$, i.e., a real polynomial matrix with the property that $Z^T(-\xi) = Z(\xi)$, together with two nonnegative integers q_+ and q_- such that $q_+ + q_- = q$, the problem is to compute a square polynomial matrix F such that $Z(\xi) = F^T(-\xi)J_{q_+,q_-}F(\xi)$. Here J_{q_+,q_-} denotes the signature matrix

$$J_{q_+,q_-} := \begin{pmatrix} I_{q_+} & 0 \\ 0 & -I_{q_-} \end{pmatrix}.$$

Depending on the problem setting at hand, one often wants the spectral factor F to satisfy certain additional properties. Often one requires F to be Hurwitz, or anti-Hurwitz. Especially in H_∞ applications, the para-hermitian matrix Z is in general a priori given to be of the form $Z(\xi) = M^T(-\xi)J'M(\xi)$, where J' is some signature matrix. The spectral factor F to be computed is then often required to satisfy the additional property that MF^{-1} is a matrix of proper rational functions. This property is closely connected to the property that F is a canonical factor (see [9] and [2]). Another important property that F is often required to satisfy is that the rational matrix MF^{-1} is J'-lossless (see [10]).

This paper is organized as follows. Section 2 contains the basic background material on quadratic differential forms and two-variable polynomial matrices. Section 3 connects the problem of polynomial J-spectral factorization with the problem of factoring a constant matrix that can be obtained from the coefficient matrices of the polynomial matrix Z to be factored. In Section 4 we propose our first J-spectral factorization algorithm. The algorithm is based on finding a particular solution of an algebraic Riccati equation of suitably high dimension. The coefficients of this algebraic Riccati equation are immediately obtained from the coefficients of the polynomial matrix to be factored, *without any intermediate state space realization step*. In Section 5 we restrict attention to the positive semidefinite case (the case that $J = I$). Two algorithms are proposed in this section. The first one is based on solving a linear matrix inequality, with coefficients again immediately given by the coefficient matrices of the polynomial matrix to be factored. The second algorithm is based on solving an algebraic Riccati equation, this time of a priori known dimension (in contrast with the algorithm in Section 4 for the indefinite factorization problem).

In Section 6 a completely different type of algorithm is developed. Here we specifically address the problem of computing J-spectral factors in the H_∞

context, so Hurwitz J-spectral factors F such that MF^{-1} is a J'-lossless, proper rational matrix. In this section we first give a necessary and sufficient condition for the existence of such J-spectral factors. Next, we propose an algorithm that uses the notion of a Pick matrix associated with a given two-variable polynomial matrix.

The paper has been written in the hope of making its content accessible to potential users of the algorithms. Therefore, the proof of the main result of Section 6 is given in a separate section, Section 9. This proof is strongly based on results from Sections 7 and 8 on dissipativity and on duality of quadratic differential forms. Our algorithms are put in the form of step-by-step procedures, easily implementable in Matlab or Mathematica, and with concrete examples.

The following notation is used in this paper. $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural, of real, and of complex numbers, respectively. If $\lambda \in \mathbb{C}$, then $\bar{\lambda}$ denotes its complex conjugate. $\mathbb{C}^-$ and $\mathbb{C}^+$ denote the open left-half, respectively the open right-half, of the complex plane. $\mathbb{R}^n$ and $\mathbb{C}^n$ denote the n-dimensional Euclidean spaces over $\mathbb{R}$, respectively, $\mathbb{C}$. $\mathbb{R}^{p\times q}$ (resp. $\mathbb{C}^{p\times q}$) denotes the set of real (resp. complex) $p \times q$ matrices. The set of real matrices with a finite number q of rows (resp. columns) and an infinite number of columns (resp. rows) is denoted as $\mathbb{R}^{q\times\infty}$ (resp. $\mathbb{R}^{\infty\times q}$). The set of real matrices with an infinite number of rows and of columns is denoted by $\mathbb{R}^{\infty\times\infty}$. For given matrices A and B with the same number of columns, $\mathrm{col}(A, B)$ denotes the matrix obtained by stacking A over B. Likewise, given two column vectors x and y, the column vector obtained by stacking x over y is denoted by $\mathrm{col}(x, y)$. For a given matrix $A \in \mathbb{C}^{q\times q}$, A^* denotes its complex conjugate transpose. A is called hermitian if $A = A^*$. If A is a hermitian matrix, then we define its signature as the ordered triple of integers $\mathrm{sign}(A) = (n_-(A), n_0(A), n_+(A))$, where $n_-(A)$ denotes the number of negative eigenvalues of A (counting multiplicities), $n_0(A)$ denotes the multiplicity of zero as an eigenvalue of A, and $n_+(A)$ denotes the number of positive eigenvalues of A (counting multiplicities). Of course, $n_-(A) + n_+(A) = \mathrm{rank}(A)$. $\mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ denotes the set of infinitely differentiable $\mathbb{R}^q$-valued functions. The set of $p \times q$ polynomial matrices with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}^{p\times q}[\xi]$. If $Z \in \mathbb{R}^{p\times q}[\xi]$, then $Z(\xi) = \sum_{i=1}^{M} Z_i \xi^i$, with $Z_M \neq 0$. The integer M is called the degree of Z. $Z \in \mathbb{R}^{q\times q}[\xi]$ is called para-hermitian if $Z^T(-\xi) = Z(\xi)$. The set of two-variable $p \times q$ polynomial matrices with real coefficients in the indeterminates ζ and η is denoted as $\mathbb{R}^{p\times q}[\zeta, \eta]$.

2. Quadratic Differential Forms

Many control problems, for example, in linear quadratic optimal control and H_∞-control, require the solution of an optimization problem in which a quadratic functional of the system variables is to be minimized. It has been shown in [19] that such quadratic functionals can be effectively represented by *quadratic differential forms* (QDFs in the following). This section is devoted to an exposition of the notational conventions for QDFs and to a discussion of the results which are relevant to the problem of J-spectral factorization.

2.1. Basics

Let $\Phi \in \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$ be a real polynomial matrix in the indeterminates ζ and η, i.e., an expression of the form

$$\Phi(\zeta, \eta) = \sum_{k,j} \Phi_{k,j} \zeta^k \eta^j, \tag{2.1}$$

where $\Phi_{k,j} \in \mathbb{R}^{q_1 \times q_2}$, $\forall k, j$. The sum in (2.1) is a finite one, and $k,\ j \in \mathbb{N}$. We call Φ *symmetric* if $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^T$. In this paper we restrict attention to the symmetric elements in $\mathbb{R}^{q \times q}[\zeta, \eta]$, and we denote this subset by $\mathbb{R}_s^{q \times q}[\zeta, \eta]$.

Each $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$ induces a QDF, i.e., a map $Q_\Phi : \mathscr{C}^\infty(\mathbb{R}, \mathbb{R}^q) \to \mathscr{C}^\infty(\mathbb{R}, \mathbb{R})$ defined by

$$Q_\Phi(w) := \sum_{k,j} \left(\frac{d^k w}{dt^k}\right)^T \Phi_{k,j} \left(\frac{d^j w}{dt^j}\right).$$

With every $\Phi \in \mathbb{R}^{q \times q}[\zeta, \eta]$ there is associated the infinite matrix with a finite number of nonzero elements

$$\tilde{\Phi} := \begin{pmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots & \Phi_{0,N} & \cdots \\ \Phi_{1,0} & \Phi_{1,1} & \cdots & \Phi_{1,N} & \cdots \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \Phi_{N,0} & \Phi_{N,1} & \cdots & \Phi_{N,N} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

in the sense that $\Phi(\zeta, \eta)$ is equal to the product

$$\Phi(\zeta, \eta) = (I_q \quad \zeta I_q \quad \cdots \quad \zeta^N I_q \quad \cdots) \tilde{\Phi} \begin{pmatrix} I_q \\ \eta I_q \\ \vdots \\ \eta^N I_q \\ \vdots \end{pmatrix}.$$

$\tilde{\Phi}$ is called the *coefficient matrix* of Φ. Note that $\Phi(\zeta, \eta)$ is symmetric if and only if $\tilde{\Phi}$ is symmetric. This correspondence between $\mathbb{R}_s^{q \times q}[\zeta, \eta]$ and the set of infinite real symmetric matrices with a finite number of nonzero elements is bijective.

Factorizations of the coefficient matrix give rise to factorizations of the corresponding two-variable polynomial matrix. The relevant facts are summarized in the following result (see Section 3 of [19]).

Proposition 2.1. *Let $\Phi \in \mathbb{R}_s^{q \times q}[\zeta, \eta]$, and let $\tilde{\Phi}$ be its coefficient matrix. Let p_- and p_+ be two nonnegative integers such that $p := p_- + p_+ = \operatorname{rank}(\tilde{\Phi})$. Then the following statements are equivalent:*

(i) $n_-(\tilde{\Phi}) = p_-$ *and* $n_+(\tilde{\Phi}) = p_+$;

(ii) *there exists a matrix* $\tilde{F} \in \mathbb{R}^{p\times\infty}$ *with finitely many elements unequal to zero such that* $\tilde{\Phi} = \tilde{F}^T J_{p_+,p_-}\tilde{F}$;

(iii) *there exist a polynomial matrix* $F \in \mathbb{R}^{p\times q}[\xi]$ *such that* $\Phi(\zeta,\eta) = F^T(\zeta)J_{p_+,p_-}F(\eta)$.

If these conditions hold, then a factor F as in (iii) *can be obtained as follows: factor* $\tilde{\Phi}$ *as in* (ii) *and define* $F(\xi) := \tilde{F}\ \mathrm{col}(I, \xi I, \xi^2 I, \ldots)$.

The correspondence between QDFs and two-variable polynomial matrices allows us to develop a calculus that has applications in stability theory, optimal control, and H_∞-control (see [19] and [16]). We do not illustrate all the features of this calculus here, but we now introduce two concepts that are used extensively in this paper. One of these is the map $\partial : \mathbb{R}_s^{q\times q}[\zeta,\eta] \to \mathbb{R}^{q\times q}[\xi]$, that associates a one-variable polynomial matrix with a two-variable one. Given $\Phi \in \mathbb{R}_s^{q\times q}[\zeta,\eta]$, we define

$$\partial\Phi(\xi) := \Phi(-\xi,\xi).$$

Observe that $\partial\Phi$ is para-hermitian. Another notion that is used extensively in what follows is that of a *derivative* of a QDF. A QDF Q_Φ is called the derivative of Q_Ψ if $(d/dt)Q_\Psi = Q_\Phi$. Here $(d/dt)Q_\Psi$ is defined as $((d/dt)Q_\Psi)(\ell) := (d/dt)Q_\Psi(\ell)$. In terms of the underlying two-variable polynomial matrices, this relationship can be expressed as follows: $(d/dt)Q_\Psi = Q_\Phi$ if and only if, for the corresponding two-variable polynomial matrices, $(\zeta+\eta)\Psi(\zeta,\eta) = \Phi(\zeta,\eta)$ holds. In terms of the underlying coefficient matrices $\tilde{\Phi}$ and $\tilde{\Psi}$ this relationship is equivalent to

$$\tilde{\Phi} = \sigma_{\mathrm{R}}(\tilde{\Psi}) + \sigma_{\mathrm{D}}(\tilde{\Psi}),$$

where the *right-shift*, respectively *downward-shift, operators* $\sigma_{\mathrm{R}}\colon \mathbb{R}^{\infty\times\infty} \to \mathbb{R}^{\infty\times\infty}$ and $\sigma_{\mathrm{D}}\colon \mathbb{R}^{\infty\times\infty} \to \mathbb{R}^{\infty\times\infty}$ are defined as

$$\sigma_{\mathrm{R}}(\tilde{\Psi}) := \begin{pmatrix} 0_{\infty\times q} & \tilde{\Psi} \end{pmatrix} \quad \text{and} \quad \sigma_{\mathrm{D}}(\tilde{\Psi}) := \begin{pmatrix} 0_{q\times\infty} \\ \tilde{\Psi} \end{pmatrix}.$$

2.2. Pick Matrices

The notion of a Pick matrix plays an important role in system and in circuit theory; for example, it has connections with metric interpolation problems [20] and with model matching in the H_∞-norm [5]. Recently, it has been connected with QDFs and the notion of half-line positivity of a QDF (see Section 9 of [19]). In Section 6 of the present paper we use the Pick matrix associated with a two-variable polynomial matrix to devise an algorithm to perform J-spectral factorization.

The Pick matrix associated with a two-variable polynomial matrix Φ is most easily introduced in the case that $\partial\Phi$ is semisimple. We briefly review the notion of semisimplicity of a polynomial matrix here. Let $F \in \mathbb{R}^{q\times q}[\xi]$ be nonsingular, i.e., $\det(F) \neq 0$. We call F *semisimple* if for all $\lambda \in \mathbb{C}$ the dimension of $\ker(F(\lambda))$ is equal to the multiplicity of λ as a root of $\det(F)$. If $\det(F)$ has only simple roots, then F is certainly semisimple. We introduce the notion of a Pick matrix associated with a two-variable polynomial matrix in the semisimple case only. Let $\Phi(\zeta,\eta) \in \mathbb{R}_s^{q\times q}[\zeta,\eta]$, and assume that $\det(\partial\Phi)$ has no roots on the imaginary axis. Since $\partial\Phi$ is para-hermitian, $\deg(\det\partial(\Phi))$ is even, say $2n$, and its roots can be grouped into two disjoint subsets, $\{\lambda_1,\dots,\lambda_k\} \subset \mathbb{C}^-$ and $\{-\lambda_1,\dots,-\lambda_k\} \subset \mathbb{C}^+$. ($\lambda_i \neq \lambda_j$ for $i \neq j$.) Denote by n_i the multiplicity of λ_i as a root of $\det(\partial\Phi)$. Let $V_i \in \mathbb{C}^{q\times n_i}$ be full column rank matrices such that $\ker(\partial\Phi)(\lambda_i) = \mathrm{im}(V_i)$ $(i = 1,2,\dots,k)$. Define $T_{\Phi,i,j} \in \mathbb{C}^{n_i\times n_j}$ by

$$T_{\Phi,i,j} := \frac{1}{\bar{\lambda}_i + \lambda_j} V_i^* \Phi(\bar{\lambda}_i,\lambda_j) V_j \tag{2.2}$$

and define the Pick matrix $T_\Phi \in \mathbb{C}^{n\times n}$ associated with Φ by

$$T_\Phi := (T_{\Phi,i,j}).$$

Note that T_Φ is a hermitian matrix. Also note that the matrices V_i are not uniquely defined, and therefore talking about "the" Pick matrix associated with Φ is not entirely correct: depending on the particular choice of V_i's whose columns span $\ker(\partial\Phi(\lambda_i))$, different matrices are obtained in (2.2). However, if T_Φ and T'_Φ correspond to different choices of the V_i's, there exists a nonsingular $n \times n$ matrix S such that $S^T T_\Phi S = T'_\Phi$. Therefore, the different matrices defined by (2.2) are congruent.

3. A Matrix Signature Condition for *J*-Spectral Factorization

It is well known that a nonsingular para-hermitian $q \times q$ polynomial matrix Z has a J-spectral factorization for some signature matrix J if and only if the number $n_+(Z(i\omega))$ of positive eigenvalues of $Z(i\omega)$ is constant over all $\omega \in \mathbb{R}$ for which $i\omega$ is not a root of $\det(Z)$ (see Theorem 5.1 of [13]). In particular, for given nonnegative integers q_+ and q_- such that $q_+ + q_- = q$ such factorization exists with $J = J_{q_+,q_-}$ if and only if $n_+(Z(i\omega)) = q_+$ and $n_-(Z(i\omega)) = q_-$ for all $\omega \in \mathbb{R}$ for which $i\omega$ is not a root of $\det(Z)$.

In this section we show that the existence of a J_{q_+,q_-}-spectral factor of Z is equivalent to a condition *on the coefficient matrix* $\tilde{Z} = (Z_0\ Z_1\ \cdots\ Z_M)$ of $Z(\xi)$. We also explain how a J_{q_+,q_-}-spectral factorization can be obtained by performing a factorization of a *constant* matrix obtained from this coefficient matrix. This result will be achieved by lifting the problem of J-spectral factorization to a two-variable polynomial context, i.e., by associating with the to-be-factored *one*-variable polynomial matrix $Z(\xi)$ a particular *two*-variable polynomial matrix. Lifting the problem to a two-variable polynomial matrix context allows us to formulate the problem of J-spectral factorization in terms of QDFs and the related concepts.

Let $Z(\xi)$ be a para-hermitian polynomial matrix, say of degree M. Write $Z(\xi) = Z_0 + Z_1\xi + Z_2\xi^2 + \cdots + Z_M\xi^M$, with $Z_M \neq 0$. In the following, we associate with $Z(\xi)$ a symmetric, two-variable polynomial matrix $\Phi(\zeta,\eta)$ with the property that

$$\partial\Phi = Z. \tag{3.1}$$

For a given Z there are always infinitely many Φ's such that this property holds. For example, whenever Φ satisfies (3.1) then $\Phi'(\zeta,\eta) := \Phi(\zeta,\eta) + (\zeta+\eta)\Gamma(\zeta,\eta)$ also satisfies (3.1), for any choice of Γ. In fact, in the following lemma we characterize all Φ's that, for a given Z, satisfy (3.1):

Lemma 3.1. *Let $\Phi(\zeta,\eta) = \sum_{k,j} \Phi_{k,j}\zeta^k\eta^j$ be a symmetric two-variable polynomial matrix. Then $\partial\Phi = Z$ iff $\Phi_{0,k} - \Phi_{1,k-1} + \Phi_{2,k-2} - \cdots + (-1)^k\Phi_{k,0} = Z_k$ for all $k = 0,1,\ldots,M$.*

In terms of the coefficient matrix $\tilde{\Phi}$, this requires that the $M+1$ antidiagonals, with an appropriate sign-pattern, add up to the coefficients of $Z(\xi)$. The proof of this lemma follows by inspection.

We now give two examples of a possible choice of Φ such that (3.1) holds. As a first example, we consider

$$\Phi(\zeta,\eta) := \tfrac{1}{2}(Z^T(\zeta) + Z(\eta)).$$

Since $\Phi(\eta,\zeta)^T = \frac{1}{2}(Z(\eta) + Z^T(\zeta)) = \Phi(\zeta,\eta)$, $\Phi(\zeta,\eta)$ is symmetric. Using $Z^T(-\xi) = Z(\xi)$, this Φ indeed satisfies (3.1). Also from the assumption that Z is para-hermitian, it follows that $Z_k^T = (-1)^k Z_k$ for all $k = 0,1,2,\ldots,M$. The coefficient matrix $\tilde{\Phi}$ of Φ is equal to

$$\tilde{\Phi} = \frac{1}{2}\begin{pmatrix} 2Z_0 & Z_1 & Z_2 & \cdots & Z_M & 0 & \cdots \\ Z_1^T & 0 & 0 & \cdots & 0 & 0 & \cdots \\ Z_2^T & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ Z_M^T & 0 & 0 & \cdots & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.2}$$

Note that the degree of Φ, i.e., the highest power of ζ and η occurring in $\Phi(\zeta,\eta)$ is equal to M.

We now give a second example of a possible choice of Φ. It is only valid for the case that M is even. Assume this to be the case. Then we define Φ in terms of its coefficient matrix as follows:

$$\tilde{\Phi} = \frac{1}{2}\begin{pmatrix} 2Z_0 & Z_1 & Z_2 & \cdots & Z_{M/2-1} & Z_{M/2} & 0 & \cdots \\ Z_1^T & 0 & 0 & \cdots & 0 & -Z_{M/2+1} & 0 & \cdots \\ Z_2^T & 0 & 0 & \cdots & 0 & Z_{M/2+2} & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \\ Z_{M/2-1}^T & 0 & 0 & \cdots & 0 & (-1)^{M/2-1}Z_{M-1} & 0 & \cdots \\ Z_{M/2}^T & -Z_{M/2+1}^T & Z_{M/2+2}^T & \cdots & (-1)^{M/2-1}Z_{M-1}^T & 2(-1)^{M/2}Z_M & 0 & \cdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}. \tag{3.3}$$

Note that the two-variable polynomial matrix Φ defined in this way has degree $M/2$. A related Φ, with degree $(M+1)/2$, can be defined for the case that M is odd. We omit the details here.

We now state the main result of this section.

Proposition 3.1. *Let Z be a $q \times q$ para-hermitian polynomial matrix with* $\det(Z) \neq 0$. *Let Φ be any symmetric two-variable polynomial matrix such that $\partial\Phi = Z$. Let q_+, q_- be two nonnegative integers such that $q_- + q_+ = q$. Then the following three statements are equivalent*:

(i) *there exists a $q \times q$ polynomial matrix F such that*

$$Z(\xi) = F^T(-\xi) J_{q_+,q_-} F(\xi); \tag{3.4}$$

(ii) $\operatorname{sign}(Z(i\omega)) = (q_-, 0, q_+)$ *for all $\omega \in \mathbb{R}$ such that $i\omega$ is not a root of* $\det(Z)$;

(iii) *there exists a symmetric matrix $\tilde{\Psi} \in \mathbb{R}^{\infty\times\infty}$, with finitely many elements unequal to zero, such that*

$$\begin{aligned} n_-(\tilde{\Phi} - \sigma_{\mathrm{R}}(\tilde{\Psi}) - \sigma_{\mathrm{D}}(\tilde{\Psi})) &= q_-, \\ n_+(\tilde{\Phi} - \sigma_{\mathrm{R}}(\tilde{\Psi}) - \sigma_{\mathrm{D}}(\tilde{\Psi})) &= q_+. \end{aligned} \tag{3.5}$$

Assume that any of these conditions hold. Then an F such that (3.4) *holds can be computed as follows: find $\tilde{\Psi} \in \mathbb{R}^{\infty\times\infty}$, with finitely many elements unequal to zero, such that* (3.5) *holds, and factorize*

$$\tilde{\Phi} - \sigma_{\mathrm{R}}(\tilde{\Psi}) - \sigma_{\mathrm{D}}(\tilde{\Psi}) = \tilde{F}^T J_{q_+,q_-} \tilde{F} \tag{3.6}$$

with $\tilde{F} \in \mathbb{R}^{q\times\infty}$ having finitely many elements unequal to zero. Then the polynomial matrix F defined by $F(\xi) := \tilde{F}\ \mathrm{col}(I, \xi I, \xi^2 I, \ldots)$ satisfies (3.4).

Proof. We first prove the equivalence of (i), (ii), and (iii) by running the circle (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i). To prove (i) $\Rightarrow$ (ii), let $i\omega$ be such that $\det(Z(i\omega)) \neq 0$. Note that from (3.4) it follows that $\det(F(i\omega)) \neq 0$. This, together with (3.4), immediately yields (ii). We now prove (ii) $\Rightarrow$ (iii). From the fact that $\partial\Phi(i\omega) = \Phi(-i\omega, i\omega) = Z(i\omega)$, $\forall \omega \in \mathbb{R}$, and the fact that $\operatorname{sign}(Z(i\omega)) = (q_-, 0, q_+)$ for all $\omega \in \mathbb{R}$ such that $\det(Z(i\omega)) \neq 0$, there follows the existence (see Theorem 5.1 of [13]) of a $q \times q$ Hurwitz polynomial matrix $F(\xi)$ such that

$$\Phi(-\xi, \xi) = F^T(-\xi) J_{q_+,q_-} F(\xi).$$

Define now

$$\Psi(\zeta, \eta) := \frac{1}{\zeta + \eta} (\Phi(\zeta, \eta) - F^T(\zeta) J_{q_+,q_-} F(\eta)). \tag{3.7}$$

Observe that since $\Phi(\zeta, \eta)$ and $F^T(\zeta) J_{q_+,q_-} F(\eta)$ are symmetric polynomial matrices, it follows that $\Psi(\zeta, \eta)$ is also symmetric. Therefore, $\tilde{\Psi}$ is symmetric. We now show that $\tilde{\Phi} - \sigma_{\mathrm{R}}(\tilde{\Psi}) - \sigma_{\mathrm{D}}(\tilde{\Psi})$ has q_- negative eigenvalues and q_+ positive eigenvalues. Denote with N the degree of the polynomial matrix F, and denote with L the highest power of ζ and η in $\Phi(\zeta, \eta)$. From (3.7) it follows that $\Psi(\zeta,\eta) = \sum_{k,l=0}^{\max\{L-1,N-1\}} \Psi_{k,l}\zeta^k\eta^l$, and that the coefficient matrix of $\Phi(\zeta,\eta) - (\zeta+\eta)\Psi(\zeta,\eta)$ satisfies (3.6), where $\tilde{F} = (F_0 \; F_1 \; \cdots \; F_N \; 0 \; \cdots)$. Observe now that $\operatorname{rank}(\tilde{\Phi} - \sigma_{\mathrm{R}}(\tilde{\Psi}) - \sigma_{\mathrm{D}}(\tilde{\Psi})) \geq \operatorname{rank}\Phi_{0,0}$. Also note that $\operatorname{rank}(\Phi_{0,0}) = \operatorname{rank}\Phi(0,0) = \operatorname{rank}(Z_0) = \operatorname{rank}(Z(0)) = q_+ + q_- = q$. Then $\operatorname{rank}(\tilde{\Phi} - \sigma_{\mathrm{R}}(\tilde{\Psi}) - \sigma_{\mathrm{D}}(\tilde{\Psi})) \geq q$ follows. From (3.6) it follows that $n_+(\tilde{\Phi} - \sigma_{\mathrm{R}}(\tilde{\Psi}) - \sigma_{\mathrm{D}}(\tilde{\Psi})) \leq q_+$, and that $n_-(\tilde{\Phi} - \sigma_{\mathrm{R}}(\tilde{\Psi}) - \sigma_{\mathrm{D}}(\tilde{\Psi})) \leq q_-$. Therefore,

$$\begin{aligned} q &\leq \operatorname{rank}(\tilde{\Phi} - \sigma_{\mathrm{R}}(\tilde{\Psi}) - \sigma_{\mathrm{D}}(\tilde{\Psi})) \\ &= n_+(\tilde{\Phi} - \sigma_{\mathrm{R}}(\tilde{\Psi}) - \sigma_{\mathrm{D}}(\tilde{\Psi})) + n_-(\tilde{\Phi} - \sigma_{\mathrm{R}}(\tilde{\Psi}) - \sigma_{\mathrm{D}}(\tilde{\Psi})) \\ &\leq q_+ + q_- = q. \end{aligned}$$

This implies that $\operatorname{rank}(\tilde{\Phi} - \sigma_{\mathrm{R}}(\tilde{\Psi}) - \sigma_{\mathrm{D}}(\tilde{\Psi})) = q$ and that the number of negative (resp. positive) eigenvalues of $\tilde{\Phi} - \sigma_{\mathrm{R}}(\tilde{\Psi}) - \sigma_{\mathrm{D}}(\tilde{\Psi})$ is q_- (resp. q_+). This concludes the proof of (ii) $\Rightarrow$ (iii). To prove (iii) $\Rightarrow$ (i), observe that by (3.5) the symmetric matrix $\tilde{\Phi} - \sigma_{\mathrm{R}}(\tilde{\Psi}) - \sigma_{\mathrm{D}}(\tilde{\Psi})$ has q_+ positive and q_- negative eigenvalues. Hence there exists a matrix $\tilde{F}$ such that $\tilde{\Phi} - \sigma_{\mathrm{R}}(\tilde{\Psi}) - \sigma_{\mathrm{D}}(\tilde{\Psi}) = \tilde{F}^T J_{q_+,q_-}\tilde{F}$. Multiplying this equality on the left by $(I_q \; \zeta I_q \; \cdots \; \zeta^k I_q \; \cdots)$ and on the right by $(I_q \; \eta I_q \; \cdots \; \eta^k I_q \; \cdots)^T$ yields $\Phi(\zeta,\eta) - (\zeta+\eta)\Psi(\zeta,\eta) = F^T(\zeta)J_{q_+,q_-}F(\eta)$. Substituting $-\xi$ for ζ and ξ for η yields (3.4). This concludes the proof of the equivalence of statements (i), (ii), and (iii). The second part of the claim of the proposition follows immediately. ■

Proposition 3.1 brings the problem of finding a J_{q_+,q_-}-spectral factor for Z down to finding a matrix $\tilde{\Psi}$ with a finite number of nonzero elements such that after substracting its right-shifted version $\sigma_{\mathrm{R}}(\tilde{\Psi})$ and its downward-shifted version $\sigma_{\mathrm{D}}(\tilde{\Psi})$ from the coefficient matrix $\tilde{\Phi}$ (for example, given in terms of the coefficients Z_k by (3.2) or (3.3)), the resulting matrix has q_+ positive eigenvalues and q_- negative eigenvalues. It is important to note that the search for such a matrix $\tilde{\Psi}$ is complicated by two facts. Observe first that while the conditions (3.5) involve infinite matrices, the largest power of ζ and η in $\Psi(\zeta,\eta)$ is finite. Define N to be the *effective size* of $\tilde{\Psi}$ if $N := \min\{N' \mid \text{such that } \Psi_{k,l} = 0, \forall k,l > N'\}$. The first problem that arises is that if $q_- \neq 0$, then it is unknown what we should take as the effective size of $\tilde{\Psi}$ in order to get the right signature in the matrix $\tilde{\Phi} - \sigma_{\mathrm{R}}(\tilde{\Psi}) - \sigma_{\mathrm{D}}(\tilde{\Psi})$. In other words, *the number of unknown variables we should solve for is not known a priori*. In general, it is an open problem to get an upper bound in terms of $\tilde{\Phi}$ on the effective size of $\tilde{\Psi}$ so that (3.5) is satisfied. A second problem arises from the fact that not all $\tilde{\Psi}$ such that (3.5) hold yield a Hurwitz spectral factor. This is illustrated in Example 3.2 below.

We now illustrate the application of Proposition 3.1 with two examples.

Example 3.1. In this example we apply the method of Proposition 3.1 to the matrix Z defined by

$$Z(\xi) := \begin{pmatrix} 0 & 1-\xi \\ 1+\xi & 1-\xi^2 \end{pmatrix}.$$

This example was studied before in [8]. The signature of $Z(i\omega)$ equals $(1,0,1)$ for all $\omega \in \mathbb{R}$. A matrix Φ such that $\partial\Phi = Z$ is

$$\Phi(\zeta,\eta) = \tfrac{1}{2}\begin{pmatrix} 0 & 2+\zeta-\eta \\ 2+\eta-\zeta & 2-\zeta^2-\eta^2 \end{pmatrix},$$

with coefficient matrix

$$\tilde{\Phi} = \tfrac{1}{2}\begin{pmatrix} 0 & 2 & 0 & -1 & 0 & 0 & 0 & \cdots \\ 2 & 2 & 1 & 0 & 0 & -1 & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Consider the (constant) two-variable polynomial matrix

$$\Psi(\zeta,\eta) = \tfrac{1}{2}\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}.$$

Its coefficient matrix is

$$\tilde{\Psi} = \tfrac{1}{2}\begin{pmatrix} 0 & -1 & 0 & \cdots \\ -1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Observe that

$$\tilde{\Phi} - \sigma_{\mathrm{R}}(\tilde{\Psi}) - \sigma_{\mathrm{D}}(\tilde{\Psi}) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 1 & 1 & 0 & 0 & -\frac{1}{2} & 0 & \cdots \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & -\frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{3.8}$$

and that this matrix has exactly one positive eigenvalue and one negative eigenvalue. The right-hand side of (3.8) can be factored as $\tilde{F}^T J_{1,1}\tilde{F}$, with

$$\tilde{F} = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & \frac{3}{2} & 1 & 0 & 0 & -\frac{1}{2} & 0 & \cdots \\ 1 & -\frac{1}{2} & 1 & 0 & 0 & -\frac{1}{2} & 0 & \cdots \end{pmatrix}.$$

The corresponding $J_{1,1}$-spectral factor $F(\xi)$ is given by

$$F(\xi) = \frac{\sqrt{2}}{2}\begin{pmatrix} 1+\xi & \frac{3}{2}-\xi^2/2 \\ 1+\xi & -\frac{1}{2}-\xi^2/2 \end{pmatrix}.$$

Note that $\det(F(\xi)) = -(\xi+1)$ so F is, in fact, Hurwitz.

Example 3.2. The point of this example is to show that a $\tilde{\Psi}$ such that (3.5) holds does not necessarily yield a *Hurwitz* J-spectral factor. Consider the polynomial $z(\xi) = \xi^4 - 5\xi^2 + 4$. Clearly, $z(i\omega) > 0$ for all $\omega \in \mathbb{R}$. The two-variable polynomial $\Phi(\zeta,\eta) = \zeta^2\eta^2 - \frac{5}{2}\eta^2 - \frac{5}{2}\zeta^2 - \frac{1}{2}\zeta\eta^2 - \frac{1}{2}\zeta^2\eta + 4$ satifies $\partial\Phi = z$. Define $\Psi(\zeta,\eta) = \frac{1}{2}\zeta\eta - \frac{1}{2}\zeta - \frac{1}{2}\eta - 2$. Then

$$\tilde{\Phi} - \sigma_{\mathrm{R}}(\tilde{\Psi}) - \sigma_{\mathrm{D}}(\tilde{\Psi}) = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 1 & -1 \\ -2 & -1 & 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} -2 & -1 & 1 \end{pmatrix}.$$

However, $F(\xi) = (-2 \quad -1 \quad 1)\,\mathrm{col}(1,\xi,\xi^2) = \xi^2 - \xi - 2$ is not Hurwitz.

As suggested in the previous example, the property that the J-spectral factor F computed via the factorization of $\tilde{\Phi} - \sigma_{\mathrm{R}}(\tilde{\Psi}) - \sigma_{\mathrm{D}}(\tilde{\Psi})$ is Hurwitz, depends on the choice of an appropriate $\tilde{\Psi}$. It is of interest to devise procedures to compute a $\tilde{\Psi}$ such that the corresponding factor F is Hurwitz. In the next section we develop such a procedure, based on algebraic Riccati equations.

4. A Novel Algebraic Riccati Equation Approach to *J*-Spectral Factorization

It is well known that the solvability of algebraic Riccati equations (called AREs in what follows) is closely related to polynomial spectral factorization and J-spectral factorization (see, for example, [1] and [9]). The algorithms based on this connection essentially consist of turning the problem into a J-spectral factorization problem for a proper rational matrix, associating a state-space system to the rational matrix to be factored, solving an optimal control problem, and finally translating the result back to obtain a polynomial spectral factorization.

In this section we uncover a previously unknown aspect of the relationship between AREs and J-spectral factorization, by connecting the signature condition of Proposition 3.1 with the solvability of an ARE. We show that the existence of a J-spectral factorization is equivalent to the existence of a real symmetric solution of an ARE *of suitably high dimension*. The coefficients of this ARE are immediately obtained from the coefficient matrix of the polynomial matrix to be factored.

In the following, let Z be given, and let Φ be a symmetric two-variable polynomial matrix such that $\partial\Phi = Z$. Let $\tilde{\Phi}$ be the coefficient matrix of Φ. As before, denote the degree of Z by M. Assume that the effective size L of $\tilde{\Phi}$ satisfies $L \leq M$. Note that the coefficient matrices (3.2) and (3.3) satisfy this assumption.

In the following, for any given integer $K > 1$, define matrices $A(K) \in \mathbb{R}^{Kq\times Kq}$

and $B(K) \in \mathbb{R}^{Kq\times q}$ by

$$A(K) := \begin{pmatrix} 0_{q\times q} & 0_{q\times q} & \cdots & \cdots & 0_{q\times q} \\ I_q & 0_{q\times q} & \cdots & \cdots & 0_{q\times q} \\ 0_{q\times q} & I_q & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_{q\times q} & \cdots & 0_{q\times q} & I_q & 0_{q\times q} \end{pmatrix}, \qquad B(K) := \begin{pmatrix} I_q \\ 0_{q\times q} \\ 0_{q\times q} \\ \vdots \\ 0_{q\times q} \end{pmatrix}. \tag{4.1}$$

Furthermore, for $K \geq L$, denote by $\tilde{\Phi}^{(0,1)}(K)$ the $q \times Kq$ matrix

$$\tilde{\Phi}^{(0,1)}(K) := (\Phi_{0,1}\ \Phi_{0,2}\ \cdots\ \Phi_{0,L}\ 0_{q\times(K-L)q}) \tag{4.2}$$

and by $\tilde{\Phi}^{(1,1)}(K)$ the $Kq \times Kq$ matrix

$$\tilde{\Phi}^{(1,1)}(K) := \begin{pmatrix} \Phi_{1,1} & \Phi_{1,2} & \cdots & \Phi_{1,L} & 0_{q\times(K-L)q} \\ \Phi_{2,1} & \Phi_{2,2} & \cdots & \Phi_{2,L} & 0_{q\times(K-L)q} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \Phi_{L,1} & \Phi_{L,2} & \cdots & \Phi_{L,L} & 0_{q\times(K-L)q} \\ 0_{(K-L)q\times q} & 0_{(K-L)q\times q} & \cdots & 0_{(K-L)q\times q} & 0_{(K-L)q\times q} \end{pmatrix}. \tag{4.3}$$

Obviously, the original coefficient matrix $\tilde{\Phi}$ can then be written as

$$\tilde{\Phi} = \begin{pmatrix} \Phi_{0,0} & \tilde{\Phi}^{(0,1)}(K) & 0_{q\times\infty} \\ \tilde{\Phi}^{(0,1)}(K)^T & \tilde{\Phi}^{(1,1)}(K) & 0_{Kq\times\infty} \\ 0_{\infty\times q} & 0_{\infty\times Kq} & 0_{\infty\times\infty} \end{pmatrix}.$$

In the following, for notational reasons we suppress the dependence of A, B, $\tilde{\Phi}^{(0,1)}$, and $\tilde{\Phi}^{(1,1)}$ on K. The following result connects the existence of a J-spectral factorization of Z with the solvability of an ARE.

Proposition 4.1. *Let Z be a $q \times q$ para-hermitian polynomial matrix of degree M with* $\det(Z(0)) \neq 0$, *and let Φ be a symmetric two-variable polynomial matrix such that $\partial\Phi = Z$ and $\tilde{\Phi}$ has effective size $L \leq M$. Let q_- and q_+ be nonnegative integers such that $q_- + q_+ = q$. Then the following statements are equivalent*:

(i) *there exists a $q \times q$ polynomial matrix F such that*

$$Z(\xi) = F^T(-\xi)J_{q_+,q_-}F(\xi); \tag{4.4}$$

(ii) $\operatorname{sign}(\Phi_{0,0}) = (q_-, 0, q_+)$ *and there exists an integer $K \geq L$ such that the ARE*

$$A^TP + PA + \tilde{\Phi}^{(1,1)} - (\tilde{\Phi}^{(0,1)} + B^TP)^T\Phi_{0,0}^{-1}(\tilde{\Phi}^{(0,1)} + B^TP) = 0 \tag{4.5}$$

has a symmetric solution $P \in \mathbb{R}^{Kq\times Kq}$.

Assume that any of these conditions hold. Then an F such that (4.4) *holds can be computed as follows: First, factorize $\Phi_{0,0} = F_0^T J_{q_+,q_-} F_0$, with $F_0 \in \mathbb{R}^{q\times q}$ nonsingular. Next, find a symmetric solution P of ARE* (4.5). *Then $F(\xi)$ defined as*

$$F(\xi) := F_0 \begin{pmatrix} I_q & \Phi_{0,0}^{-1}(\tilde{\Phi}^{(0,1)} + B^T P) \end{pmatrix} \begin{pmatrix} I_q \\ \xi I_q \\ \vdots \\ \xi^K I_q \end{pmatrix} \tag{4.6}$$

satisfies (4.4).

Proof. From Proposition 3.1 we have that (i) is equivalent to the existence of a $\tilde{\Psi}$ such that $\tilde{\Phi} - \sigma_{\mathrm{R}}(\tilde{\Psi}) - \sigma_{\mathrm{D}}(\tilde{\Psi})$ has q_- negative eigenvalues and q_+ positive eigenvalues. Obviously, the effective size of $\tilde{\Psi}$ must be finite. Call it K. Then we have

$$\tilde{\Psi} = \begin{pmatrix} -P & 0_{Kq\times\infty} \\ 0_{\infty\times Kq} & 0_{\infty\times\infty} \end{pmatrix},$$

with P a symmetric $Kq \times Kq$ matrix. This shows that (ii) holds if and only if there exists a symmetric matrix P such that

$$\begin{pmatrix} \Phi_{0,0} & \tilde{\Phi}^{(0,1)} \\ \tilde{\Phi}^{(0,1)T} & \tilde{\Phi}^{(1,1)} \end{pmatrix} + \begin{pmatrix} 0_{Kq\times q} & P \\ 0_{q\times q} & 0_{q\times Kq} \end{pmatrix} + \begin{pmatrix} 0_{q\times Kq} & 0_{q\times q} \\ P & 0_{Kq\times q} \end{pmatrix} \tag{4.7}$$

has q_- negative eigenvalues and q_+ positive eigenvalues. Next, observe that (4.7) can be written as

$$\begin{pmatrix} \Phi_{0,0} & \tilde{\Phi}^{(0,1)} + B^T P \\ \tilde{\Phi}^{(0,1)T} + PB & \tilde{\Phi}^{(1,1)} + A^T P + PA \end{pmatrix}, \tag{4.8}$$

with A and B the matrices defined in (4.1). Observe now that if $\mathrm{sign}(Z(i\omega)) = (q_-, 0, q_+)$ for all $\omega \in \mathbb{R}$ such that $\det(Z(i\omega)) \neq 0$, then in particular $Z(0) = Z_0 = \Phi(0,0) = \Phi_{0,0}$ has signature $(q_-, 0, q_+)$. Clearly, under this condition, the signature condition on (4.7) is equivalent to the condition that the Schur complement of $\Phi_{0,0}$ in matrix (4.8) is zero. This Schur complement is equal to $\tilde{\Phi}^{(1,1)} + A^T P + PA - (\tilde{\Phi}^{(0,1)} + B^T P)^T \Phi_{0,0}^{-1}(\tilde{\Phi}^{(0,1)} + B^T P)$. We conclude that (i) holds if and only if $\mathrm{sign}(\Phi_{0,0}) = (q_-, 0, q_+)$ and there exists a symmetric solution P to ARE (4.5). The remaining statement follows immediately from the fact that, with $\tilde{F}$ defined by $\tilde{F} := F_0(I_q \quad \Phi_{0,0}^{-1}(\tilde{\Phi}^{(0,1)} + B^T P))$, matrix (4.7) equals $\tilde{F}^T J_{q_+,q_-} \tilde{F}$. ■

In general, one is not interested in just *any* J-spectral factor of a given Z, but one wants to obtain a factor that is Hurwitz. It follows from Theorem 1 on p. 66 of [4] that for a given nonsingular para-hermitian $q \times q$ polynomial matrix Z, and nonnegative integers q_- and q_+ such that $q_- + q_+ = q$, there exists a Hurwitz J_{q_+,q_-}-spectral factorization, i.e., a factorization $Z(\xi) = H^T(-\xi) J_{q_+,q_-} H(\xi)$ with H Hurwitz, iff $\mathrm{sign}(Z(i\omega)) = (q_-, 0, q_+)$ for all $\omega \in \mathbb{R}$.

It turns out that Hurwitz spectral factors are obtained from solutions of ARE (4.5) with the property that $\sigma(A - B\Phi_{0,0}^{-1}(\tilde{\Phi}^{(0,1)} + B^T P)) \subset \mathbb{C}^- \cup \{0\}$. Note that eigenvalues in 0 are allowed. Indeed, our following result states that the *J*-spectral factor F given by (4.6) is Hurwitz iff the underlying P is a "stabilizing" (in the sense that the "closed-loop eigenvalues" lie in $\mathbb{C}^- \cup \{0\}$) solution of ARE (4.5).

Theorem 4.1. *Let* $\operatorname{sign}(\Phi_{0,0}) = (q_-, 0, q_+)$. *For a given integer K, let $P \in \mathbb{R}^{Kq \times Kq}$ be a symmetric solution of ARE* (4.5). *Let F_0 be such that $\Phi_{0,0} = F_0^T J_{q_+,q_-} F_0$. Define*

$$F(\xi) := F_0 \begin{pmatrix} I_q & \Phi_{0,0}^{-1}(\tilde{\Phi}^{(0,1)} + B^T P) \end{pmatrix} \begin{pmatrix} I_q \\ \xi I_q \\ \vdots \\ \xi^K I_q \end{pmatrix}.$$

Then F is Hurwitz iff $\sigma(A - B\Phi_{0,0}^{-1}(\tilde{\Phi}^{(0,1)} + B^T P)) \subset \mathbb{C}^- \cup \{0\}$.

Proof. Partition $P \in \mathbb{R}^{Kq \times Kq}$ as $P = (P_{i,j})$ with $P_{i,j} \in \mathbb{R}^{q \times q}$, $i,j = 0,1,2,\ldots,K-1$. Define $\tilde{G} \in \mathbb{R}^{q \times Kq}$ by $\tilde{G} := \Phi_{0,0}^{-1}(\tilde{\Phi}^{(0,1)} + B^T P)$. Observe that the first q rows of $A - B\Phi_{0,0}^{-1}(\tilde{\Phi}^{(0,1)} + B^T P)$ coincide with $\tilde{G}$. Define a $q \times q$ polynomial matrix $G(\xi)$ by

$$G(\xi) := \xi^K I + \tilde{G} \begin{pmatrix} \xi^{K-1} I \\ \xi^{K-2} I \\ \vdots \\ \xi I \\ I \end{pmatrix}.$$

It can be shown that the characteristic polynomial of $A - B\Phi_{0,0}^{-1}(\tilde{\Phi}^{(0,1)} + B^T P)$ is equal to $\det(G)$. Also, $F(\xi) = \xi^K F_0 G(1/\xi)$. We show that F is Hurwitz iff G has all its zeros in $\mathbb{C}^- \cup \{0\}$. Assume F is Hurwitz and let λ be a zero of G, i.e., $\det(G(\lambda)) = 0$. Since $G(\xi) = \xi^K F_0^{-1} F(1/\xi)$, we must have $\lambda = 0$ or $\det(F(1/\lambda)) = 0$. This implies that $\lambda = 0$ or $\Re e(\lambda) < 0$. Conversely, assume that G has its zeros in $\mathbb{C}^- \cup \{0\}$. Let $\det(F(\lambda)) = 0$. Then we must have $\lambda = 0$ or $\det(G(1/\lambda)) = 0$. Again, this implies that $\lambda = 0$ or $\Re e(\lambda) < 0$. Since however $F(0) = F_0$ is nonsingular, we conclude that $\Re e(\lambda) < 0$. ■

Putting together the results of Proposition 4.1 and Theorem 4.1, we obtain the following proposition, which states that there exists a Hurwitz *J*-spectral factor for Z iff ARE (4.5) of suitably high dimension has a stabilizing solution.

Proposition 4.2. *Let Z be a $q \times q$ para-hermitian polynomial matrix of degree M, and let Φ be a symmetric two-variable polynomial matrix such that $\partial\Phi = Z$ and $\tilde{\Phi}$*

has effective size $L \leq M$. *Let* q_- *and* q_+ *be nonnegative integers such that* $q_- + q_+ = q$. *Then the following statements are equivalent:*

(i) *there exists a* $q \times q$ *Hurwitz polynomial matrix* H *such that*

$$Z(\xi) = H^T(-\xi)J_{q_+,q_-}H(\xi); \tag{4.9}$$

(ii) $\mathrm{sign}(\Phi_{0,0}) = (q_-, 0, q_+)$ *and there exists an integer* $K \geq L$ *such that ARE* (4.5) *has a symmetric solution* $P \in \mathbb{R}^{Kq \times Kq}$ *with the property that* $\sigma(A - B\Phi_{0,0}^{-1}(\tilde{\Phi}^{(0,1)} + B^T P)) \subset \mathbb{C}^- \cup \{0\}$.

Assume that any of these conditions hold. Then a Hurwitz polynomial matrix H *such that* (4.9) *holds can be computed as follows: First, factorize* $\Phi_{0,0} = H_0^T J_{q_+,q_-} H_0$, *with* $H_0 \in \mathbb{R}^{q \times q}$ *nonsingular. Next, find a symmetric solution* P *of ARE* (4.5) *as in* (ii). *Then* $H(\xi)$ *defined as*

$$H(\xi) := H_0(I_q \quad \Phi_{0,0}^{-1}(\tilde{\Phi}^{(0,1)} + B^T P)) \begin{pmatrix} I_q \\ \xi I_q \\ \vdots \\ \xi^K I_q \end{pmatrix}$$

is Hurwitz and satisfies (4.9).

Proof. (i) $\Rightarrow$ (ii) Proceeding as in the proof of Proposition 3.1 we conclude that (i) implies that the coefficient matrix $\tilde{\Psi}$ associated with

$$\Psi(\zeta, \eta) := \frac{\Phi(\zeta, \eta) - H^T(\zeta)J_{q_+,q_-}H(\eta)}{\zeta + \eta}$$

satisfies the signature conditions (3.5). As in the proof of Proposition 4.1, we have that, for some K,

$$\tilde{\Psi} = \begin{pmatrix} -P & 0_{Kq \times \infty} \\ 0_{\infty \times Kq} & 0_{\infty \times \infty} \end{pmatrix}$$

with $P \in \mathbb{R}^{Kq \times Kq}$ symmetric, and that P satisfies ARE (4.5). Our aim is to show that this P is a stabilizing solution. Let F_0 be such that $\Phi_{0,0} = F_0^T J_{q_+,q_-} F_0$ and define $\tilde{F} := F_0(I_q \ \Phi_{0,0}^{-1}(\tilde{\Phi}^{(0,1)} + B^T P))$. Let $F(\xi)$ be the polynomial matrix corresponding to the coefficient matrix $\tilde{F}$. We claim that F is Hurwitz. Indeed, $\tilde{F}^T J_{q_+,q_-} \tilde{F} = \tilde{\Phi} - \sigma_{\mathrm{R}}(\tilde{\Psi}) - \sigma_{\mathrm{D}}(\tilde{\Psi})$, so we get $H^T(\zeta)J_{q_+,q_-}H(\eta) = F^T(\zeta)J_{q_+,q_-}F(\eta)$. In terms of the associated bilinear differential forms this implies that for all $w, w' \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q)$ we have $(H(d/dt)w')^T J_{q_+,q_-} H(d/dt)w = (F(d/dt)w')^T J_{q_+,q_-} \cdot F(d/dt)w$. Assume now that w is such that $F(d/dt)w = 0$. Then for all w' we have $(F(d/dt)w')^T J_{q_+,q_-} F(d/dt)w = 0$, so $(H(d/dt)w')^T J_{q_+,q_-} H(d/dt)w = 0$. This can be seen to imply that $H(d/dt)w = 0$. Since H is Hurwitz, this yields $w(t) \to 0$ as $t \to \infty$, proving that F is indeed Hurwitz. By Theorem 4.1 we now conclude that $\sigma(A - B\Phi_{0,0}^{-1}(\tilde{\Phi}^{(0,1)} + B^T P)) \subset \mathbb{C}^- \cup \{0\}$. The implication (ii) $\Rightarrow$ (i) and the remaining statements of the proposition follow immediately from Theorem 4.1 and Proposition 4.1. ■

Remark 4.1. Note that Propositions 4.1 and 4.2 do not claim that the existence of suitable J_{q_+,q_-}-spectral factors is equivalent to the existence of suitable solutions to an a priori given ARE. Instead, there should be suitable solutions to an ARE *of sufficiently high dimension*. It is a drawback of our results that this dimension is not known a priori (see also the remarks following Proposition 3.1). In the case where we deal with positive semidefinite factorization, i.e., $J_{q_+,q_-} = I_q$, see Section 5, this drawback disappears. It turns out that in this case we can always take the dimension equal to the degree of the polynomial matrix to be factored.

The previous proposition can be applied to obtain algorithms to compute a Hurwitz J-spectral factor for a given polynomial matrix. As noted in Section 3, for a given Z there exist many two-variable polynomial matrices Φ such that $\partial\Phi = Z$. Of course, different choices of Φ lead to different Riccati equations (4.5). As an example, below we state an algorithm that is based on the choice (3.3) of Φ.

Algorithm 4.1.

Input: A para-hermitian polynomial matrix $Z \in \mathbb{R}^{q\times q}[\xi]$, $Z(\xi) = Z_0 + Z_1\xi + \cdots + Z_M\xi^M$, with $Z_M \neq 0$, and two nonnegative integers q_- and q_+ such that $q_- + q_+ = q$ and such that $\operatorname{sign}(Z(i\omega)) = (q_-, 0, q_+)$ for all $\omega \in \mathbb{R}$.
Output: A Hurwitz polynomial matrix $H \in \mathbb{R}^{q\times q}[\xi]$ such that $Z(\xi) = H^T(-\xi)J_{q_+,q_-}H(\xi)$.

For a given positive integer K, let $A(K)$ and $B(K)$ be given by (4.1). Also, define $S(K) := \frac{1}{2}(Z_1 \;\; Z_2 \;\; \cdots \;\; Z_M \;\; 0_{q\times(K-M)q})$.

Step 1. Calculate $H_0 \in \mathbb{R}^{q\times q}$ such that $H_0^T J_{q_+,q_-} H_0 = Z_0$.
Step 2. Assign the value M to the variable K.
Step 3. If is exists, find a real symmetric solution $P \in \mathbb{R}^{Kq\times Kq}$ of the ARE

$$A(K)^T P + PA(K) - (S(K) + B(K)^T P)^T Z_0^{-1}(S(K) + B(K)^T P) = 0 \tag{4.10}$$

such that $\sigma(A(K) - B(K)Z_0^{-1}(S(K) + B(K)^T P)) \subset \mathbb{C}_- \cup \{0\}$.
Step 4. If such a P does not exist, then let $K = K + 1$ and go to Step 3. Else, partition the first q rows of P into $(P_{0,0} \;\; P_{0,1} \;\; \cdots \;\; P_{0,K-1})$, with $P_{0,j} \in \mathbb{R}^{q\times q}$ and let

$$H(\xi) = H_0 Z_0^{-1}\left(Z_0 + \sum_{i=1}^{M}(\tfrac{1}{2}Z_i + P_{0,i-1})\xi^i + \sum_{i=M+1}^{K} P_{0,i-1}\xi^i\right).$$

Note that, due to Proposition 4.2, ARE (4.10) indeed has a required solution *for some K*. Thus the above algorithm terminates after finitely many iterations.

Example 4.1. Consider the polynomial matrix of Example 3.1:

$$Z(\xi) = \begin{pmatrix} 0 & 1-\xi \\ 1+\xi & 1-\xi^2 \end{pmatrix}.$$

We have $\operatorname{sign}(Z(i\omega)) = (1,0,1)$ for all $\omega \in \mathbb{R}$. Furthermore, $M = 2$, $q = 2$, and

$$Z_0 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \qquad Z_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad Z_2 = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have

$$A(2) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \qquad B(2) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}, \qquad S(2) = \tfrac{1}{2}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

It can be calculated that

$$H_0 = \tfrac{1}{2}\sqrt{2}\begin{pmatrix} 1 & \frac{3}{2} \\ 1 & -\frac{1}{2} \end{pmatrix}.$$

We find that

$$P = \begin{pmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

is a solution of the ARE, and that it satisfies $\sigma(A(2) - B(2)Z_0^{-1}(S(2) + B(2)^T P))$ $= \{0,0,0,-1\}$. The algorithm then yields

$$H(\xi) = H_0 + H_0 Z_0^{-1}\left(\tfrac{1}{2}Z_1 + \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}\right)\xi + H_0 Z_0^{-1}\tfrac{1}{2}Z_2\xi^2,$$

which yields

$$H(\xi) = \tfrac{1}{2}\sqrt{2}\begin{pmatrix} 1+\xi & \dfrac{3-\xi^2}{2} \\ 1+\xi & \dfrac{1-\xi^2}{2} \end{pmatrix}.$$

5. Positive Semidefinite Spectral Factorization

Let Z be a $q \times q$ nonsingular para-hermitian polynomial matrix. In this section we restrict attention to the case where Z is positive definite on the imaginary axis, except in its singularities. Obviously, this requirement is equivalent to the condition $Z(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$, and in this case the number of negative eigenvalues, q_-, equals 0, and the number of positive eigenvalues, q_+, equals q. Also, $J_{q_+,q_-} = I_q$, the $q \times q$ identity matrix. It follows from Proposition 3.1 that there exists a $q \times q$ polynomial matrix F such that $Z(\xi) = F^T(-\xi)F(\xi)$. It is easy to show that this requires the degree, M, of Z to be *even*, and that any spectral factor F has degree $\frac{1}{2}M$. This a priori knowledge of the degree of the spectral factor translates into information on the effective size of the unknown matrix $\tilde{\Psi}$ in (3.5), and on the dimension of the AREs to be solved.

The first result of this section is an extension of Proposition 3.1 to the case at hand.

Proposition 5.1. *Let Z be a $q \times q$ para-hermitian matrix of degree M with* $\det(Z) \neq 0$, *and let Φ be a symmetric two-variable polynomial matrix such that $\partial\Phi = Z$. Denote the effective size of $\tilde{\Phi}$ by L. Let $\tilde{\Phi}_{\text{trunc}}$ denote the $(L+1)q \times (L+1)q$ matrix consisting of the first $(L+1)q$ rows and columns of $\tilde{\Phi}$. The following three statements are equivalent*:

(i) *there exists a $q \times q$ polynomial matrix F such that*

$$Z(\xi) = F^T(-\xi)F(\xi); \tag{5.1}$$

(ii) $Z(i\omega) \geq 0$ *for all* $\omega \in \mathbb{R}$;
(iii) *there exists a real symmetric solution $P \in \mathbb{R}^{qL \times qL}$ to the linear matrix inequality* (LMI)

$$L(P) := \tilde{\Phi}_{\text{trunc}} + \begin{pmatrix} 0_{Lq\times q} & P \\ 0_{q\times q} & 0_{q\times Lq} \end{pmatrix} + \begin{pmatrix} 0_{q\times Lq} & 0_{q\times q} \\ P & 0_{Lq\times q} \end{pmatrix} \geq 0. \tag{5.2}$$

Assume that any of these conditions hold. Then there exists a solution P with the additional property that $\text{rank}(L(P)) = q$. *An F such that* (5.1) *holds can then be computed as follows: find a real symmetric solution $P \in \mathbb{R}^{qL\times qL}$ of the LMI such that* $\text{rank}(L(P)) = q$, *and factorize*

$$L(P) = \tilde{F}^T\tilde{F} \tag{5.3}$$

with $\tilde{F} \in \mathbb{R}^{q\times(L+1)q}$. Then the polynomial matrix F defined by $F(\xi) := \tilde{F}\,\text{col}(I, \xi I, \xi^2 I, \ldots \xi^L I)$ *satisfies* (5.1).

Remark 5.1. Note that, for example, (3.3) yields $\tilde{\Phi}$ with effective size $L = \frac{1}{2}M$. This choice of $\tilde{\Phi}$ thus yields an LMI with unknown P of size $(q/2)M \times (q/2)M$.

Proof. The equivalence of (i) and (ii) is already known. We prove (i) $\Rightarrow$ (iii). From $\partial\Phi = Z$, we obtain $L \geq \frac{1}{2}M$. Hence, certainly the effective size of $\tilde{\Phi} - \tilde{F}^T\tilde{F}$ is less than or equal to L. Define $\Psi(\zeta,\eta) := (1/(\zeta+\eta))(\Phi(\zeta,\eta) - F^T(\zeta)F(\eta))$. Then the coefficient matrix of $\Phi(\zeta,\eta) - F^T(\zeta)F(\eta)$ equals $\sigma_{\text{R}}(\tilde{\Psi}) + \sigma_{\text{D}}(\tilde{\Psi})$, so the latter also has effective size less than or equal to L. Obviously, the matrix obtained by taking the first $(L+1)q$ rows and columns must then be of the form

$$\begin{pmatrix} 0_{Lq\times q} & -P \\ 0_{q\times q} & 0_{q\times Lq} \end{pmatrix} + \begin{pmatrix} 0_{q\times Lq} & 0_{q\times q} \\ -P & 0_{Lq\times q} \end{pmatrix}$$

for some real symmetric matrix $P \in \mathbb{R}^{qL\times qL}$. Since also $\tilde{\Phi} - \sigma_{\text{R}}(\tilde{\Psi}) - \sigma_{\text{D}}(\tilde{\Psi}) = \tilde{F}^T\tilde{F}$, for this P we obtain $L(P) \geq 0$. We now prove (iii) $\Rightarrow$ (i). Let $\Psi(\zeta,\eta)$ be the two-variable polynomial matrix corresponding to the coefficient matrix

$$\tilde{\Psi} := \begin{pmatrix} -P & 0_{Lq\times q} \\ 0_{q\times Lq} & 0_{q\times q} \end{pmatrix}.$$

Since $L(P) \geq 0$, it can be factored as $L(P) = \tilde{F}^T\tilde{F}$ for some matrix $\tilde{F}$. Next, premultiply $L(P)$ by $(I_q\ \zeta I_q\ \cdots\ \zeta^L I_q)$ and postmultiply by $(I_q\ \eta I_q\ \cdots\ \eta^L I_q)^T$, and use (5.2) to obtain $\Phi(\zeta,\eta) - (\zeta+\eta)\Psi(\zeta,\eta) = F^T(\zeta)F(\eta)$, with F the polynomial matrix associated with the coefficient matrix $\tilde{F}$. Applying the ∂-operator, we then obtain $Z(\xi) = F^T(-\xi)F(\xi)$, which immediately implies $Z(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. To prove the remaining statement, note that if F satisfies (5.1), then $\det(F) \neq 0$. This can be seen to imply that the rows of $\tilde{F}$ are linearly independent. Consequently, the P constructed above must satisfy $\operatorname{rank}(L(P)) = q$. ■

Of course, in the positive semidefinite case an extension of Proposition 4.1 also holds, in which the order of ARE (4.5) is equal to the effective size of $\tilde{\Phi}$. We do not explicitly formulate this result here, but instead concentrate on methods to obtain Hurwitz spectral factors. First note that for a given para-hermitian matrix Z there exists a $q \times q$ Hurwitz polynomial matrix H such that $Z(\xi) = H^T(-\xi)H(\xi)$ iff $Z(i\omega) > 0$ for all $\omega \in \mathbb{R}$. It was shown in [17] that if $Z(i\omega) \geq 0$ for all ω, then LMI (5.2) has real symmetric solutions P_- and P_+ such that any real symmetric solution P satisfies $P_- \leq P \leq P_+$. It was also shown in [17] that if $Z(i\omega) > 0$ for all ω, then the choice $P = P_+$, i.e., the largest real symmetric solution of the LMI, yields a Hurwitz spectral factor. In fact, we quote from [17]:

Proposition 5.2. *Let Z be a $q \times q$ para-hermitian matrix of degree M with* $\det(Z) \neq 0$, *and let Φ be a symmetric two-variable polynomial matrix such that $\partial\Phi = Z$. Denote the effective size of $\tilde{\Phi}$ by L. Let $\tilde{\Phi}_{\text{trunc}}$ denote the $(L+1)q \times (L+1)q$ matrix consisting of the first $(L+1)q$ rows and columns of $\tilde{\Phi}$. Assume that $Z(i\omega) > 0$. Let $P_+ \in \mathbb{R}^{Lq\times Lq}$ be the largest real symmetric solution of LMI* (5.2). *Then* $\operatorname{rank}(L(P_+)) = q$. *Furthermore, any factorization*

$$L(P_+) = \tilde{H}^T\tilde{H}$$

with $\tilde{H} \in \mathbb{R}^{q\times(L+1)q}$ yields a Hurwitz spectral factor $H(\xi) := \tilde{H}$ $\operatorname{col}(I, \xi I, \xi^2 I, \ldots, \xi^L I)$.

As before, different choices of Φ such that $\partial\Phi = Z$ yield different LMIs. Below we state an algorithm for Hurwitz spectral factorization that is based on the particular choice (3.3) of Φ:

Algorithm 5.1.

Input: A para-hermitian polynomial matrix $Z \in \mathbb{R}^{q\times q}[\xi]$, $Z(\xi) = Z_0 + Z_1\xi + \cdots + Z_M\xi^M$, with $Z_M \neq 0$, such that $Z(i\omega) > 0$ for all $\omega \in \mathbb{R}$.

Output: A Hurwitz polynomial matrix $H \in \mathbb{R}^{q\times q}[\xi]$ such that $Z(\xi) = H^T(-\xi)H(\xi)$.

Define a $\left(\frac{1}{2}M+1\right)q \times \left(\frac{1}{2}M+1\right)q$ matrix Q by

$$Q := \frac{1}{2}\begin{pmatrix} 2Z_0 & Z_1 & Z_2 & \cdots & Z_{M/2-1} & Z_{M/2} \\ Z_1^T & 0 & 0 & \cdots & 0 & -Z_{M/2+1} \\ Z_2^T & 0 & 0 & \cdots & 0 & Z_{M/2+2} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \\ Z_{M/2-1}^T & 0 & 0 & \cdots & 0 & (-1)^{M/2-1}Z_{M-1} \\ Z_{M/2}^T & -Z_{M/2+1}^T & Z_{M/2+2}^T & \cdots & (-1)^{M/2-1}Z_{M-1}^T & 2(-1)^{M/2}Z_M \end{pmatrix}.$$

Step 1. Find the largest real symmetric solution $P_+ \in \mathbb{R}^{Mq/2 \times Mq/2}$ of the LMI

$$L(P) := Q + \begin{pmatrix} 0_{Mq/2\times q} & P \\ 0_{q\times q} & 0_{q\times Mq/2} \end{pmatrix} + \begin{pmatrix} 0_{q\times Mq/2} & 0_{q\times q} \\ P & 0_{Mq/2\times q} \end{pmatrix} \geq 0.$$

Step 2. Factor $L(P_+) = \tilde{H}^T\tilde{H}$ with $\tilde{H} \in \mathbb{R}^{q\times(M/2+1)q}$.
Step 3: Define

$$H(\xi) := \tilde{H}\begin{pmatrix} I_q \\ \xi I_q \\ \vdots \\ \xi^{M/2}I_q \end{pmatrix}.$$

Remark 5.2. The computation of the largest real symmetric solution of the LMI in Step 1 amounts to maximizing the convex function $P \mapsto \operatorname{trace}(P)$ over the convex set of all real symmetric solutions of the LMI. This compuation can be performed using the LMI toolbox of Matlab. For this, one should first rewrite $L(P)$ in the usual state space (A, B, Q, R, S)-format as (4.8), with A and B given by (4.1). Of course, once we have found P_+, a factorization $L(P_+) = \tilde{H}^T\tilde{H}$ can be computed using any numerical algorithm to compute a Schur decomposition of the symmetric matrix $L(P_+)$ (see [6]).

We now extend Proposition 4.2 to the positive semidefinite case. As stated before, in this case the dimension of the corresponding ARE can be taken to be equal to the effective size L of the underlying $\tilde{\Phi}$. In the following, let $A := A(L)$ and $B := B(L)$ with $A(L)$ and $B(L)$ given by (4.1) with $K = L$. Also, let $\tilde{\Phi}^{(0,1)}$ and $\tilde{\Phi}^{(1,1)}$ be given by (4.2) and (4.3) with $K = L$, so without the zero-blocks. Then, obviously, the truncated coefficient matrix $\tilde{\Phi}_{\text{trunc}}$ is equal to

$$\tilde{\Phi}_{\text{trunc}} = \begin{pmatrix} \Phi_{0,0} & \tilde{\Phi}^{(0,1)} \\ \tilde{\Phi}^{(0,1)T} & \tilde{\Phi}^{(1,1)} \end{pmatrix}.$$

Proposition 5.3. *Let Z be a $q \times q$ para-hermitian polynomial matrix of degree M, and let Φ be a symmetric two-variable polynomial matrix such that $\partial\Phi = Z$ and $\tilde{\Phi}$ has effective size $L \leq M$. Then the following statements are equivalent*:

(i) *there exists a* $q \times q$ *Hurwitz polynomial matrix* H *such that*

$$Z(\xi) = H^T(-\xi)H(\xi); \tag{5.4}$$

(ii) $\Phi_{0,0} > 0$ *and ARE* (4.5) *has a symmetric solution* $P \in \mathbb{R}^{Lq \times Lq}$ *with the property that* $\sigma(A - B\Phi_{0,0}^{-1}(\tilde{\Phi}^{(0,1)} + B^T P)) \subset \mathbb{C}^- \cup \{0\}$.

Assume that any of these conditions hold. Then a Hurwitz polynomial matrix H *such that* (5.4) *holds can be computed as follows: First, factorize* $\Phi_{0,0} = H_0^T H_0$, *with* $H_0 \in \mathbb{R}^{q \times q}$ *nonsingular. Next, find a real symmetric solution* P *of ARE* (4.5) *as in* (ii). *Then* H *defined as*

$$H(\xi) := H_0(I_q \quad \Phi_{0,0}^{-1}(\tilde{\Phi}^{(0,1)} + B^T P)) \begin{pmatrix} I_q \\ \xi I_q \\ \vdots \\ \xi^L I_q \end{pmatrix}$$

is Hurwitz and satisfies (5.4).

Proof. The proof is completely analogous to the proof of Proposition 4.2, the only difference being that we can now take the ARE to be of dimensions $qL \times qL$, with L the effective size of $\tilde{\Phi}$. ■

As before, different choices of Φ such that $\partial\Phi = Z$ yield different AREs. Below we describe an algorithm for Hurwitz spectral factorization that is based on the particular choice (3.3):

Algorithm 5.2.

Input: A para-hermitian polynomial matrix $Z \in \mathbb{R}^{q \times q}[\xi]$, $Z(\xi) = Z_0 + Z_1\xi + \cdots + Z_M\xi^M$, with $Z_M \neq 0$, such that $Z(i\omega) > 0$ for all $\omega \in \mathbb{R}$.
Output: A Hurwitz polynomial matrix $H \in \mathbb{R}^{q \times q}[\xi]$ such that $Z(\xi) = H^T(-\xi)H(\xi)$.

Let $A := A\left(\frac{1}{2}M\right)$ and $B := B\left(\frac{1}{2}M\right)$, given by (4.1). Define a (symmetric) $\frac{1}{2}Mq \times \frac{1}{2}Mq$ matrix Q by

$$Q := \frac{1}{2}\begin{pmatrix} 0 & 0 & \cdots & 0 & -Z_{M/2+1} \\ 0 & 0 & \cdots & 0 & Z_{M/2+2} \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 0 & (-1)^{M/2-1}Z_{M-1} \\ -Z_{M/2+1}^T & Z_{M/2+2}^T & \cdots & (-1)^{M/2-1}Z_{M-1}^T & 2(-1)^{M/2}Z_M \end{pmatrix}$$

and a $q \times \frac{1}{2}Mq$ matrix S by

$$S := \frac{1}{2}(Z_1 \quad Z_2 \quad \cdots \quad Z_{M/2-1} \quad Z_{M/2}).$$

Step 1. Calculate $H_0 \in \mathbb{R}^{q\times q}$ such that $H_0^T H_0 = Z_0$.
Step 2. Find a real symmetric solution $P \in \mathbb{R}^{Mq/2\times Mq/2}$ of the ARE

$$A^T P + PA + Q - (S + B^T P)^T Z_0^{-1}(S + B^T P) = 0,$$

such that $\sigma(A - BZ_0^{-1}(S + B^T P)) \subset \mathbb{C}^- \cup \{0\}$.
Step 3. Partition the first q rows of P into $(P_{0,0}\ P_{0,1}\ \cdots\ P_{0,M/2-1})$, with $P_{0,j} \in \mathbb{R}^{q\times q}$ and put

$$H(\xi) = H_0 + H_0 Z_0^{-1} \sum_{i=1}^{M/2} (\tfrac{1}{2} Z_i + P_{0,i-1})\xi^i.$$

We now illustrate the application of this algorithm with two examples.

Example 5.1. Consider the polynomial $z(\xi) = 4 - 5\xi^2 + \xi^4$ used in Example 3.2. In this case, $M = 4$, $Z_0 = 4$, $Z_1 = 0$, $Z_2 = -5$, $Z_3 = 0$, and $Z_4 = 1$. Hence,

$$Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \qquad S = (0 \quad -\tfrac{5}{2}), \qquad A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

The matrix

$$P = \begin{pmatrix} 6 & \tfrac{9}{2} \\ \tfrac{9}{2} & 3 \end{pmatrix}$$

satisfies the corresponding ARE, and, in fact, satisfies $\sigma(A - BZ_0^{-1}(S + B^T P)) \subset \mathbb{C}^-$. The algorithm yields $H(\xi) = 2 + 3\xi + \xi^2$.

Example 5.2. We now consider an example which was also studied in Section 8 of [7]. In this example, $q = 3$ and $M = 3$. Let $Z(\xi)$ be given by

$$Z(\xi) = \begin{pmatrix} \tfrac{1}{4} - \xi^2 & \tfrac{1}{2} - \xi & 0 \\ \tfrac{1}{2} + \xi & 2 - \xi^2 & -\tfrac{1}{4} - \tfrac{1}{2}\xi \\ 0 & -\tfrac{1}{4} + \tfrac{1}{2}\xi & \tfrac{1}{4} - \xi^2 \end{pmatrix}.$$

Therefore we have

$$Z_0 = \begin{pmatrix} \tfrac{1}{4} & \tfrac{1}{2} & 0 \\ \tfrac{1}{2} & 2 & -\tfrac{1}{4} \\ 0 & -\tfrac{1}{4} & \tfrac{1}{4} \end{pmatrix}, \qquad Z_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -\tfrac{1}{2} \\ 0 & \tfrac{1}{2} & 0 \end{pmatrix}, \qquad Z_2 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Thus we have $A = 0_{3\times 3}$, $B = I_{3\times 3}$, $Q = -Z_2$, and $S = \tfrac{1}{2} Z_1$. It turns out that

$$P = \begin{pmatrix} 0.5000 & 0.5000 & 0.000 \\ 0.5000 & 0.8660 & -0.2500 \\ 0.0000 & -0.2500 & 0.5000 \end{pmatrix}$$

is a solution of the ARE, and that $\sigma(A - BZ_0^{-1}(S + B^T P)) \subset \mathbb{C}^-$. This yields a Hurwitz spectral factor

$$H(\xi) = \begin{pmatrix} 0.4095 + 0.8191\xi & 0.2832 - 0.5665\xi & 0.0452 + 0.0905\xi \\ 0.2832 + 0.5665\xi & 1.3783 + 0.7739\xi & -0.1416 - 0.2832\xi \\ 0.0452 + 0.0905\xi & -0.1416 + 0.2832\xi & 0.4774 + 0.9548\xi \end{pmatrix}.$$

6. A Pick Matrix Approach to *J*-Spectral Factorization

One of the areas in which the problem of polynomial J-spectral factorization arises is the polynomial approach to H_∞ control (see [16], [8], [10], and [11]). In that context the para-hermitian polynomial matrix $Z \in \mathbb{R}^{q\times q}[\xi]$ to be factored is usually given as

$$Z(\xi) = M^T(-\xi) J_{p_+,p_-} M(\xi), \tag{6.1}$$

where $M \in \mathbb{R}^{p\times q}[\xi]$ is given, $p > q$, and p_- and p_+ are given positive integers such that $p_- + p_+ = p$. We can partition M according to (6.1) as

$$M = \begin{pmatrix} M_+ \\ M_- \end{pmatrix},$$

where M_+ has p_+ rows and M_- has p_- rows, so $Z(\xi) = M_+^T(-\xi)M_+(\xi) - M_-^T(-\xi)M_-(\xi)$. In the polynomial H_∞ control context, the J-spectral factorization problem is to find a Hurwitz polynomial matrix $H \in \mathbb{R}^{q\times q}[\xi]$ such that $Z(\xi) = H^T(-\xi)J_{q-p_-,p_-}H(\xi)$ and such that the following two additional requirements are satisfied:

(1) MH^{-1} is a matrix of *proper* rational functions.
(2) The matrix

$$\begin{pmatrix} M_- \\ H_+ \end{pmatrix}$$

is Hurwitz. Here, H_+ is the polynomial matrix consisting of the first $q - p_-$ rows of H.

Note that the algorithms described in Sections 3 and 4 do not address the issue of finding H such that the additional two properties (1) and (2) hold. A J-spectral factor H of Z is called *regular* if (1) holds, and *stabilizing* if (2) holds.[1] Of course, a *necessary* condition for the existence of such a J-spectral factor H is that $\operatorname{sign}(Z(i\omega)) = (p_-, 0, q - p_-)$ for all $\omega \in \mathbb{R}$. However, the signature condition on $Z(i\omega)$ is certainly not sufficient for the existence of a regular, stabilizing, Hurwitz J-spectral factor H. In [16], necessary and sufficient conditions for the existence of

[1] It can be shown that a Hurwitz J-spectral factor H of Z satisfies (1) and (2) iff $G := MH^{-1}$ is a J_{p_+,p_-}-*lossless* proper rational matrix, i.e., it is proper and satisfies $G(\bar{\lambda})^T J_{p_+,p_-} G(\lambda) \le J_{q-p_-,p_-}$ for $\Re e(\lambda) \ge 0$, with equality at infinity (see [11]).

such *J*-spectral factors were given. Below, we briefly review these conditions and state an algorithm to compute such *J*-spectral factors.

Let Z be given by (6.1). In the remainder of this section we assume that the matrix M in (6.1) has the property that $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. We also assume that $\det(Z)$ has no roots on the imaginary axis and that Z is semisimple. Define now the symmetric two-variable polynomial matrix Φ by $\Phi(\zeta,\eta) := M^T(\zeta)J_{p_+,p_-}M(\eta)$. Let $\lambda_1, \lambda_2, \dots, \lambda_k$ be the distinct roots of $\det(Z)$ in $\mathbb{C}^-$, and let V_i be full column rank matrices such that $\mathrm{im}(V_i) = \ker(Z(\lambda_i))$ $(i = 1, 2, \dots, k)$. Let T_Φ be the corresponding Pick matrix associated with Φ. The following result from [16] gives necessary and sufficient conditions for the existence of a regular, stabilizing, Hurwitz J_{q-p_-,p_-}-spectral factorization of Z:

Proposition 6.1. *There exists* $H \in \mathbb{R}^{q\times q}[\xi]$ *such that*

1. $M^T(-\xi)J_{p_+,p_-}M(\xi) = H^T(-\xi)J_{q-p_-,p_-}H(\xi)$,
2. *H is Hurwitz,*
3. *MH^{-1} is a matrix of proper rational functions,*
4. $\begin{pmatrix} M_- \\ H_+ \end{pmatrix}$ *is Hurwitz (here, M_- is the polynomial matrix consisting of the last p_- rows of M, and H_+ is the polynomial matrix consisting of the first $q-p_-$ rows of H)*

if and only if the following two conditions are satisfied:

1. *there exists $\varepsilon > 0$ such that for all $\omega \in \mathbb{R}$ it holds that*
$$\mathrm{sign}(M^T(-i\omega)J_{p_+,p_-}M(i\omega) + \varepsilon M^T(-i\omega)M(i\omega)) = (p_-, 0, q-p_-); \quad (6.2)$$
2. $T_\Phi < 0$.

In the following we refer to (6.2) as the *strict signature condition.*

Remark 6.1. We note that if Z is unimodular, then of course $\det(Z)$ has no roots, so the Pick matrix does not exist. In that case, the condition $T_\Phi < 0$ in Proposition 6.1 should be interpreted to be automatically satisfied.

Remark 6.2. It can be shown that there exists $\varepsilon > 0$ such that (6.2) holds for all $\omega \in \mathbb{R}$ iff there exists ε_0 such that (6.2) holds for all $0 \le \varepsilon \le \varepsilon_0$, for all $\omega \in \mathbb{R}$ (this will follow, e.g., from Lemma 8.3 in this paper). Thus, this condition requires not only that $M^T(-i\omega)J_{p_-,p_+}M(i\omega)$ has signature $(p_-, 0, q-p_-)$, but also that the signature does not change by adding a sufficiently small positive multiple of $M^T(-i\omega)M(i\omega)$.

Remark 6.3. In [16] it was proven that the strict signature condition (6.2) and $T_\Phi < 0$ hold if and only if the full information H_∞ control problem associated with the system in image representation $w = M(d/dt)\ell$ admits a solution. The fact that solvability of this H_∞ control problem is equivalent to the existence of a regular, stabilizing, Hurwitz spectral factorization of $M^T(-\xi)J_{p_+,p_-}M(\xi)$ was already established in [11].

In the J-spectral factorization algorithm that we give in this section, an important role is played by the notion of a *state map* of a linear differential system (see [14]). We briefly review this notion here. To the polynomial matrix $M(\xi) = M_0 + M_1\xi + \cdots + M_k\xi^k$ we associate the linear differential operator $M(d/dt) := M_0 + M_1(d/dt) + \cdots + M_k(d^k/dt^k)$. This operator defines a linear differential system $\Sigma(M) = (\mathbb{R}, \mathbb{R}^p, \mathfrak{B}(M))$ with behavior equal to the image of $M(d/dt)$:

$$\mathfrak{B}(M) := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^p) \mid \text{there exists } \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \text{ such that } w = M(d/dt)\ell\}. \tag{6.3}$$

Given a polynomial matrix $M \in \mathbb{R}^{p\times q}[\xi]$, a polynomial matrix $X \in \mathbb{R}^{n\times q}[\xi]$ is said to define a *state map* for $\Sigma(M)$ or, simply, for M, if the system $\Sigma_s = (\mathbb{R}, \mathbb{R}^n \times \mathbb{R}^p, \mathfrak{B}_s)$ with $\mathfrak{B}_s$ defined by

$$\mathfrak{B}_s := \left\{(w, x) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^p \times \mathbb{R}^n) \mid \text{there exists } \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^q) \text{ such that } w = M\left(\frac{d}{dt}\right)\ell \text{ and } x = X\left(\frac{d}{dt}\right)\ell\right\}$$

is a state representation of $\Sigma(M)$, i.e., x satisfies the axiom of state (see [18]). Given M, a state map X for M can be computed as follows. If necessary, permute the components of w so that M is of the form

$$M = \begin{pmatrix} U \\ Y \end{pmatrix}$$

with $U \in \mathbb{R}^{q\times q}[\xi]$, $\det(U) \neq 0$, and YU^{-1} a proper rational matrix (such a permutation exists and can be found with the procedure outlined in [18]). It can be shown that the set of polynomial row vectors

$$\{r \in \mathbb{R}^{1\times q}[\xi] \mid rU^{-1} \text{ is strictly proper}\} \tag{6.4}$$

is a vector space over $\mathbb{R}$. Moreover, $X(d/dt)$ is a state map for M if and only if the rows of $X(\xi)$ span (6.4); it is a minimal state map if and only if the rows of $X(\xi)$ form a basis for (6.4). In this case the number of rows of X is called the McMillan degree of (the system defined by) M. This number is denoted by $n(M)$, or simply by n. It can be shown that $n(M) = \deg(\det(U))$.

The following theorem is the main result of this section. It serves as the theoretical background for our J-spectral factorization algorithm. We distinguish between the case that the Z to be factored is not unimodular, and the case that it is unimodular. First, we treat the non-unimodular case.

Theorem 6.1. *Assume that* $\det(Z)$ *has at least one root and that* Z *is semisimple. Assume that* (6.2) *holds and that* $\det(T_\Phi) \neq 0$. *Then we have* $n\,(= n(M)) = \frac{1}{2}\deg(\det(Z))$. *Let* $X \in \mathbb{R}^{n\times q}[\xi]$ *define a minimal state map for* M. *Then the matrix* $W \in \mathbb{C}^{n\times n}$ *defined by*

$$W := \begin{pmatrix} X(\lambda_1)V_1 & X(\lambda_2)V_2 & \cdots & X(\lambda_k)V_k \end{pmatrix}$$

is nonsingular. Furthermore, there exists $H \in \mathbb{R}^{q\times q}[\xi]$ *such that*

$$M^T(\zeta)J_{p_+,p_-}M(\eta) - (\zeta+\eta)X^T(\zeta)(W^*)^{-1}T_\Phi W^{-1}X(\eta) = H^T(\zeta)J_{q-p_-,p_-}H(\eta). \tag{6.5}$$

Moreover, for any $H \in \mathbb{R}^{q\times q}[\xi]$ *such that* (6.5) *holds we have* $\det(H) \neq 0$ *and* MH^{-1} *is a matrix of proper rational functions. If, in addition,* $T_\Phi < 0$, *then for any* H *such that* (6.5) *holds,* H *and* $\mathrm{col}(M_-, H_+)$ *are Hurwitz.*

Proof. The proof of this result uses intermediate results on dissipativeness and duality of QDFs. These topics are discussed in Sections 7 and 8. In Section 9 a proof of Theorem 6.1 is given. ■

Note that any H as in the above theorem immediately yields a regular, stabilizing, Hurwitz *J*-spectral factor. Indeed, by taking $\zeta = -\xi$ and $\eta = \xi$ in (6.5) we obtain the *J*-spectral factorization $M^T(-\xi)J_{p_+,p_-}M(\xi) = H^T(-\xi)J_{q-p_-,p_-}H(\xi)$. The importance of the theorem is that formula (6.5) can be used in an algorithm to calculate a desired *J*-spectral factorization. Below we list the steps of this algorithm. In the remainder, for a given polynomial matrix P, $\tilde{P}$ denotes its coefficient matrix. As before, σ_{R} and σ_{D} denote the right and downward shift operators.

Algorithm 6.1.

Input: A $p \times q$ real polynomial matrix M, and positive integers p_- and p_+ such that $p_- + p_+ = p$. It is assumed that $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. Also, it is assumed that $Z(\xi) := M^T(-\xi)J_{p_-,p_+}M(\xi)$ is semisimple, and $\det(Z(i\omega)) \neq 0$ for all $\omega \in \mathbb{R}$.

Output: A $q \times q$ polynomial matrix H such that:

1. $M^T(-\xi)J_{p_+,p_-}M(\xi) = H^T(-\xi)J_{q-p_-,p_-}H(\xi)$;
2. H is Hurwitz;
3. MH^{-1} is a matrix of proper rational functions;
4. $\begin{pmatrix} M_- \\ H_+ \end{pmatrix}$ is Hurwitz (here, M_- is the polynomial matrix consisting of the last p_- rows of M, and H_+ is the polynomial matrix consisting of the first $q - p_-$ rows of H).

Step 1. Check whether there exists $\varepsilon > 0$ such that

$$\mathrm{sign}(M^T(-i\omega)J_{p_+,p_-}M(i\omega) + \varepsilon M^T(-i\omega)M(i\omega)) = (p_-, 0, q-p_-)$$

for all $\omega \in \mathbb{R}$. If yes, go to Step 2, otherwise terminate: no suitable H exists.

Step 2. Compute the distinct roots $\lambda_1, \lambda_2, \ldots, \lambda_k$ of $\det(Z)$ in $\mathbb{C}^-$, and compute matrices $V_i \in \mathbb{C}^{q\times n_i}$ of full column rank such that $\mathrm{im}(V_i) = \ker(Z(\lambda_i))$. Compute the Pick matrix

$$T_\Phi = \left(\frac{1}{\bar{\lambda}_i + \lambda_j} V_i^* M^T(\bar{\lambda}_i)J_{p_+,p_-}M(\lambda_j)V_j\right).$$

Check whether $T_\Phi < 0$. If yes, go to Step 3, otherwise terminate: no suitable H exists.

Step 3. Compute an $n \times q$ polynomial matrix X such that X is a minimal state map for M. Let $\tilde{X}$ denote the coefficient matrix of X.

Step 4. Compute the complex $n \times n$ matrix $W = (X(\lambda_1)V_1 \;\; X(\lambda_2)V_2 \;\; \cdots \;\; X(\lambda_k)V_k)$.

Step 5. The hermitian matrix

$$\tilde{M}^T J_{p_+,p_-} \tilde{M} - \sigma_{\mathrm{R}}(\tilde{X}^T (W^*)^{-1} T_\Phi W^* \tilde{X}) - \sigma_{\mathrm{D}}(\tilde{X}^T (W^*)^{-1} T_\Phi W^* \tilde{X})$$

has p_- negative eigenvalues and $q - p_-$ positive eigenvalues. Factorize it as $\tilde{H}^T J_{q-p_-,p_-} \tilde{H}$.

Step 6. Let $H(\xi) := \tilde{H} \operatorname{col}(I_q, I_q\xi, I_q\xi^2, \ldots)$.

Remark 6.4. In Lemma 8.3 it is proven that if $R \in \mathbb{R}^{(p-q)\times q}[\xi]$ is such that $RM = 0$ and $R(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$, then the strict signature condition of Step 1 in Algorithm 6.1 is equivalent to: there exists $\varepsilon > 0$ such that $R(i\omega)(J_{p_+,p_-} - \varepsilon I)R^T(-i\omega) \geq 0$ for all $\omega \in \mathbb{R}$. This condition seems in general easier to check than the original one.

We now illustrate the application of this algorithm with an example.

Example 6.1. We consider the case of a mixed sensitivity problem from Example 4.4.3 of [10], with parameters $r = 0$, $c = 1$, and $\gamma = 2$. In this example we have

$$M(\xi) := \begin{pmatrix} 1 & -1-\xi \\ -1 & 0 \\ 2 & -2\xi \end{pmatrix}$$

so $p = 3$ and $q = 2$. We take $p_- = 2$ and $p_+ = 1$, and we want to find a $J_{1,1}$-spectral factorization of

$$Z(\xi) = M^T(-\xi)J_{2,1}M(\xi) = \begin{pmatrix} -2 & -1+3\xi \\ -1-3\xi & 1+3\xi^2 \end{pmatrix}.$$

The relevant two-variable polynomial matrix Φ is given by

$$M^T(\zeta)J_{2,1}M(\eta) = \begin{pmatrix} -2 & -1+3\eta \\ -1+3\zeta & 1+\zeta+\eta-3\zeta\eta \end{pmatrix}.$$

It can be verified that $Z(i\omega) + \varepsilon M^T(-i\omega)M(i\omega)$ has constant signature $(1, 0, 1)$ for $\varepsilon > 0$ sufficiently small. Also, $\det(Z(\xi)) = 3(\xi^2 - 1)$, so the only left half-plane root of $\det(Z)$ is -1, with multiplicity 1. It can also be seen that Z is semisimple. The kernel of $Z(-1)$ is spanned by $(-2 \;\; 1)^T$, and the Pick matrix is therefore the 1×1 matrix given by

$$T_\Phi = \frac{1}{-2}(-2 \quad 1)\begin{pmatrix} -2 & -4 \\ 2 & 4 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \end{pmatrix} = -2.$$

Note that T_Φ is indeed negative definite. A minimal state map for M can be chosen to be $X(\xi) = (0 \quad 1)$, so

$$W = (0 \quad 1)\begin{pmatrix} -2 \\ 1 \end{pmatrix} = 1.$$

We now compute

$$\tilde{M}^T J_{2,1}\tilde{M} - \sigma_{\mathrm{R}}(\tilde{X}^T(W^*)^{-1}T_\Phi W^*\tilde{X}) - \sigma_{\mathrm{D}}(\tilde{X}^T(W^*)^{-1}T_\Phi W^*\tilde{X})$$
$$= \begin{pmatrix} -2 & 1 & 0 & 3 \\ -1 & 1 & 0 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 3 & 0 & -3 \end{pmatrix}.$$

This matrix has $p_1 = 1$ negative eigenvalue (-6.69) and $q - p_- = 1$ positive eigenvalue (2.69). Therefore it can be factored as $\tilde{H}^T J_{1,1}\tilde{H}$, with

$$\tilde{H} := \begin{pmatrix} -0.26 & -1.37 & 0 & -0.86 \\ 1.44 & 0.94 & 0 & -1.93 \end{pmatrix}.$$

A regular, stabilizing, Hurwitz $J_{1,1}$-spectral factor $H(\xi)$ can then be computed as

$$H(\xi) = \begin{pmatrix} -0.26 & -1.37 - 0.86\xi \\ 1.44 & 0.94 - 1.93\xi \end{pmatrix}.$$

For the sake of completeness, we also treat the case when the polynomial matrix Z is unimodular. Recall that in this case the Pick matrix does not exist. We then get the following:

Theorem 6.2. *Assume that Z is unimodular, and that* (6.2) *holds. Then there exists $H \in \mathbb{R}^{q\times q}[\xi]$ such that*

$$M^T(\zeta)J_{p_+,p_-}M(\eta) = H^T(\zeta)J_{q-p_-,p_-}H(\eta). \tag{6.6}$$

Under these assumptions, any $H \in \mathbb{R}^{q\times q}[\xi]$ such that (6.6) *holds is unimodular, MH^{-1} is a matrix of constants, and* $\mathrm{col}(M_-, H_+)$ *is unimodular.*

This theorem immediately yields an algorithm to compute a regular, stabilizing, Hurwitz J-spectral factorization of the unimodular Z. Indeed, in this case the coefficient matrix $\tilde{M}^T J_{p_-,p_+}\tilde{M}$ has p_- negative eigenvalues and $q - p_-$ positive eigenvalues, and can hence be factored as $\tilde{H}^T J_{q-p_-,p_-}\tilde{H}$. The corresponding polynomial matrix $H(\xi)$ is a regular, stabilizing, Hurwitz J-spectral factor. The proof of Theorem 6.2 is given in Remark 9.1 below.

7. Dissipativity and Storage Functions

In this section we illustrate the concepts connected to dissipativity of systems that we need in our proof of Theorem 6.1 and Algorithmn 6.1. The notions of dissipativity and storage function in the context of linear differential systems and QDFs were treated extensively in [19] and [15]. We start this section with a brief review of this material. After that, we present a new result stating that the smallest storage function can be computed in terms of the Pick matrix associated with the dissipative system.

Let $M \in \mathbb{R}^{p\times q}[\xi]$ and consider the associated behavior $\mathfrak{B}(M)$, described in image representation by (6.3). Assume that $M(\lambda)$ has full column rank for every $\lambda \in \mathbb{C}$. Let Φ be a symmetric two-variable polynomial matrix, and let Q_Φ be the associated QDF. This QDF is called the *supply rate*. The system $\mathfrak{B}(M)$ is called *dissipative* with respect to the supply rate Q_Φ, if for all $w \in \mathfrak{B}(M)$ of compact support, $\int_{-\infty}^{\infty} Q_\Phi(w)\,dt \geq 0$ holds. The system $\mathfrak{B}(M)$ is called *strictly dissipative* with respect to the supply rate Q_Φ if there exists an $\varepsilon > 0$ such that, for all $w \in \mathfrak{B}(M)$ of compact support, $\int_{-\infty}^{\infty} Q_\Phi(w)\,dt \geq \varepsilon \int_{-\infty}^{\infty} \|w\|^2\,dt$ holds. A system can be effectively tested for (strict) dissipativity as follows. Define $\Phi'(\zeta,\eta) := M^T(\zeta)\Phi(\zeta,\eta)M(\eta)$. It can be shown (see Proposition 5.2 of [19]) that dissipativity of $\mathfrak{B}(M)$ with respect to Q_Φ is equivalent to $\partial\Phi'(i\omega) \geq 0$ for all $\omega \in \mathbb{R}$; strict dissipativity is equivalent to the existence of $\varepsilon > 0$ such that $\partial\Phi'(i\omega) \geq \varepsilon M^T(-i\omega)M(i\omega)$ for all $\omega \in \mathbb{R}$.

We now review the concept of storage function. A QDF Q_Ψ is a *storage function* for $(\mathfrak{B}(M), Q_\Phi)$ if

$$\frac{d}{dt} Q_\Psi(w) \leq Q_\Phi(w) \tag{7.1}$$

for all $w \in \mathfrak{B}(M)$. It can be shown (see Proposition 5.4 of [19]) that a system is dissipative with respect to a QDF if and only if there exists a storage function. It was also shown in [19] that the set of all storage functions of a given dissipative systems contains a smallest and a largest element, in the sense that there exist storage functions Q_{Ψ_-} and Q_{Ψ_+} for $(\mathfrak{B}(M), Q_\Phi)$ such that any storage fuction Q_Ψ satisfies the inequality $Q_{\Psi_-} \leq Q_\Psi \leq Q_{\Psi_+}$.

A possible interpretation of supply rate is the power going into a physical system, and a storage function can be thought of as measuring the quantity of energy which is stored instantaneously in the system. Therefore, it is to be expected that storage functions are related to the memory, i.e., the state, of a system. Indeed, if the two-variable polynomial matrix Φ is constant, equivalently, if the supply rate $Q_\Phi(w)$ is a quadratic function of w (and not of its derivatives), then it can be shown (see Theorem 5.5 of [19]) that if $\mathfrak{B}(M)$ is dissipative with respect to Q_Φ, and if Q_Ψ is a storage function for $\mathfrak{B}(M)$, then for every $X \in \mathbb{R}^{n\times q}[\xi]$ that defines a state map for $\mathfrak{B}(M)$ there exists a symmetric matrix $K \in \mathbb{R}^{n\times n}$ such that $\Psi(\zeta,\eta) = X^T(\zeta)KX(\eta)$.

In the remainder of this section we assume that the supply rate is a quadratic function of w, i.e., $Q_\Phi(w) = w^T S w$, with S a given symmetric matrix. Define $\Phi'(\zeta,\eta) := M^T(\zeta)SM(\eta)$. Again, we only treat the semisimple case: in the remainder of this section we assume that $\partial\Phi'$ is a semisimple polynomial matrix. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct roots of $\det(\partial\Phi')$ in $\mathbb{C}^-$, and let $V_i \in \mathbb{C}^{q\times n_i}$ be full column rank matrices such that $\mathrm{im}(V_i) = \ker(\partial\Phi'(\lambda_i))$. Let $T_{\Phi'}$ be the associated Pick matrix. We now prove a lemma that states that if the system $\mathfrak{B}(M)$ is strictly dissipative, then the size of the Pick matrix is equal to the McMillan degree of M:

Lemma 7.1. *Let $\mathfrak{B}(M)$ be strictly dissipative with respect to $Q_\Phi(w) = w^T Sw$. Then* $\deg(\det(\partial\Phi')) = 2n(M)$.

Proof. There exists a permutation matrix Π and polynomial matrices $U \in \mathbb{R}^{q\times q}[\xi]$ and $Y \in \mathbb{R}^{(p-q)\times q}[\xi]$ such that $\Pi M = \mathrm{col}(U, Y)$, with $G := YU^{-1}$ a proper rational matrix. For any such U we have $n(M) = \deg(\det(U))$. Define $G_\infty := \lim_{|\lambda|\to\infty} G(\lambda)$. Obviously, $U^T(-\xi)^{-1}\partial\Phi'(\xi)U(\xi)^{-1}$ is a proper rational matrix, so $\deg(\det(\partial\Phi')) \leq 2n(M)$. We show that, in fact, equality holds. Because of strict dissipativity, there exists $\varepsilon > 0$ such that for all $\omega \in \mathbb{R}$ we have $M^T(-i\omega)SM(i\omega) \geq \varepsilon M^T(-i\omega)M(i\omega)$. This implies that

$$\lim_{|\omega|\to\infty} U^T(-i\omega)^{-1}\partial\Phi'(i\omega)U^{-1}(i\omega) = M_\infty^T SM_\infty \geq \varepsilon M_\infty^T M_\infty$$

with $M_\infty := \lim_{|\omega|\to\infty} M(i\omega)U^{-1}(i\omega)$. Clearly, $M_\infty^T M_\infty = I + G_\infty^T G_\infty > 0$. This implies that $U^T(-\xi)^{-1}\partial\Phi'(\xi)U(\xi)^{-1}$ has a proper inverse, which yields our claim. ∎

We now formulate the main result of this section. Recall that Ψ_- denotes the two-variable polynomial matrix associated with the smallest storage function of the system $\mathfrak{B}(M)$ with supply rate Q_Φ.

Theorem 7.1. *Assume that $\mathfrak{B}(M)$ is strictly dissipative with respect to $Q_\Phi(w) = w^T Sw$. Let $X \in \mathbb{R}^{n\times q}[\xi]$ define a minimal state map for $\mathfrak{B}(M)$. Define $W \in \mathbb{C}^{n\times n}$ by*

$$W := \begin{pmatrix} X(\lambda_1)V_1 & X(\lambda_2)V_2 & \cdots & X(\lambda_k)V_k \end{pmatrix}.$$

Then $\det(W) \neq 0$ *and*

$$\Psi_-(\zeta,\eta) = X^T(\zeta)(W^*)^{-1}T_{\Phi'}W^{-1}X(\eta). \tag{7.2}$$

Proof. There exists $\varepsilon > 0$ such that $\partial\Phi'(i\omega) \geq \varepsilon M^T(-i\omega)M(i\omega) > 0$ for all $\omega \in \mathbb{R}$. Hence (see [19]) Ψ_- is given by

$$\Psi_-(\zeta,\eta) = \frac{\Phi'(\zeta,\eta) - H^T(\zeta)H(\eta)}{\zeta+\eta},$$

where $H \in \mathbb{R}^{q\times q}[\xi]$ is Hurwitz, and a spectral factor of $\partial\Phi'$, i.e., $\partial\Phi'(\xi) = H^T(-\xi)H(\xi)$. Since $H^T(-\xi)$ is anti-Hurwitz, the roots of $\det(H)$ are $\lambda_1, \lambda_2, \ldots, \lambda_k$, and $H(\lambda_i)V_i = 0$ $(\mathrm{i} = 1, 2, \ldots, k)$. We first prove that if $X(\xi)$ is a minimal state map for $\mathfrak{B}(M)$, then W is a nonsingular matrix. Assume, on the contrary, that $\sum_{i=1}^k X(\lambda_i)V_i a_i = 0$, $\mathrm{col}(a_1, a_2, \ldots, a_k) \neq 0$. Define ℓ by $\ell(t) := \sum_{i=1}^k e^{\lambda_i t}V_i a_i$. Then $\ell \neq 0$. We have $(X(d/dt)\ell)(t) = \sum_{i=1}^k e^{\lambda_i t}X(\lambda_i)V_i a_i$, so $(X(d/dt)\ell)(0) = 0$. Also, $H(d/dt)\ell = 0$. We now prove that this implies that $\ell = 0$, in contradiction with what has been concluded above. In order to do this, let U be such that, possibly after rearranging the components of w, $M = \mathrm{col}(U, Y)$, and YU^{-1} is proper.

Using the inequality $\partial\Phi'(i\omega) \geq \varepsilon M^T(-i\omega)M(i\omega)$, it can be shown that MH^{-1}, UH^{-1}, and HU^{-1} are proper. Consider the extended system $\mathfrak{B}(M,H)$, given in image representation by

$$\begin{pmatrix} w \\ h \end{pmatrix} = \begin{pmatrix} M(d/dt) \\ H(d/dt) \end{pmatrix} \ell. \tag{7.3}$$

Since MH^{-1} is proper, in (7.3) $\operatorname{col}(w,h)$ is in fact an input/output partition, with input h and output w (see Section 3.3 of [12]). We claim that $X(d/dt)$ is a minimal state map also for $\mathfrak{B}(M,H)$. This follows immediately from the fact that, for any polynomial row-vector $r \in \mathbb{R}^{1\times q}[\xi]$, rU^{-1} is strictly proper iff rH^{-1} is strictly proper. Thus, in the system $\mathfrak{B}(M,H)$ we have input $h = H(d/dt)\ell = 0$, and initial state $(X(d/dt)\ell)(0) = 0$. This implies that the output variable w satisfies $w = M(d/dt)\ell = 0$. From the assumption that $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$ we then conclude that $\ell = 0$, which is a contradiction. This proves that W is nonsingular.

Next we prove (7.2). First note that there exists a symmetric $n \times n$ matrix, say K_-, such that $\Psi_-(\zeta,\eta) = X^T(\zeta)K_-X(\eta)$. Thus we have

$$\Phi'(\zeta,\eta) - H^T(\zeta)H(\eta) = (\zeta+\eta)X^T(\zeta)K_-X(\eta).$$

By taking $\zeta = \bar{\lambda}_i$ and $\eta = \lambda_j$, and by premultiplying (7.4) by V_i^* and postmultiplying by V_j we get

$$V_i^*\Phi'(\bar{\lambda}_i,\lambda_j)V_j = (\bar{\lambda}_i + \lambda_j)V_i^*X^T(\bar{\lambda}_i)K_-X(\lambda_j)V_j.$$

This immediately yields $T_{\Phi'} = W^*K_-W$, equivalently, $K_- = (W^*)^{-1}T_{\Phi'}W^*$. This completes the proof of the theorem. ∎

8. Duality of Quadratic Differential Forms

We now review the notion of duality of QDFs. This concept and the related notion of a matched pair of state maps was introduced in [19].

Let $M \in \mathbb{R}^{p\times q}[\xi]$ be such that $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. As in the previous section denote by $\mathfrak{B}(M)$ the system with image representation $w = M(d/dt)\ell$. Let $S \in \mathbb{R}^{p\times p}$ be nonsingular and symmetric, and define $\Phi(\zeta,\eta) := M^T(\zeta)SM(\eta)$. In what follows we deal with the dual of the system $\mathfrak{B}(M)$. This is defined as the system with time axis $\mathbb{R}$, signal space $\mathbb{R}^q$, and with behavior $\mathfrak{B}(M)^\perp$ defined by

$$\mathfrak{B}(M)^\perp := \left\{ v \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^q) \;\middle|\; \int_{-\infty}^{\infty} w^T(t)v(t)\,dt = 0 \right.$$
$$\left. \text{for all } w \in \mathfrak{B}(M) \text{ with compact support} \right\}.$$

It can be shown that if $R \in \mathbb{R}^{(p-q)\times q}[\xi]$ is such that $RM = 0$ and $R(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$, then an image representation of the dual system is given by $v = R^T(-(d/dt))\ell'$, i.e., $\mathfrak{B}(M)^\perp = \mathfrak{B}(R^\sim)$, with $R^\sim(\xi) := R^T(-\xi)$. The latent variable ℓ' takes its values in $\mathbb{R}^{p-q}$. It can be shown that $n(M) = n(R^\sim)$, i.e., the

McMillan degrees of $\mathfrak{B}(M)$ and $\mathfrak{B}(M)^{\perp}$ are the same. Associated with the dual system we define a QDF $\Phi^{\perp}(\zeta,\eta) := R(-\zeta)S^{-1}R^{T}(-\eta)$, and we say that $\Phi^{\perp}$ is the *dual* of *phi*. The main result of this section states that, under certain conditions, the Pick matrices associated with Φ and $\Phi^{\perp}$ coincide. As a first step toward this result, we prove that $\partial\Phi$ and $\partial\Phi^{\perp}$ have the same roots. In fact:

Lemma 8.1. *Let p and q be positive integers such that $p > q$. Let $M \in \mathbb{R}^{p\times q}[\xi]$ be such that $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$, and let $R \in \mathbb{R}^{(p-q)\times q}[\xi]$ be such that $RM = 0$ and $R(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Let $S \in \mathbb{R}^{p\times p}$ be nonsingular and symmetric and let $\Phi(\zeta,\eta) := M^{T}(\zeta)SM(\eta)$ and $\Phi^{\perp}(\zeta,\eta) := R(-\zeta)S^{-1}R^{T}(-\eta)$. Then for all $\lambda \in \mathbb{C}$ we have* $\det(\partial\Phi(\lambda)) = 0$ *iff* $\det(\partial\Phi^{\perp}(\lambda)) = 0$. *For any such λ we have* $\dim(\ker(\partial\Phi(\lambda))) = \dim(\ker(\partial\Phi^{\perp}(\lambda)))$.

Proof. It is easily seen that for all $\lambda \in \mathbb{C}$ we have $\ker(R(\lambda)) = \operatorname{im}(M(\lambda))$ and $\operatorname{im}(R^{T}(\lambda)) = \ker(M^{T}(\lambda))$. Now assume that $\det(\partial\Phi(\lambda)) = 0$. Then there exists $0 \neq v \in \mathbb{C}^{q}$ such that $\partial\Phi(\lambda)v = 0$, so $SM(\lambda) \in \ker(M^{T}(-\lambda))$. Consequently, there exists $v' \in \mathbb{C}^{p-q}$ such that $SM(\lambda)v = R^{T}(-\lambda)v'$. Note that, since $M(\lambda)$ has full column rank, $v' \neq 0$. Since also $SM(\lambda)v \in \ker(R(\lambda))S^{-1}$, we have $R(\lambda)S^{-1}R^{T}(-\lambda)v' = 0$, so $\det(\partial\Phi^{\perp}(\lambda)) = 0$. Because of symmetry, the converse implication also holds. The fact that the dimensions are equal follows from the equality $R^{T}(-\lambda)\ker(\partial\Phi^{\perp}(\lambda)) = SM(\lambda)\ker(\partial\Phi(\lambda))$. ■

Our next lemma states that if Φ and $\Phi^{\perp}$ are dual, then the signature along the imaginary axis of $\partial\Phi^{\perp}$ is completely determined by that of $\partial\Phi$:

Lemma 8.2. *Let R, M, S, Φ, and $\Phi^{\perp}$ be as in Lemma* 8.1. *Then for all $\omega \in \mathbb{R}$ such that* $\det(\partial\Phi(i\omega)) \neq 0$ *we have*

$$\operatorname{sign}(\partial\Phi(i\omega)) + \operatorname{sign}(\partial\Phi^{\perp}(i\omega)) = \operatorname{sign}(S).$$

Proof. Let ω be such that $\det(\partial\Phi(i\omega)) \neq 0$. Define $W(i\omega) \in \mathbb{C}^{p\times p}$ by

$$W(i\omega) := \begin{pmatrix} M^{T}(-i\omega)S \\ R(i\omega) \end{pmatrix}.$$

We claim that $W(i\omega)$ is nonsingular. Indeed,

$$W(i\omega)(M(i\omega) \quad R^{T}(-i\omega)) = \begin{pmatrix} \partial\Phi(i\omega) & M^{T}(-i\omega)SR^{T}(-i\omega) \\ 0 & R(i\omega)R^{T}(-i\omega) \end{pmatrix}.$$

The claim follows from the fact that the right-hand side of this equation is a non-singular matrix. The proof of the lemma is then completed by noting that

$$W(i\omega)SW^{T}(-i\omega) = \begin{pmatrix} \partial\Phi(i\omega) & 0 \\ 0 & \partial\Phi^{\perp}(i\omega) \end{pmatrix}.$$

In our next lemma we apply this result to the case in which S is a signature matrix. It turns out that, along the imaginary axis, $\partial\Phi$ satisfies a strict signature condition with negativity equal to the negativity of the given signature matrix iff

$\partial\Phi^{\perp}$ is strictly positive definite. Under this condition, the degrees of the determinants of $\partial\Phi$ and $\partial\Phi^{\perp}$ are the same.

Lemma 8.3. *Let R and M be as in Lemma* 8.1. *Let p_-* and *p_+ be positive integers such that $p_- + p_+ = p$. Define $\Phi(\zeta,\eta) := M^T(\zeta)J_{p_+,p_-}M(\eta)$ and $\Phi^{\perp}(\zeta,\eta) := R(-\zeta)J_{p_+,p_-}R^T(-\eta)$. Let $\varepsilon > 0$. Then the following two statement are equivalent*:

(i) $\operatorname{sign}(\partial\Phi(i\omega) + \varepsilon M^T(-i\omega)M(i\omega)) = (p_-, 0, q - p_-)$ *for all* $\omega \in \mathbb{R}$;
(ii) $\partial\Phi^{\perp}(i\omega) > \varepsilon R(i\omega)R^T(-i\omega)$ *for all* $\omega \in \mathbb{R}$.

Assume that there exists $\varepsilon > 0$ such that this condition holds. Then we have $\deg(\det(\partial\Phi)) = \deg(\det(\partial\Phi^{\perp}))$. *Consequently, $\partial\Phi$ is semisimple iff $\partial\Phi^{\perp}$ is semisimple.*

Proof. Define $S_\varepsilon := J_{p_+,p_-} + \varepsilon I$. For all $\varepsilon > 0$, $\varepsilon \neq 1$, and for all $\omega \in \mathbb{R}$ we have

$$\operatorname{sign}(M^T(-i\omega)S_\varepsilon M(i\omega)) + \operatorname{sign}(R(i\omega)S_\varepsilon^{-1}R^T(-i\omega)) = \operatorname{sign}(S_\varepsilon).$$

As a consequence, for all such ε we have $n_-(M^T(-i\omega)S_\varepsilon M(i\omega)) \le n_-(S_\varepsilon) \le p_-$ for all ω. Now assume that (i) holds. Then $\varepsilon \neq 1$, for otherwise $S_\varepsilon \geq 0$, which would imply $M^T(-i\omega)S_\varepsilon M(i\omega) \geq 0$. Hence we get $n_-(S_\varepsilon) = p_-$, so $\varepsilon < 1$ and $\operatorname{sign}(S_\varepsilon) = (p_-, 0, p_+)$. This implies that for all $\omega \in \mathbb{R}$ we have $\operatorname{sign}(R(i\omega)S_\varepsilon^{-1}R^T(-i\omega)) = (0, 0, p - q)$. By noting that $S_\varepsilon^{-1} = (1/(1-\varepsilon^2)) \cdot (J_{p_+,p_-} - \varepsilon I)$, we conclude (ii). To prove the converse, if (ii) holds, then we must have $\varepsilon < 1$. Reversing the previous argument then yields (i).

We now prove the second part of the lemma. First note that if (ii) holds, then the system $\mathfrak{B}(R^{\sim})$ is strictly dissipative with respect to the supply rate $v^T J_{p_+,p_-} v$. From Lemma 7.1 it then follows that $\deg(\det(\partial\Phi^{\perp})) = 2n(R^{\sim})$. There exist permutation matrices Π and Π' such that $R = (P \quad Q)\Pi$, with $Q^{-1}P$ a proper rational matrix, and $M = \Pi' \operatorname{col}(Y, U)$, with YU^{-1} a proper rational matrix. In that case we have $n(M) = \deg(\det(U))$. Define $M_\infty := \lim_{|\lambda|\to\infty} M(\lambda)U^{-1}(\lambda)$, and $R_\infty := \lim_{|\lambda|\to\infty} Q^{-1}(\lambda)R(\lambda)$. Note that R_∞ has full row rank. Now, from (ii) it is easily seen that $R_\infty J_{p_+,p_-} R_\infty^T \geq \varepsilon R_\infty R_\infty^T > 0$, so $R_\infty J_{p_+,p_-} R_\infty^T$ is nonsingular. From this it follows that the matrix

$$\begin{pmatrix} M_\infty^T \\ R_\infty J_{p_+,p_-} \end{pmatrix}$$

is nonsingular. Indeed, we have that

$$\begin{pmatrix} M_\infty^T \\ R_\infty J_{p_+,p_-} \end{pmatrix} (M_\infty \quad R_\infty^T) = \begin{pmatrix} M_\infty^T M_\infty & 0 \\ R_\infty J_{p_+,p_-} M_\infty & R_\infty J_{p_+,p_-} R_\infty^T \end{pmatrix}$$

is nonsingular. We claim that $M_\infty^T J_{p_+,p_-} M_\infty$ is also nonsingular. This follows from the matrix equality

$$\begin{pmatrix} M_\infty^T \\ R_\infty J_{p_+,p_-} \end{pmatrix} J_{p_+,p_-} (M_\infty \quad J_{p_+,p_-} R_\infty^T) = \begin{pmatrix} M_\infty^T J_{p_+,p_-} M_\infty & 0 \\ 0 & R_\infty J_{p_+,p_-} R_\infty^T \end{pmatrix}.$$

From the fact that $M_\infty^T J_{p_+,p_-} M_\infty$ is nonsingular, it finally follows that the proper rational matrix $(U^T(-\xi))^{-1} M^T(-\xi) J_{p_+,p_-} M(\xi) U(\xi)^{-1}$ has a proper inverse. This implies that $\deg(\det(\partial\Phi)) = 2\deg(U) = 2n(M)$. By recalling that $n(M) = n(R^\sim)$ the claim on the degrees follows. The remaining equivalence follows by combining the fact that the degrees are equal with the fact that for all roots λ we have $\dim(\ker(\partial\Phi(\lambda))) = \dim(\ker(\partial\Phi^\perp(\lambda)))$. ■

We now apply the previous lemmas to obtain the main result of this section. Again assume that S is a signature matrix. The next theorem states that if $\partial\Phi$ satisfies a strict signature condition along the imaginary axis, with negativity equal to the negativity of the given signature matrix, then the Pick matrices of Φ and $\Phi^\perp$ coincide.

Theorem 8.1. *Let* R, M, Φ, and $\Phi^\perp$ *be as in Lemma* 8.3. *Assume that there exists* $\varepsilon > 0$ *such that for all* $\omega \in \mathbb{R}$ *we have* $\operatorname{sign}(\partial\Phi(i\omega) + \varepsilon M^T(-i\omega)M(i\omega)) = (p_-, 0, q - p_-)$ *and assume that* $\partial\Phi$ *is semisimple. Let* $\lambda_1, \lambda_2, \ldots, \lambda_k$ *be the distinct roots of* $\det(\partial\Phi)$ *in* $\mathbb{C}^-$, *and let* $V_i \in \mathbb{C}^{q\times n_i}$ *be full column rank matrices such that* $\operatorname{im}(V_i) = \ker(\partial\Phi(\lambda_i))$ $(i = 1, 2, \ldots, k)$. *Then there exist full column rank matrices* $V_i' \in \mathbb{C}^{(p-q)\times n_i}$ *such that*

$$J_{p_+,p_-} R^T(-\lambda_i) V_i' = M(\lambda_i) V_i. \tag{8.1}$$

For such matrices V_i' *we have* $\operatorname{im}(V_i') = \ker(\partial\Phi^\perp(\lambda_i))$ $(i = 1, 2, \ldots, k)$. *Furthermore, for* $i, j = 1, 2, \ldots, k$ *we have*

$$\frac{V_i^* \Phi(\bar{\lambda}_i, \lambda_j) V_j}{\bar{\lambda}_i + \lambda_j} = \frac{V_i^* \Phi^\perp(\bar{\lambda}_i, \lambda_j) V_j}{\bar{\lambda}_i + \lambda_j},$$

i.e., the corresponding Pick matrices T_Φ *and* $T_{\Phi^\perp}$ *coincide.*

Proof. The strict signature condition on $\partial\Phi$, together with the assumption that $\partial\Phi$ is semisimple, implies that the sets of distinct roots of $\det(\partial\Phi)$ and $\det(\partial\Phi^\perp)$ are the same, and equal to $\{\lambda_1, \lambda_2, \ldots, \lambda_k, -\lambda_1, -\lambda_2, \ldots, -\lambda_k\}$, with $\{\lambda_1, \lambda_2, \ldots, \lambda_k\} \subset \mathbb{C}^-$. Furthermore, the multiplicity n_i of λ_i as a root of $\det(\partial\Phi)$ is equal to the multiplicity of λ_i as a root of $\det(\partial\Phi^\perp)$; this number n_i is equal to the dimension of $\ker(\partial\Phi(\lambda_i))$, which is equal to that of $\ker(\partial\Phi^\perp(\lambda_i))$.

Let $V_i \in \mathbb{C}^{q\times n_i}$ be a full column rank matrix such that $\operatorname{im}(V_i) = \ker(\partial\Phi(\lambda_i))$. Then $\operatorname{im}(J_{p_+,p_-} M(\lambda_i) V_i) \subset \ker(M^T(-\lambda_i)) = \operatorname{im}(R^T(-\lambda_i))$. Hence there exists $V_i' \in \mathbb{C}^{(p-q)\times n_i}$ such that

$$J_{p_+,p_-} M(\lambda_i) V_i = R^T(-\lambda_i) V_i'. \tag{8.2}$$

Since $M(\lambda_i)$ has full column rank, V_i' has full column rank, and since $R(\lambda_i) J_{p_+,p_-} R^T(-\lambda_i) V_i' = 0$, we have $\operatorname{im}(V_i') = \ker(\partial\Phi^\perp(\lambda_i))$. The claim that the Pick matrices coincide then follows immediately from (8.2). ■

The notion of a *matched pair of state maps* is connected to that of duality of QDFs. We now review this concept. Assume that M and R are as above. Let

$X \in \mathbb{R}^{n\times q}$ define a minimal state map for $\mathfrak{B}(M)$. It can be shown (see [19]) that there exists a polynomial matrix $X' \in \mathbb{R}^{n\times(p-q)}[\xi]$ defining a minimal state map for the dual system $\mathfrak{B}(R^{\sim})$, such that for all infinitely often differentiable ℓ, ℓ' it holds that

$$\frac{d}{dt}\left(X'\left(\frac{d}{dt}\right)\ell'\right)^T X\left(\frac{d}{dt}\right)\ell = \left(R^T\left(-\frac{d}{dt}\right)\ell'\right)^T M\left(\frac{d}{dt}\right)\ell. \tag{8.3}$$

A pair (X, X') of state maps satisfying (8.3) is called a *matched pair* of state maps; equivalently, we say that X and X' are matched. Note that (8.3) is equivalent to $(\zeta+\eta)X'^T(\zeta)X(\eta) = R(-\zeta)M(\eta)$. Our next theorem states that the Pick matrix of Φ can be computed in terms of a matched pair of state maps of M and $R^{\sim}$.

Theorem 8.2. *Let R, M, Φ, and $\Phi^{\perp}$ be as in Lemma 8.3. Assume there exists $\varepsilon > 0$ such that for all $\omega \in \mathbb{R}$ we have $\operatorname{sign}(\partial\Phi(i\omega) + \varepsilon M^T(-i\omega)M(i\omega)) = (p_-, 0, q-p_-)$ and assume that $\partial\Phi$ is semisimple. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ be the distinct roots of $\det(\partial\Phi)$ in $\mathbb{C}^-$, and let $V_i \in \mathbb{C}^{q\times n_i}$ be full column rank matrices such that $\operatorname{im}(V_i) = \ker(\partial\Phi(\lambda_i))$ $(i = 1, 2, \ldots, k)$. Let $V_i' \in \mathbb{C}^{(p-q)\times n_i}$ be full column rank matrices such that (8.1) holds. Then for any matched pair of state maps (X, X') of M and $R^{\sim}$ it holds that*

$$W'^* W = T_\Phi, \tag{8.4}$$

where W and W' are the complex $n\times n$ matrices defined by

$$W := (X(\lambda_1)V_1 \quad X(\lambda_2)V_2 \cdots X(\lambda_k)V_k),$$

$$W' := (X'(\lambda_1)V_1' \quad X'(\lambda_2)V_2' \cdots X'(\lambda_k)V_k').$$

Proof. In the equality $(\zeta+\eta)X'^T T(\zeta)X(\eta) = R(-\zeta)M(\eta)$, take $\zeta = \bar{\lambda}_i$ and $\eta = \lambda_j$. Next, pre- and postmultiply by $V_i'^*$ and V_j to obtain

$$(\bar{\lambda}_i + \lambda_j)V_i'^* X'^T(\bar{\lambda}_i)X(\lambda_j)V_j = V_i'^* R(-\bar{\lambda}_i)M(\lambda_j)V_j.$$

Finally, by (8.1) we have $V_i'^* R(-\bar{\lambda}_i) = V_i^* M(\bar{\lambda}_i)J_{p_+,p_-}$, which yields

$$V_i'^* X'^T(\bar{\lambda}_i)X(\lambda_j)V_j = \frac{V_i^* M(\bar{\lambda}_i)J_{p_+,p_-}M(\lambda_j)V_j}{\bar{\lambda}_i+\lambda_j} = \frac{V_i^*\Phi(\bar{\lambda}_i,\lambda_j)V_j}{\bar{\lambda}_i+\lambda_j}.$$

9. Proof of Theorem 6.1

In this section we give a proof of Theorem 6.1. The proof is based on the results on dissipativity illustrated in Section 7, and on the duality results that were derived in Section 8. Again let p and q be positive integers such that $p > q$, and let $M \in \mathbb{R}^{p\times q}[\xi]$ be such that $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. Let $R \in \mathbb{R}^{(p-q)\times q}[\xi]$ be such that $RM = 0$, and $R(\lambda)$ has full row rank for all $\lambda \in \mathbb{C}$. Recall from Section 8 that $\mathfrak{B}(R^{\sim})$ is the dual of $\mathfrak{B}(M)$. Recall also that if $\mathfrak{B}(R^{\sim})$ is dissipative with respect to a given supply rate, and if $\Psi(\zeta,\eta)$ yields a storage function, then for each minimal state map $X'(\xi)$ of $\mathfrak{B}(R^{\sim})$ there exists $K = K^T \in$

$\mathbb{R}^{n\times n}$ such that $\Psi(\zeta,\eta) = X'^T(\zeta)KX'(\eta)$. As before, we denote by Ψ_- the two-variable polynomial matrix associated with the smallest storage function. Our proof of Theorem 6.1 is heavily based on the following result from Lemma 9.3 of [16]:

Proposition 9.1. *Let p_- and p_+ be positive integers such that $p_- + p_+ = p$. Assume that $\mathfrak{B}(R^\sim)$ is strictly dissipative with respect to the supply rate $v^T J_{p_+,p_-} v$ (equivalently: there exists $\varepsilon > 0$ such that $R(i\omega)J_{p_+,p_-}R^T(-i\omega) \geq \varepsilon R(i\omega)R^T(-i\omega)$). Assume that $n(M) \neq 0$, and let (X, X') be a matched pair of minimal state maps of M and $R^\sim$. Let $K_- = K_-^T \in \mathbb{R}^{n\times n}$ be such that $\Psi_-(\zeta,\eta) = X'^T(\zeta)K_-X'(\eta)$. If $\det(K_-) \neq 0$, then there exists $H \in \mathbb{R}^{q\times q}[\xi]$ such that*

$$M^T(\zeta)J_{p_+,p_-}M(\eta) - (\zeta+\eta)X^T(\zeta)K_-^{-1}X(\eta) = H^T(\zeta)J_q - p_-,p_-H(\eta). \tag{9.1}$$

Under these assumptions, for every H such that (9.1) holds we have $\det(H) \neq 0$ and the rational matrix MH^{-1} is proper. If, in addition, $K_- < 0$, then for any H such that (9.1) holds, $\mathrm{col}(M_-, H_+)$ and H are Hurwitz. Here M_- and H_+ are obtained by taking the last p_- and the first $q - p_-$ rows of M and H, respectively.

Proof of Theorem 6.1. We have $\Phi(\zeta,\eta) = M^T(\zeta)J_{p_+,p_-}M(\eta)$ and $Z = \partial\Phi$. Now assume that (6.2) holds. Then, according to Lemma 8.3, $\mathfrak{B}(R^\sim)$ is strictly dissipative with respect to $v^T J_{p_+,p_-} v$. Also, $\deg(\det(\partial\Phi)) = \deg(\det(\partial\Phi^\perp))$. Applying Lemma 7.1 to the system $\mathfrak{B}(R^\sim)$, we get $\deg(\det(\partial\Phi^\perp)) = 2n(R^\sim) = 2n(M)$. This yields $\frac{1}{2}\deg(\det(Z)) = n(M)$ as claimed.

Let X' be a minimal state map of $R^\sim$ such that (X, X') is a matched pair. Also, let V_i' be full column rank matrices such that (8.1) holds. Define

$$W' := \begin{pmatrix} X'(\lambda_1)V_1' & X'(\lambda_2)V_2' & \cdots & X'(\lambda_k)V_k' \end{pmatrix}.$$

According to Theorem 8.2, $W'^* W = T_\Phi$. Hence, if $\det(T_\Phi) \neq 0$, then W is nonsingular.

Now apply Theorem 7.1 to the system $\mathfrak{B}(R^\sim)$. The theorem says that $\Psi_-(\zeta,\eta)$, the two-variable polynomial matrix associated with the smallest storage function of $\mathfrak{B}(R^\sim)$, is equal to

$$\Psi_-(\zeta,\eta) = X'^T(\zeta)(W'^*)^{-1}T_{\Phi^\perp}(W')^{-1}X'(\eta),$$

i.e., we can take $K_- = (W'^*)^{-1}T_{\Phi^\perp}(W')^{-1}$. Since by Theorem 8.1 $T_\Phi = T_{\Phi^\perp}$, we conclude that $K_-^{-1} = W'T_\Phi^{-1}W'^*$. Since also from Theorem 8.2 $W'^* = T_\Phi W^{-1}$, we obtain

$$K_-^{-1} = (W^*)^{-1}T_\Phi T_\Phi^{-1}T_\Phi W^{-1} = (W^*)^{-1}T_\Phi W^{-1}.$$

Plugging this expression for K_-^{-1} in (9.1), the existence of H such that (6.5) holds follows. The remaining statements follow from the fact that $K_- < 0$ iff $T_\Phi < 0$. ■

Remark 9.1. In Proposition 9.1 it is assumed that the McMillan degree $n(M)$ of M is not zero. If $n(M)(= n(R^\sim)) = 0$, then it can be shown that the following holds: Let p_- and p_+ be positive integers such that $p_- + p_+ = p$. Assume that

$\mathfrak{B}(R^{\sim})$ is strictly dissipative with respect to the supply rate $v^T J_{p_+,p_-} v$. Then there exists $H \in \mathbb{R}^{q\times q}[\xi]$ such that

$$M^T(\zeta) J_{p_+,p_-} M(\eta) = H^T(\zeta) J_{q-p_-,p_-} H(\eta). \tag{9.2}$$

Under these assumptions, any H such that (9.2) holds is unimodular, MH^{-1} is a constant matrix, and $\mathrm{col}(M_-, H_+)$ is unimodular. Note that this immediately yields a proof of Theorem 6.2, dealing with the case that Z is unimodular.

10. Conclusions

In this paper we have developed new algorithms for J-spectral factorization of polynomial matrices. These algorithms are all based on the calculus of two-variable polynomial matrices and their associated quadratic differential forms. The algorithms developed in the first part of this paper originate from the idea of associating with the (one-variable) polynomial matrix Z to be factored, a two-variable polynomial matrix Φ such that $\partial\Phi = Z$. The problem of computing J-spectral factors for Z is then translated into the problem of factoring a *constant* matrix that can be obtained from the coefficient matrix of Φ. In Section 4 this problem is formulated in terms of solvability of AREs whose coefficients can be obtained immediately from the coefficient matrix of Z, without any intermediate state space realization step. An interesting feature of this approach is that different choices of Φ yield different algorithms and different AREs; this aspect can probably be exploited in order to improve the numerical stability and the computational efficiency of the procedures. In Section 5 the results of Section 4 are applied to the special case in which the polynomial matrix to be factored is positive semidefinite on the imaginary axis. In the second part of the paper, starting in Section 6, a different type of algorithm is introduced, which is especially aimed at computing J-spectral factors for polynomial matrices appearing in the context of the polynomial approach to H_∞ control. A crucial role in this algorithm is played by the Pick matrix associated with the two-variable polynomial matrix Φ. The algorithm is based on several original results connected with dissipativity and duality of QDFs which are of independent interest.

References

[1] B. D. O. Anderson, K. L. Hitz, and N. D. Diem, Recursive algorithm for spectral factorization, *IEEE Trans. Circuits and Systems*, vol. 21, no. 6, pp. 453–464, Nov. 1976.

[2] A. E. Barabanov, Canonical matrix factorization and polynomial Riccati equations, *European J. Control*, vol. 3, no. 1, pp. 47–67, 1997.

[3] F. M. Callier, On polynomial spectral factorization by symmetric factor extraction, *IEEE Trans. Automat Control*, vol. 30, pp. 453–464, 1985.

[4] W. A. Coppel, *Linear Systems*, Notes in Pure Mathematics, vol. 6, Australian National University, Canberra, 1972.

[5] J. C. Doyle, B. A. Francis, and A. R. Tannenbaum, *Feedback Control Theory*, McMillan, New York, 1992.

[6] G. H. Golub and C. F. Van Loan, *Matrix Computations*, The Johns Hopkins University Press, Baltimore, MD, 1983.
[7] J. Ježek and V. Kučera, Efficient algorithm for matrix spectral factorization, *Automatica*, vol. 21, no. 6, pp. 663–669, 1985.
[8] H. Kwakernaak, The polynomial approach to H_∞-optimal regulation, in *H_∞-Control Theory*, E. Mosca and L. Pandolfi, eds., Lecture Notes in Mathematics 1496, Springer-Verlag, Berlin, 1991.
[9] H. Kwakernaak, and M. Šebek, Polynomial J-spectral factorization, *IEEE Trans. Automat Control*, vol. 39, no. 2, pp. 315–328, 1994.
[10] G. Meinsma, Frequency Domain Methods in $\mathscr{H}_\infty$ Control, Doctoral Dissertation, University of Twente, 1993.
[11] G. Meinsma, Polynomial solutions to H_∞ problems, *Internat. J. Robust Nonlinear Control*, vol. 4, pp. 323–351, 1994.
[12] J. W. Polderman and J. C. Willems, *Introduction to Mathematical Systems Theory: A Behavioral Approach*, Texts in Applied Mathematics 26, Springer-Verlag, New York, 1997.
[13] A. C. M. Ran and L. Rodman, Factorization of matrix polynomials with symmetries, IMA Preprint Series, no. 993, 1992.
[14] P. Rapisarda and J. C. Willems, State maps for linear systems, *SIAM J. Control Optim.*, vol. 35, no. 3, pp. 1053–1091, 1997.
[15] H. L. Trentelman and J. C. Willems, Every storage function is a state function, *Systems Control Lett.*, vol. 32, pp. 249–259, 1997.
[16] H. L. Trentelman and J. C. Willems, H_∞-control in a behavioral context: the full information case, *IEEE Trans. Automat Control*, to appear.
[17] R. A. B. van der Geest and H. L. Trentelman, The Kalman–Yakubovich–Popov lemma in a behavioural framework, *System Control Lett.*, vol. 32, pp. 283–290, 1997.
[18] J. C. Willems, Models for dynamics, *Dynamics Reported*, vol. 2, pp. 171–269, 1989.
[19] J. C. Willems and H. L. Trentelman, On quadratic differential forms, *SIAM J. Control Optim.*, vol. 36, no. 5, pp. 1703–1749, 1998.
[20] D. C. Youla and M. Saito, Interpolation with positive-real functions, *J. Franklin Inst.*, vol. 284, pp. 77–108, 1967.