Palindromic polynomials, time-reversible systems, and conserved quantities*

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Abstract

The roots of palindromic and antipalindromic polynomials can be grouped in pairs $(\lambda, 1/\lambda)$. A polynomial with such root pattern is palindromic/antipalindromic if, in addition, it has a root at 1 of an even/odd multiplicity. The result has applications in system theory: 1) any kernel representation of a discrete-time, time-reversible, scalar, autonomous LTI system is either palindromic or antipalindromic. (Similar statement holds for systems with inputs.) 2) LTI systems with palindromic or antipalindromic kernel representations have nontrivial conserved quantities.

Key words: palindromic polynomials, time-reversible systems, conserved quantities, behaviors.

1 Introduction

Links between patterns of the roots and coefficients of polynomials have been extensively studied in the context of dynamical systems and control. A famous result of this type is the Routh–Hurwitz stability test, which allows to check the stability of a single input single output (SISO) linear time-invariant system without computing its poles, i.e., by a *finite* number of operations on the coefficients of a differential equation representation of the system.

We study the root location of palindromic and antipalindromic polynomials, i.e., polynomials whose coefficients are respectively symmetric and antisymmetric with respect to the middle coefficient. It turns out that an autonomous discrete-time linear time-invariant (LTI) system defined by a difference equation whose coefficients are palindromic or antipalindromic is time-reversible in the sense that any trajectory of that system reversed in time is also a trajectory of the system. The continuous-time analogue of the palindromic and antipalindromic polynomials are even and odd polynomials.

Furthermore, we show that time-reversible systems possess conserved quantities. These are quadratic functionals of the system variables that remain constant in time along any trajectory of the system. Time-reversible systems have been studied by Fagnani and Willems [FW91]. In this paper, we give more details (see, Theorems 13 and 14 in Section 4) about the structure of scalar autonomous and SISO time-reversible systems.

For both palindromic and antipalindromic polynomials a root λ has a corresponding root $1/\lambda$ of the same multiplicity. The property that distinguishes palindromic from antipalindromic polynomials is the multiplicity of the root at 1. In the case of simple roots, a palindromic polynomial has no root at 1, while an antipalindromic polynomial has a root at 1. The system theoretic implication of this fact is that the multiplicity of the root at 1 determines the coefficient pattern of a difference equation representation of a time-reversible system.

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$\mathbb{R}$	real numbers, $\mathbb{C}$ — complex numbers, $\mathbb{Z}$ — integers
$\Re(\lambda)$	real part of λ, $\Im(\lambda)$ — imaginary part of λ
$\overline{\lambda}$	complex conjugate of λ, $\mathbf{i} := \sqrt{-1}$ — the imaginary unit
$\mathbb{R}[\xi]$	ring of polynomials with real coefficients in the indeterminate ξ
$\mathbb{R}_s[\zeta,\eta]$	space of symmetric two-variable polynomials in the indeterminates ζ and η
$\operatorname{col}(p_0, p_1, \dots, p_n)$	the column vector with elements $p_0, p_1, \dots, p_n$
$\operatorname{diag}(a_0, a_1, \dots, a_n)$	the diagonal matrix with diagonal elements $a_0, a_1, \dots, a_n$
$\mathbb{R}^{\mathbb{Z}}$	the set of functions mapping $\mathbb{Z}$ to $\mathbb{R}$ (i.e., scalar sequences)

Table 1: Notation.

2 Palindromic and antipalindromic polynomials

In this section, we study the root location of palindromic and antipalindromic polynomials.

Definition 1 (Palindromic and antipalindromic polynomials)**.** A polynomial $p \in \mathbb{R}[\xi]$ of degree n

$$p(\xi) = p_0 + p_1\xi + \cdots + p_n\xi^n, \qquad p_n \neq 0$$

is palindromic if its coefficients p_0, p_1, ..., p_n form a palindrome, i.e.,

$$p_i = p_{n-i}, \qquad \text{for } i = 0, 1, \dots, n.$$

A polynomial p is antipalintromic if $p_i = -p_{n-i}$, for $i = 0, 1, \dots, n$.

Our main result is stated in the following theorem.

Theorem 2 (Root location of palindromic and antipalindromic polynomials)**.** *The polynomial $p \in \mathbb{R}[\xi]$ is palindromic/antipalindromic if and only if*

1. *every root $\lambda \in \mathbb{C}$ of p is either on the unit circle or p has a root $1/\lambda$ with the same multiplicity as λ, and*
2. *$1 + \mathbf{i}0$ is a root of p of even/odd multiplicity (multiplicity 0 means that $1 + \mathbf{i}0$ is not a root of p).*

In addition, if p has an odd degree, then $-1 + \mathbf{i}0$ is a root of p of an odd/even multiplicity.

Let $\overline{\lambda}$ be the complex conjugate of λ. Theorem 2 shows that the complex roots of palindromic and antipalindromic polynomials that are not on the unit circle can be divided into four-tuples $(\lambda, 1/\lambda, \overline{\lambda}, 1/\overline{\lambda})$. The complex roots that are on the unit circle and the real roots of p, except possibly roots at ± 1, can be divided into tuples $(\lambda, 1/\lambda)$, see Figure 1. These properties are common for palindromic and antipalindromic polynomials. The distinguishing property of antipalindromic polynomials is that they have a root at $+1$ with an odd multiplicity. Conversely, provided that any root λ, except possibly roots at ± 1, has a corresponding root $1/\lambda$ of the same multiplicity, the polynomial is either palindromic or antipalindromic. The existence of a root at $+1$ and its multiplicity determine whether the polynomial is palindromic (even multiplicity of a root at $+1$) or antipalindromic (odd multiplicity of a root at $+1$).

Let $\operatorname{rev}(p)$ denote the "reversed" polynomial of p, i.e.,

$$\operatorname{rev}(p)(\xi) := p_n\xi^0 + p_{n-1}\xi^1 + \cdots + p_0\xi^n, \qquad \text{where } p(\xi) = p_0\xi^0 + p_1\xi^1 + \cdots + p_n\xi^n. \tag{1}$$

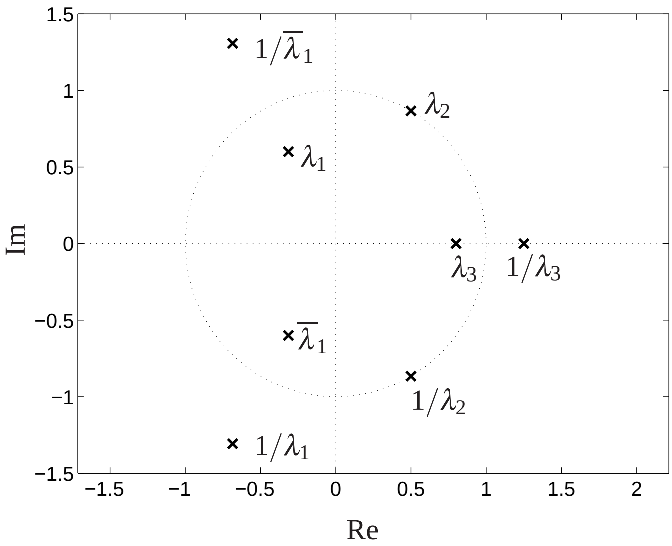

Figure 1: The roots of $p(\xi) = \xi^0 - 1.05\xi^1 + 1.45\xi^2 - 3.63\xi^3 + 3.98\xi^4 - 3.63\xi^5 5 + 1.45\xi^6 - 1.05\xi^7 + \xi^8$ can be grouped in the four-tuple $(\lambda_1, 1/\lambda_1, \overline{\lambda}_1, 1/\overline{\lambda}_1)$, $\lambda_1 = -0.32 + \mathbf{i}0.6$, the tuple $(\lambda_2, 1/\lambda_2)$, $\lambda_2 = 0.5 + \mathbf{i}0.866$, and the tuple $(\lambda_3, 1/\lambda_3)$, $\lambda_3 = 0.8$.

Then, obviously p is palindromic/antipalindromic if and only if $p = \pm\text{rev}(p)$ ($+$ refers to the palindromic case and $-$ refers to the antipalindromic case). Note that the rev operator obeys the following property

$$\text{rev}(p)(\xi) = \begin{cases} \xi^n p(1/\xi), & \text{for } \xi \neq 0, \\ p_n, & \text{for } \xi = 0, \end{cases} \tag{2}$$

which gives an algebraic characterisation of palindromic and antipalindromic polynomials

$$p \text{ is palindromic/antipalindromic} \quad \Longleftrightarrow \quad \begin{cases} p(\xi) \mp \xi^n p(1/\xi) = 0, & \text{for all } \xi \in \mathbb{C},\ \xi \neq 0, \text{ and} \\ p_0 \mp p_n = 0. \end{cases} \tag{3}$$

Next we state three corollaries of Theorem 2.

Corollary 3 (Elementary palindromic and antipalindromic polynomials)**.** *Any palindromic or antipalindromic polynomial p can be represented as a product of the following five elementary polynomials:*

- $e_1(\xi) := \xi - 1$,
- $e_2(\xi) := \xi + 1$,
- $e_3(\alpha, \xi) := \xi^2 - \frac{1+\alpha^2}{\alpha}\xi + 1$, *where* $\alpha \in \mathbb{R}$,
- $e_4(\omega, \xi) := \xi^2 - 2\cos(\omega)\xi + 1$, *where* $\omega \in \mathbb{R}$, *and*
- $e_5(\lambda, \xi) := \xi^4 - 2\Re(\lambda)\frac{1+|\lambda|^2}{|\lambda|^2}\xi^3 + \left(2 + |\lambda|^2 + \frac{2\Re(\lambda)^2 - \Im(\lambda)^2}{|\lambda|^2}\right)\xi^2 - 2\Re(\lambda)\frac{1+|\lambda|^2}{|\lambda|^2}\xi + 1$, *where* $\lambda \in \mathbb{C}$ *is neither purely real nor purely imaginary.*

i.e., there exist unique scalar $c \in \mathbb{R}$ *and parameters* $\alpha_i \in \mathbb{R}$ *for* $i = 1, \ldots, k_3$, $\omega_i \in \mathbb{R}$ *for* $i = 1, \ldots, k_4$, $\lambda_i \in \mathbb{C}$ *for* $i = 1, \ldots, k_5$, *such that*

$$p(\xi) = c e_1^{k_1}(\xi) e_2^{k_2}(\xi) \prod_{i=1}^{k_3} e_3(\alpha_i, \xi) \prod_{i=1}^{k_4} e_4(\omega_i, \xi) \prod_{i=1}^{k_5} e_5(\lambda_i, \xi) \tag{4}$$

($k_i = 0$ means that there is no elementary polynomial of the ith type). If p is palindromic k_1 is even and if p antipalindromic k_1 is odd. Conversely, any product (4) of elementary polynomials is palindromic if k_1 is even and antipalindromic if k_1 is odd.

Proof. Observe that

- the root of $e_1(\xi)$ is $+1$,
- the root of $e_2(\xi)$ is -1,
- the roots of $e_3(\alpha,\xi)$ are α and $1/\alpha$,
- the roots of $e_4(\omega,\xi)$ are $e^{\mathbf{i}\omega}$ and $e^{-\mathbf{i}\omega}$, and
- the roots of $e_5(\lambda,\xi)$ are λ, $1/\lambda$, $\overline{\lambda}$, and $1/\overline{\lambda}$,

so that the factorisation (4) corresponds to the grouping of the roots of p into roots at ± 1, tuples on the real axis, tuples on the unit circle, and four-tuples in the complex plane, as done in the proof of Theorem 2. Therefore, Corollary 3 is a restatement of Theorem 2 in terms of factor polynomials instead of groups of roots. □

Corollary 4 (Algebraic properties of the palindromic and antipalindromic polynomials)**.** *The following hold:*

- *"palindromic" × "palindromic" is palindromic,*
- *"palindromic" × "antipalindromic" is antipalindromic,*
- *"antipalindromic" × "antipalindromic" is palindromic.*

Proof. Consider the factorisations of palindromic or antipalindromic polynomials p and q into elementary polynomials. Obviously, the polynomial pq is also a product of elementary polynomials, so that it is either palindromic or antipalindromic. Let $k_{1,p}$, $k_{1,q}$, and $k_{1,pq}$ be the powers of e_1 in the factorisations (4) of respectively p, q, and pq. We have that $k_{1,p} + k_{1,q} = k_{1,pq}$. According to Corollary 3 the power of e_1 is even if and only if the corresponding polynomial is palindromic and odd if and only if the polynomial is antipalindromic. For the three cases in the statement of Corollary 4, we have

- even $k_{1,p}$ + even $k_{1,q}$ = even $k_{1,pq}$, so that pq is palindromic,
- even $k_{1,p}$ + odd $k_{1,q}$ = odd $k_{1,pq}$, so that pq is antipalindromic,
- odd $k_{1,p}$ + odd $k_{1,q}$ = even $k_{1,pq}$, so that pq is palindromic. □

Corollary 5. *If a polynomial with real coefficients $p \in \mathbb{R}[\xi]$ has all its roots on the unit circle, then it is either palindromic or antipalintromic. The antipalintromic case corresponds to the polynomials with a root at $1 + \mathbf{i}0$ of an odd multiplicity.*

Proof. A polynomial p whose roots are on the unit circle satisfies item 1 in Theorem 2, so that p is either palindromic or antipalindromic. If, in addition, p satisfies item 2, then it is palindromic. Otherwise, p is antipalindromic. □

3 Even and odd polynomials

In this section, we study the root location of even or odd polynomials, i.e., polynomials of the type

$$p(\xi) = p_0 + p_2\xi^2 + \cdots + p_{2n}\xi^{2n} =: p'(\xi^2),$$
$$p(\xi) = p_1\xi + p_3\xi^3 + \cdots + p_{2n+1}\xi^{2n+1} =: \xi p'(\xi^2).$$

Note 6 (Continuous-time systems). Palindromic/antipalindromic polynomials are related to discrete-time LTI systems, see Section 4. Even/odd polynomials are the "continuous-time equivalent" to palindromic/antipalindromic polynomials, i.e., they are related to continuous-time LTI systems. The results for the continuous-time case can be deduced from the corresponding results for the discrete-time case, so in Section 4 we consider only the discrete-time case.

An algebraic characterisation of even/odd polynomials, analogous to the characterisation (2) of palindromic/antipalindromic polynomials is

$$p \text{ is even/odd} \quad \Longleftrightarrow \quad p(\xi) \mp p(-\xi) = 0, \qquad \text{for all } \xi \in \mathbb{C}.$$

Theorem 7 (Root location of even and odd polynomials)**.** *A polynomial $p \in \mathbb{R}[\xi]$ is even/odd if and only if*

1. *for every root $\lambda \in \mathbb{C}$ of p, there is another root $-\lambda$ of p with the same multiplicity as λ and*
2. *0 is a root of p of even/odd multiplicity.*

Since we consider only polynomials with real coefficients, from Theorem 7, it can be inferred that the set of roots of an even/odd polynomial is a union of the following sets:

- set of roots at 0
- set of pairs of roots $(\mathbf{i}\omega_i, -\mathbf{i}\omega_i)$, where $\omega_i \in \mathbb{R}_+$, on the imaginary axis,
- set of pairs of roots $(\alpha_i, -\alpha_i)$, where $\alpha_i \in \mathbb{R}_+$, on the real axis, and
- set of four-tuples $(\lambda_i, \bar{\lambda}_i, -\lambda_i, -\bar{\lambda}_i)$, where $\lambda \in \mathbb{C}$ is neither purely real nor purely imaginary.

This is illustrated through an example in Figure 2. The following corollary formalises this property.

Corollary 8 (Elementary even and odd polynomials)**.** *Any even or odd polynomial $p \in \mathbb{R}[\xi]$ can be represented as a product of the following four elementary polynomials:*

- $f_1(\xi) := \xi$,
- $f_2(\omega, \xi) := \xi^2 + \omega^2$, *where* $\omega \in \mathbb{R}_+$,
- $f_3(\alpha, \xi) := \xi^2 - \alpha^2$, *where* $\alpha \in \mathbb{R}_+$,
- $f_4(\lambda, \xi) := \xi^4 + 2\big(\Im^2(\lambda) - \Re^2(\lambda)\big)\xi^2 + \big(\Im^2(\lambda) + \Re^2(\lambda)\big)^2$, *where $\lambda \in \mathbb{C}$ is neither purely real nor purely imaginary,*

i.e., there exist a unique scalar $c \in \mathbb{R}$ and parameters $\alpha_i \in \mathbb{R}$ for $i = 1, \ldots, k_2$, $\omega_i \in \mathbb{R}$ for $i = 1, \ldots, k_3$, $\lambda_i \in \mathbb{C}$ for $i = 1, \ldots, k_4$, such that

$$p(\xi) = c f_1^{k_1}(\xi) \prod_{i=1}^{k_2} f_2(\omega_i, \xi) \prod_{i=1}^{k_3} f_3(\alpha_i, \xi) \prod_{i=1}^{k_4} f_4(\lambda_i, \xi) \tag{5}$$

($k_i = 0$ means that there is no elementary polynomial of the ith type). If p is even, k_1 is even and if p is odd k_1 is odd. Conversely, any product (5) of elementary polynomials is even if k_1 is even and odd if k_1 is odd.

Proof. Observe that

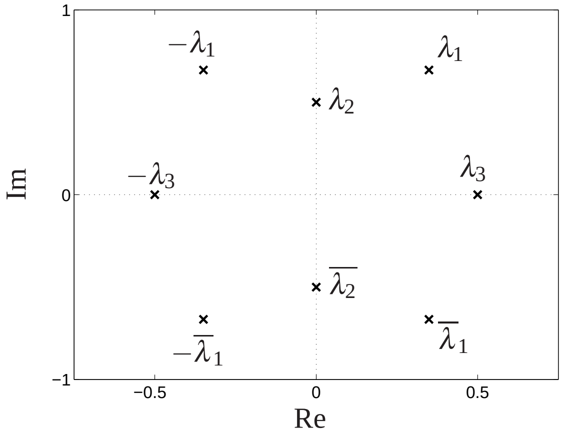

Figure 2: The roots of $p(\xi) = 3\xi^0 + 2\xi^2 + 0.8125\xi^4 - 0.125\xi^6 - 0.0625\xi^8$ can be grouped in the four-tuple $(\lambda_1, -\lambda_1, \overline{\lambda}_1, -\overline{\lambda}_1)$, $\lambda_1 = 0.3493 + \mathbf{i}0.6748$, the tuple $(\lambda_2, \overline{\lambda_2})$, $\lambda_2 = \mathbf{i}0.5$, and the tuple $(\lambda_3, -\lambda_3)$, $\lambda_3 = 0.5$.

- the root of $f_1(\xi)$ is 0,
- the roots of $f_2(\omega, \xi)$ are $\mathbf{i}\omega$ and $-\mathbf{i}\omega$,
- the roots of $f_3(\alpha, \xi)$ are α and $-\alpha$, and
- the roots of $f_4(\lambda, \xi)$ are λ, $-\lambda$, $\overline{\lambda}$, and $-\overline{\lambda}$,

so that the factorisation (5) corresponds to the grouping of the roots of p into roots at 0, pairs on the real axis, pairs on the imaginary axis, and four-tuples in the complex plane, as discussed earlier. □

Corollary 9. *If a polynomial $p \in \mathbb{R}[\xi]$ has all its roots on the imaginary axis, then it is either even or odd. The odd case corresponds to the polynomials with a root at* 0 *of an odd multiplicity.*

Proof. A polynomial p whose roots are on the imaginary axis satisfies item 1 in Theorem 7, so that p is either even or odd. If, in addition, p satisfies item 2, then it is even. Otherwise, it is odd. □

4 Palindromic/antipalindromic polynomials in system theory

In the remaining part of the paper, we study the relevance of palindromic and antipalindromic polynomials in system theory. In Section 4.1, we show that difference equation representations of LTI time-reversible systems are related to palindromic/antipalindromic polynomials. In Section 4.2, we show that palindromic/antipalindromic polynomials also have a link with representations of LTI systems that have special quadratic functionals associated with them, known as conserved quantities, i.e., functionals that remain constant along the trajectories of the system. We begin with a short description of LTI systems and quadratic difference forms. For thorough exposition of these concepts (in the continuous-time case), we refer the reader respectively to [PW98] and [WT98].

Define by σ the shift operator

$$(\sigma w)(t) := w(t+1).$$

An LTI system can always be represented in, what is called kernel form, as

$$\mathscr{B} = \ker\big(R(\sigma)\big) := \{\, w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}} \quad | \quad R(\sigma)w = 0 \,\}, \tag{6}$$

where $R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\sigma]$. The representation (6) is called minimal if R has the minimum number of rows among all the kernel representations of $\mathscr{B}$.

A quadratic difference form [Kan05] is the discrete-time analogue of quadratic differential form for continuous-time systems introduced by Willems and Trentelman [WT98]. Let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, i.e., $\Phi(\zeta,\eta) = \sum_{i,j=0}^{n} \Phi_{i,j}\zeta^i\eta^j$, where $\Phi_{i,j} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ for all $i,j \in \{0,1,\ldots,n\}$ and n is a natural number. Such a polynomial Φ induces a *quadratic difference form* (QDF) on $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ as

$$Q_\Phi : (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}} \mapsto \mathbb{R}^{\mathbb{Z}}, \qquad Q_\Phi(w)(t) := \sum_{i,j=0}^{n} w(t+i)^\top \Phi_{i,j} w(t+j).$$

We call Φ symmetric if $\Phi(\zeta,\eta) = \Phi(\eta,\zeta)^\top$. In this paper, we consider only the set of symmetric QDFs, denoted by $\mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, because every nonsymmetric QDF is equivalent to a symmetric one.

We describe the notion of the rate of change of a QDF, which will be used to obtain results about conserved quantities later on in the paper. The *rate of change* ∇Q_Φ of a QDF Q_Φ is defined as

$$\nabla Q_\Phi(w)(t) := Q_\Phi(w)(t+1) - Q_\Phi(w)(t).$$

Let $\Psi(\zeta,\eta)$ be the two-variable polynomial matrix associated with the QDF ∇Q_Φ. Then, it is easy to see that

$$\Psi(\zeta,\eta) = (\zeta\eta - 1)\Phi(\zeta,\eta).$$

A QDF Q_Φ is said to be zero along a behaviour $\mathscr{B}$ (denoted by $Q_\Phi \overset{\mathscr{B}}{=} 0$) if $Q_\Phi(w)(t) = 0$, for all $w \in \mathscr{B}$ and $t \in \mathbb{Z}$. The next proposition which is an analogue of Proposition 3.2 of [WT98] gives the condition on a two-variable polynomial under which the QDF associated with it is zero along a given scalar autonomous behaviour.

Proposition 10. *Let $\Phi \in \mathbb{R}_s[\zeta,\eta]$ and let $\mathscr{B} = \ker\big(r(\sigma)\big)$, where $r \in \mathbb{R}[\sigma]$. Then $Q_\Phi \overset{\mathscr{B}}{=} 0$ if and only if there exists $f \in \mathbb{R}[\zeta,\eta]$, such that*

$$\Phi(\zeta,\eta) = f(\eta,\zeta)r(\eta) + r(\zeta)f(\zeta,\eta).$$

Proof. The proof can be deduced from Lemma A.1, page 1734, [WT98]. □

Equip the set of QDFs associated with a behaviour $\mathscr{B}$ with the equivalence relation defined by

$$Q_\Phi \overset{\mathscr{B}}{\sim} Q_\Psi \quad \Longleftrightarrow \quad Q_\Phi(w) = Q_\Psi(w), \qquad \text{for all } w \in \mathscr{B}.$$

It is easy to see that the set of equivalence classes under $\overset{\mathscr{B}}{\sim}$ is a linear vector space over $\mathbb{R}$. With every equivalence class of QDFs associated with an autonomous behaviour $\mathscr{B}$, we associate a certain representative known as the $\mathscr{B}$-canonical representative. Below, we define the notion of $\mathscr{B}$-canonicity of QDFs.

Definition 11. Let $\mathscr{B}$ be an autonomous behaviour given by $\mathscr{B} = \ker\big(r(\sigma)\big)$, where $r \in \mathbb{R}[\sigma]$. Then a QDF Q_Φ is $\mathscr{B}$-canonical if $r(\zeta)^{-1}\Phi(\zeta,\eta)r(\eta)^{-1}$ is strictly proper.

If r has degree n, then from the definition, it follows that the two-variable polynomials associated with $\mathscr{B}$-canonical QDFs are spanned by monomials $\zeta^i\eta^j$, with $i,j \le n-1$. It is easy to see that every QDF has a $\mathscr{B}$-canonical representative.

4.1 Time-reversible systems

Next, we study kernel representations of discrete-time time-reversible scalar autonomous and SISO systems. We show that there exists an inherent link between palindromic/antipalindromic polynomials and the kernel representation of time-reversible systems. Earlier, work on time-reversible systems had been done by Fagnani and Willems [FW91].

In Section 2, we defined the reversed polynomial $\mathrm{rev}(p)$, see (1). By viewing the polynomial p as the vector $\mathrm{col}(p_0, p_1, \ldots, p_n)$ of its coefficients, we can naturally extend the "rev" operator for vectors. Thus

$$\mathrm{rev}\big(\mathrm{col}(p_0, p_1, \ldots, p_n)\big) := \mathrm{col}(p_n, p_{n-1}, \ldots, p_0).$$

We define the "rev" operator for a sequence $w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}}$ as

$$\big(\mathrm{rev}(w)\big)(t) := w(-t), \qquad \text{for all } t \in \mathbb{Z}.$$

Thus the "rev" operator acting on a sequence reverses the order of the sequence. It is easy to see that $\mathrm{rev}\big(\mathrm{rev}(w)\big) = w$.

Definition 12 (Time-reversible system)**.** A dynamical system $\mathscr{B}$ is time-reversible if $w \in \mathscr{B}$ implies $\mathrm{rev}(w) \in \mathscr{B}$.

We now give a necessary and sufficient condition in terms of a kernel representation for an autonomous scalar system to be time-reversible.

Theorem 13. *The LTI system $\mathscr{B} = \ker\big(p(\sigma)\big)$, $p \in \mathbb{R}[\xi]$, is time-reversible if and only if p is either palindromic or antipalindromic.*

Next we give a condition in terms of a kernel representation for a SISO system to be time-reversible.

Theorem 14. *Let $\mathscr{B} = \ker\big(\begin{bmatrix} q(\sigma) & -p(\sigma) \end{bmatrix}\big)$, where $p \in \mathbb{R}[\sigma]$ is of degree $\mathtt{p}$ and $q \in \mathbb{R}[\sigma]$ is of degree $\mathtt{q} \leq \mathtt{p}$. Define $\mathtt{d} := \mathtt{p} - \mathtt{q}$. The system $\mathscr{B}$ is time-reversible if and only if i) q has $\mathtt{d}$ roots at zero and ii) p and $q'(\sigma) := q(\sigma)/\sigma^{\mathtt{d}}$ are either both palindromic or both antipalindromic. If $\mathscr{B}$ is controllable, then both p and q' are palindromic.*

Using the steps followed in the proof of the above theorem, one can also deduce the structure of the kernel representation of a multi-input single-output time-reversible system. We now state the main result of [FW91] on kernel representations of discrete-time time-reversible systems and compare it with ours.

Theorem 15 ([FW91])**.** *An LTI system $\mathscr{B}$ is time-reversible if and only if it can be described by a minimal difference equation $R(\sigma, \sigma^{-1})w = 0$, where*

$$R(z^{-1}, z) = S(z)R(z, z^{-1}) \qquad \textit{and} \qquad S(z) = \mathrm{diag}(I_{\mathtt{p}_+}, -I_{\mathtt{p}_-}, I_{\mathtt{p}_{z+}}z, -I_{\mathtt{p}_{z-}}z)$$

with $I_{\mathtt{p}}$ being the identity matrix of size $\mathtt{p}$, and $\mathtt{p}_+$, $\mathtt{p}_-$, $\mathtt{p}_{z+}$, $\mathtt{p}_{z-}$ being nonnegative integers (invariants of $\mathscr{B}$), such that their sum equals the number of rows of R.

For single output systems, the kernel representation is unique up to a multiplication with a nonzero scalar. This type of nonuniqueness does not destroy the (anti)palindromic structure. Therefore, any representation of a scalar autonomous and SISO time-reversible system has the properties stated in Theorems 13 and 14. In the multiple output case, the nonuniqueness of the kernel representation can destroy the (anti)palindromic structure. As a result the statement of Theorem 15 is existential.

In the following examples, we determine the integers $\mathtt{p}_+$, $\mathtt{p}_-$, $\mathtt{p}_{z+}$, and $\mathtt{p}_{z-}$, defined in Theorems 15 for particular systems. Then we make a general statement of how $\mathtt{p}_+$, $\mathtt{p}_-$, $\mathtt{p}_{z+}$, $\mathtt{p}_{z-}$ depend on the behaviour and its kernel representation.

$\mathbf{k}_1(\mathscr{B})$	$\mathbf{n}(\mathscr{B})$
even	even
odd	even
even	odd
odd	odd

$\Longleftrightarrow$

p	$\deg(p)$
palindromic	even
antipalindromic	even
palindromic	odd
antipalindromic	odd

$\Longleftrightarrow$

$\mathtt{p}_+$	$\mathtt{p}_-$	$\mathtt{p}_{z+}$	$\mathtt{p}_{z-}$
1	0	0	0
0	1	0	0
0	0	1	0
0	0	0	1

Table 2: $\mathbf{k}_1(\mathscr{B})$ — the multiplicity of a pole of $\mathscr{B}$ at 1, $\mathbf{n}(\mathscr{B})$ — order of $\mathscr{B}$.

Example 16 (Autonomous time-reversible system). Consider a behaviour $\mathscr{B} = \ker\big(p(\sigma)\big)$, where

$$p(\sigma) = p_0 + p_1\sigma - p_1\sigma^3 - p_0\sigma^4, \qquad \text{with } p_0, p_1 \in \mathbb{R}.$$

Since p is antipalindromic, from Theorem 13, it follows that $\mathscr{B}$ is time-reversible. According to Theorem 15, there exists another representation for $\mathscr{B}$ given by $\mathscr{B} = \ker\big(R(\sigma,\sigma^{-1})\big)$, where

$$R(\sigma,\sigma^{-1}) = p_0\sigma^{-2} + p_1\sigma^{-1} - p_1\sigma - p_0\sigma^2.$$

Observe that $R(\sigma,\sigma^{-1}) = -R(\sigma^{-1},\sigma)$, so that $\mathtt{p}_- = 1$ and $\mathtt{p}_+ = \mathtt{p}_{z+} = \mathtt{p}_{z-} = 0$.

Example 17 (SISO time-reversible system). Consider a behaviour $\mathscr{B} = \ker\big(R'(\sigma)\big)$, where

$$R'(\sigma) = \begin{bmatrix} p_0 + p_1\sigma + p_1\sigma^2 + p_0\sigma^3 & -q_1\sigma - q_1\sigma^2 \end{bmatrix}, \qquad \text{with } p_0, p_1, q_1 \in \mathbb{R}.$$

From Theorem 14, it follows that $\mathscr{B}$ is time-reversible. According to Theorem 15, there exists another representation for $\mathscr{B}$ given by $R(\sigma,\sigma^{-1})w = 0$, where

$$R(\sigma,\sigma^{-1}) = \begin{bmatrix} p_0\sigma^{-1} + p_1 + p_1\sigma + p_0\sigma^2 & -q_1 - q_1\sigma \end{bmatrix}.$$

In this case, $R(z^{-1},z) = S(z)R(z,z^{-1})$, where $S(z) = z$, so that $\mathtt{p}_+ = \mathtt{p}_{z-} = \mathtt{p}_- = 0$ and $\mathtt{p}_{z+} = 1$.

It is of interest to show how the invariants $\mathtt{p}_+$, $\mathtt{p}_-$, $\mathtt{p}_{z+}$, $\mathtt{p}_{z-}$ depend on the behaviour and on the parameters of particular representations of the behaviour. For the general multivariable case, Theorem 15 only asserts that their sum is equal to the number of outputs of the system. Using Theorems 13 and 14, we can say more for the scalar autonomous and SISO cases. The dependence of the invariants $\mathtt{p}_+$, $\mathtt{p}_-$, $\mathtt{p}_{z+}$, $\mathtt{p}_{z-}$ on the behaviour $\mathscr{B}$ and on a minimal kernel representation for these cases are summarised in Table 2.

4.2 Existence of conserved quantities

The notion of a *conserved quantity* is defined by Rapisarda and Willems [RW05] for the case of continuous-time systems. Below, we give an analogous definition for the case of discrete-time systems.

Definition 18. *Let $\mathscr{B}$ be a linear autonomous behaviour. A QDF Q_Φ is a conserved quantity for $\mathscr{B}$ if*

$$\nabla Q_\Phi(w) = 0, \qquad \textit{for all } w \in \mathscr{B}. \tag{7}$$

In other words, a conserved quantity for a behaviour $\mathscr{B}$ is a QDF, which remains constant upon acting on any sequence belonging to the behaviour. In this section, we examine the conditions on the representation of a scalar autonomous behaviour under which it has a conserved quantity associated with it. We begin with the following definition.

Definition 19. The *maximal palindromic factor* of $r \in \mathbb{R}[\xi]$ is its monic palindromic factor of maximal degree.

For any given polynomial $r \in \mathbb{R}[\xi]$, it is easy to see that there exists a unique maximal palindromic factor. In the next theorem, we examine the conditions under which a linear behaviour $\mathscr{B}$ has conserved quantities associated with it.

Theorem 20. *Consider a linear behaviour $\mathscr{B} = \ker\big(r(\sigma)\big)$, where $r \in \mathbb{R}[\sigma]$ has no root at zero. There exists a nonzero conserved quantity for $\mathscr{B}$ if and only if either r has a non-unity maximal palindromic factor r' or $r(\sigma) = \sigma - 1$. Moreover if $v := r/r'$ is such that $v(1) \neq 0$, then the dimension of the space of conserved quantities is $\lfloor\big(\deg(r') - 1\big)/2\rfloor$, otherwise it is $\lfloor\big(\deg(r') + 1\big)/2\rfloor$, where $\lfloor m \rfloor$ is the greatest integer less than or equal to m.*

From the above theorem and from the discussion on time-reversible systems, it can be inferred that every reversible scalar system has conserved quantities associated with it.

A Proofs

A.1 Theorem 2

First, note that item 1 in the statement of Theorem 2 is equivalent to:

1'. For every root $\lambda \in \mathbb{C}$ of p there exists a root $1/\lambda$ with the same multiplicity as λ.

Indeed, if λ is complex and is on the unit circle, then $1/\lambda = \overline{\lambda}$ and by the assumption that p has real coefficients, it follows that $1/\lambda$ is also a root of p with the same multiplicity. If $\lambda = \pm 1$, then $1/\lambda = \lambda$ and 1' is trivially satisfied.

Let $\lambda_1, \dots, \lambda_k$ be the distinct roots of p and let $n_1, \dots, n_k$ be their respective multiplicities. Note that the assumptions $p_n \neq 0$ and p palindromic or antipalindromic imply that $\lambda_i \neq 0$ for all i. Denote the jth derivative of $p(\xi)$, evaluated at $\lambda \in \mathbb{C}$ by $\frac{\mathrm{d}^j}{\mathrm{d}\xi^j} p(\xi)|_{\xi=\lambda}$. The characterisation (3) of palindromic and antipalindromic polynomials, can be written as

$$\begin{array}{c} p \text{ is palindromic/antipalindromic} \\ \text{with distinct roots } \lambda_1, \dots, \lambda_k \\ \text{of respective multiplicities } n_1, \dots, n_k \end{array} \iff \begin{cases} \frac{\mathrm{d}^j}{\mathrm{d}\xi^j}\big(p(\xi) \mp \xi^n p(1/\xi)\big)|_{\xi=\lambda_i} = 0, \text{ for } i = 1, \dots, k,\ j = 0, \dots, n_i - 1 \\ p_0 \mp p_n = 0. \end{cases}$$

For each $i \in \{1, \dots, k\}$, the first condition gives the system of equations

$$\pm \begin{bmatrix} \lambda_i^n & 0 & \cdots & \cdots & \cdots & 0 \\ n\lambda_i^{n-1} & -\lambda_i^{n-2} & 0 & \cdots & \cdots & 0 \\ n(n-1)\lambda_i^{n-2} & -2(n-1)\lambda_i^{n-3} & \lambda_i^{n-4} & \ddots & & \vdots \\ n(n-1)(n-2)\lambda_i^{n-3} & -3(n-1)(n-2)\lambda_i^{n-4} & 3(n-2)\lambda_i^{n-5} & -\lambda_i^{n-6} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & 0 \\ * & * & * & * & \cdots & \lambda_i^0 \end{bmatrix} \begin{bmatrix} p(1/\lambda_i) \\ \frac{\mathrm{d}}{\mathrm{d}\lambda} p(1/\lambda_i) \\ \vdots \\ \frac{\mathrm{d}^{n_i-1}}{\mathrm{d}\lambda^{n_i-1}} p(1/\lambda_i) \end{bmatrix} = 0.$$

The coefficient matrix in the left hand side is lower-triangular with nonzero diagonal elements (since $\lambda_i \neq 0$). Therefore,

$$p(1/\lambda_i) = \frac{\mathrm{d}}{\mathrm{d}\lambda} p(1/\lambda_i) = \cdots = \frac{\mathrm{d}^{n_i-1}}{\mathrm{d}\lambda^{n_i-1}} p(1/\lambda_i) = 0,$$

which proves that $1/\lambda_i$ is a root of p with multiplicity n_i. We showed that

$$p \text{ is palindromic/antipalindromic} \quad \Longleftrightarrow \quad \begin{cases} \text{item 1' holds and} \\ p_0 \mp p_n = 0, \end{cases} \tag{8}$$

i.e., any polynomial satisfying item 1' is either palindromic or antipalindromic and, vice verse, any palindromic or antipalindromic polynomial satisfies item 1'. The condition $p_0 \mp p_n = 0$ distinguishes between the palindromic and the antipalindromic case. In order to complete the proof, we have to show that

$$\text{item 1' holds and } p_0 \mp p_n = 0 \quad \Longleftrightarrow \quad \text{items 1' and 2 hold.}$$

This equivalence follows from recursive application of the following lemmas.

Lemma 21. *The polynomial $p \in \mathbb{R}[\xi]$ is antipalindromic if and only if there is a palindromic polynomial q, such that*

$$p(\xi) = (\xi - 1)q(\xi). \tag{9}$$

Lemma 22. *The polynomial $p \in \mathbb{R}[\xi]$ is palindromic and has a root at $+1$ if and only if there is an antipalindromic polynomial q, such that (9) holds.*

Let p be palindromic. If p has no root at $+1$, we are done (the multiplicity of $+1$ is 0). If p has a root at $+1$, then, according to Lemma 22, it can be factored into $(\xi - 1)p^{(1)}$, where $p^{(1)}(\xi)$ is antipalindromic. According to Lemma 21, $p^{(1)}$ can be factored as $(\xi - 1)p^{(2)}(\xi)$, were $p^{(2)}$ is palindromic. At this stage, $p = (\xi - 1)^2 p^{(2)}$ and we repeat the argument above replacing p with $p^{(2)}$. In general, the procedure terminates at the kth iteration by finding that the palindromic polynomial $p^{(2k)}$ has no root at $+1$. Then $p = (\xi - 1)^{2k} p^{(2k)}$, which proves that p has a root at $+1$ of an even multiplicity. It follows by Lemma 21 that an antipalindromic p must have a root at $+1$ of an odd multiplicity.

Proof of Lemma 21

The polynomial equation (9) can be written in a matrix-vector form as

$$\begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} -1 & 0 & \cdots & 0 \\ 1 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & 1 & -1 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_{n-1} \end{bmatrix}, \tag{10}$$

where $p_0, p_1, \ldots, p_n$ are the coefficients of p and $q_0, q_1, \ldots, q_{n-1}$ are the coefficients of q. The restriction of the linear mapping $q \mapsto p$, defined by (10), to the subspace of palindromic q is

$$\begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & 0 & -1 & 1 \\ \vdots & 0 & 1 & -1 \\ 0 & \iddots & \iddots & 0 \\ 1 & -1 & \iddots & \vdots \\ -1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_m \end{bmatrix} = \begin{bmatrix} q_0 \\ q_0 - q_1 \\ \vdots \\ q_{m-1} - q_m \\ -q_{m-1} + q_m \\ \vdots \\ -q_0 + q_1 \\ -q_0 \end{bmatrix},$$

when n is even ($n =: 2m$) and

$$\begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 1 \\ \vdots & & 0 & 0 \\ \vdots & \iddots & 1 & -1 \\ 0 & \iddots & \iddots & 0 \\ 1 & -1 & \iddots & \vdots \\ -1 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_m \end{bmatrix} = \begin{bmatrix} q_0 \\ q_0 - q_1 \\ \vdots \\ q_{m-1} - q_m \\ 0 \\ -q_{m-1} + q_m \\ \vdots \\ -q_0 + q_1 \\ -q_0 \end{bmatrix},$$

when n is odd ($n =: 2m+1$). This shows that the restriction of (10) to the subspace of palindromic arguments, automatically restricts the image to the subspace of antipalindromic vectors p. Moreover, there is a one-to-one mapping between $q_0, q_1, \ldots, q_m$ and $p_0, p_1, \ldots, p_m$ given by the equation

$$\begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_m \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ -1 & 1 & \ddots & \vdots \\ 0 & \ddots & \ddots & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_m \end{bmatrix}.$$

This proves that given an antipalindromic p there is a unique solution q of (10), which is palindromic.

Proof of Lemma 22

Consider (9) with p palindromic. The "only if" implication in (8) and the assumption that p is palindromic imply that the roots of p satisfy item 1'. The roots of q are a subset of the roots of p so that they also satisfy item 1'. By the "if" implication of (8), we conclude that q is either palindromic or palindromic. Assume that it is antipalindromic. Then by Lemma 21, p is antipalindromic, which is a contradiction. Therefore, q is antipalindromic.

A.2 Theorem 7

Let $\lambda_1,\dots,\lambda_k$ be the distinct roots of p and let $n_1,\dots,n_k$ be their respective multiplicities. Note that $p_n \neq 0$ because degree of the polynomial is assumed to be n. Then the following holds:

$$\left\{\begin{array}{c} p \text{ is even/odd with distinct roots } \lambda_1,\dots,\lambda_k \\ \text{of respective multiplicities } n_1,\dots,n_k \end{array}\right\} \iff \left\{\begin{array}{c} \frac{\mathrm{d}^j}{\mathrm{d}\xi^j}\big(p(\xi)\mp p(-\xi)\big)|_{\xi=\lambda_i}=0, \\ \text{for } i=1,\dots,k,\ j=0,\dots,n_i-1 \end{array}\right\}$$

This implies that

$$\left\{\begin{array}{c} p \text{ is even/odd with distinct roots } \lambda_1,\dots,\lambda_k \\ \text{of respective multiplicities } n_1,\dots,n_k \end{array}\right\} \iff \left\{\begin{array}{c} \big(\frac{\mathrm{d}^j}{\mathrm{d}\lambda_i^j}p(\lambda_i)\mp(-1)^j\frac{\mathrm{d}^j}{\mathrm{d}\lambda_i^j}p(-\lambda_i)\big)=0, \\ \text{for } i=1,\dots,k,\ j=0,\dots,n_i-1 \end{array}\right\}$$

It follows that

$$p(-\lambda_i)=\frac{\mathrm{d}}{\mathrm{d}\lambda}p(-\lambda_i)=\cdots=\frac{\mathrm{d}^{n_i-1}}{\mathrm{d}\lambda^{n_i-1}}p(-\lambda_i)=0,$$

for $i=1,\dots,k$. Hence $-\lambda_i$ is a root of multiplicity n_i. This proves item 1 of the theorem.

By the definition of odd polynomials, any odd polynomial p can be factorised as $p(\xi)=\xi p'(\xi)$, where p' is an even polynomial. Assume that p' has a root at zero, i.e., $p'(\xi)=\xi p''(\xi)$. Since p' is even,

$$p'(\xi)=-\xi p''(-\xi)=\xi p''(\xi).$$

This implies that p'' is odd.

Now assume that p has exactly $2m$ roots at zero, where $m\in\mathbb{N}$. Let $p(\xi)=\xi^{2m}p^{(2m)}(\xi)$. From the previous argument, it follows that $p^{(2m)}$ is odd and has no root at zero, which is a contradiction. Hence any odd polynomial has odd number of roots at zero.

Consider an even polynomial p_1. Since p_2 given by $p_2(\xi)=\xi p_1(\xi)$ is odd and we have already proved that an odd polynomial has odd number of roots at zero, it follows that p_1 has even number of roots at zero. This concludes the proof.

A.3 Theorem 13

$(\Longrightarrow)$ Let $\mathscr{B}=\ker\big(p(\sigma)\big)$ and $p(\sigma)=\sum_{i=0}^n p_i\sigma^i$. Assume that $\mathrm{rev}(w)\in\mathscr{B}$. Then

$$\big(p(\sigma)\mathrm{rev}(w)\big)(t)=\sum_{i=0}^n\big(p_i\sigma^i\mathrm{rev}(w)\big)(t)=\sum_{i=0}^n p_i w(-t-i)=0, \qquad \text{for every } t\in\mathbb{Z}.$$

Putting $i=n-j$ and $t=-t_1-n$ in the above equation, we get

$$\sum_{j=0}^n p_{n-j}w(t_1+j)=0, \qquad \text{for every } t_1\in\mathbb{Z}. \tag{11}$$

Since $\mathscr{B}$ is time-reversible, $w\in\mathscr{B}$, i.e.,

$$\sum_{j=0}^n p_j w(t_1+j)=0, \qquad \text{for every } t_1\in\mathbb{Z}. \tag{12}$$

From equations (11) and (12),

$$\sum_{j=0}^n p_{n-j}w(t_1+j)=k\sum_{j=0}^n p_j w(t_1+j), \qquad \text{for every } t_1\in\mathbb{Z}.$$

where $k \in \mathbb{R}$. Since $\mathscr{B}$ is linear, it follows that $kp = \mathrm{rev}(p)$, where $p = \mathrm{col}(p_0, p_1, \ldots, p_n)$. Therefore, $k^2 = 1$ or $k = \pm 1$ and hence $p(\sigma)$ is either palindromic or antipalindromic.

($\Longleftarrow$) Assume that $p(\sigma) = \sum_{i=0}^{n} p_i \sigma^i$ is palindromic/antipalindromic. Then $p_i = \pm p_{n-i}$ for $i = 0, 1, \ldots, n$. Let $\mathscr{B} = \ker\big(p(\sigma)\big)$. For any $w \in \mathscr{B}$, we have $\sum_{i=0}^{n} p_i \sigma_i w = 0$. The left hand side can be written as

$$\sum_{i=0}^{n} p_i w(t+i) = \pm \sum_{i=0}^{n} p_{n-i} w(t+i).$$

Putting $i = n - j$ and $t = -t_1 - n$, we get

$$\pm \sum_{j=0}^{n} p_j w(-t_1 - j) = \pm \sum_{j=0}^{n} \big(p_j \sigma^j \mathrm{rev}(w)\big)(t_1) = 0, \qquad \text{for every } t_1 \in \mathbb{Z}.$$

Thus if p is palindromic/antipalindromic, $w \in \mathscr{B}$ implies $\mathrm{rev}(w) \in \mathscr{B}$, or $\mathscr{B}$ is reversible.

A.4 Theorem 14

Let $p(\sigma) = \sum_{i=0}^{\mathtt{p}} p_i \sigma^i$ and $q(\sigma) = \sum_{i=0}^{\mathtt{q}} q_i \sigma^i$. For any $\mathrm{col}(y,u) \in \mathscr{B} := \ker(\begin{bmatrix} p(\sigma) & -q(\sigma) \end{bmatrix})$, we have

$$\sum_{i=0}^{\mathtt{p}} p_i y(t+i) = \sum_{i=0}^{\mathtt{q}} q_i u(t+i), \qquad \text{for every } t \in \mathbb{Z}. \tag{13}$$

(*If*) Assume that q is divisible by $\sigma^{\mathtt{d}}$ and that p and q_1 are palindromic, where $q_1(\sigma) := q(\sigma)/\sigma^d$. Then $p_i = p_{n-i}$ for $i = 0, 1, \ldots, \mathtt{p}$, $q_i = 0$ for $i = 0, \ldots, \mathtt{d} - 1$, and $q_i = q_{\mathtt{p}-i}$ for $i = \mathtt{d}, \ldots, \mathtt{q}$. From (13),

$$\sum_{i=0}^{\mathtt{p}} p_{\mathtt{p}-i} y(t+i) = \sum_{i=\mathtt{d}}^{\mathtt{q}} q_{\mathtt{p}-i} u(t+i), \qquad \text{for every } t \in \mathbb{Z}.$$

Putting $i = \mathtt{p} - j$, and $t = -t_1 - \mathtt{p}$ in the above equation, we get

$$\sum_{j=0}^{\mathtt{p}} p_j y(-t_1 - j) = \sum_{j=\mathtt{d}}^{\mathtt{q}} q_j u(-t_1 - j), \qquad \text{for every } t_1 \in \mathbb{Z}$$

or

$$\sum_{j=0}^{\mathtt{p}} \big(p_j \sigma^j \mathrm{rev}(y)\big) = \sum_{j=\mathtt{d}}^{\mathtt{q}} \big(q_j \sigma^j \mathrm{rev}(u)\big),$$

so that

$$\begin{bmatrix} p(\sigma) & -q(\sigma) \end{bmatrix} \mathrm{rev}\big(\mathrm{col}(y,u)\big) = 0.$$

This implies that $\mathrm{rev}\big(\mathrm{col}(y,u)\big) \in \mathscr{B}$. Hence $\mathscr{B}$ is reversible. The proof for the case when p and q are both antipalindromic is similar.

(*Only If*) Assume that $\mathscr{B}$ is time-reversible and $\mathrm{rev}\big(\mathrm{col}(y,u)\big) \in \mathscr{B}$. We have

$$\sum_{i=0}^{\mathtt{p}} p_i \sigma^i \mathrm{rev}(y) = \sum_{i=0}^{\mathtt{q}} q_i \sigma^i \mathrm{rev}(u) \quad \Longrightarrow \quad \sum_{i=0}^{\mathtt{p}} p_i y(-t-i) = \sum_{i=0}^{\mathtt{q}} q_i u(-t-i), \quad \text{for every } t \in \mathbb{Z}.$$

Putting $i = \mathtt{p} - j$ and $t = -t_1 - \mathtt{p}$ in the above equation, we get

$$\sum_{j=0}^{\mathtt{p}} p_{\mathtt{p}-j} y(t_1 + j) - \sum_{j=\mathtt{d}}^{\mathtt{p}} q_{\mathtt{p}-j} u(t_1 + j) = 0, \qquad \text{for every } t_1 \in \mathbb{Z}$$

Let

$$W(t_1) := \operatorname{col}\left(y(t_1), y(t_1+1), \dots, y(t_1+p), u(t_1), \dots, u(t_1+\mathtt{d}-1), u(t_1+\mathtt{d}), u(t_1+\mathtt{d}+1), \dots, u(t_1+\mathtt{p})\right)$$

then

$$\begin{bmatrix} p_{\mathtt{p}} & p_{\mathtt{p}-1} & \cdots & p_0 & 0 & \cdots & 0 & -q_{\mathtt{q}} & \cdots & q_0 \end{bmatrix} W(t_1) = 0, \qquad \text{for every } t_1 \in \mathbb{Z}. \tag{14}$$

Since $\mathscr{B}$ is time-reversible, $\operatorname{col}(y,u) \in \mathscr{B}$, i.e.,

$$\sum_{j=0}^{\mathtt{p}} p_j y(t_1+j) - \sum_{j=0}^{\mathtt{q}} q_j u(t_1+j) = 0, \qquad \text{for every } t_1 \in \mathbb{Z}$$

or

$$\begin{bmatrix} p_0 & p_1 & \cdots & p_{\mathtt{p}} & -q_0 & \cdots & -q_{\mathtt{q}} & 0 & \cdots & 0 \end{bmatrix} W(t_1) = 0, \qquad \text{for every } t_1 \in \mathbb{Z}. \tag{15}$$

Since $\mathscr{B}$ is linear, from equations (14) and (15), it follows that

$$\begin{bmatrix} p_{\mathtt{p}} & p_{\mathtt{p}-1} & \cdots & p_0 & 0 & \cdots & 0 & -q_{\mathtt{q}} & \cdots & q_0 \end{bmatrix} = k \begin{bmatrix} p_0 & p_1 & \cdots & p_{\mathtt{p}} & -q_0 & \cdots & -q_{\mathtt{q}} & 0 & \cdots & 0 \end{bmatrix}$$

where $k \in \mathbb{R}$. It is easy to see that $k = \pm 1$. When $k = 1$, p is palindromic, $q_i = 0$ for $i = 0, \dots, \mathtt{d}-1$, and $q_i = q_{\mathtt{p}-i}$ for $i = \mathtt{d}, \dots, \mathtt{q}$, i.e., $q(\sigma)$ is divisible by σ^d and $q'(\sigma) = q(\sigma)/\sigma^d$ is palindromic. When $k = -1$, p is antipalindromic, $q_i = 0$ for $i = 0, \dots, \mathtt{d}-1$, and $q_i = -q_{\mathtt{p}-i}$ for $i = \mathtt{d}, \dots, \mathtt{q}$, i.e., $q(\sigma)$ is divisible by σ^d and $q'(\sigma) = q(\sigma)/\sigma^d$ is antipalindromic.

If in addition, $\mathscr{B}$ is controllable, then p and q are co-prime. If both p and q' are antipalindromic, then from Lemma 21, both p and q' have at least one root at 1. Since q is divisible by q', this implies that p and q are not co-prime which is a contradiction. Hence if $\mathscr{B}$ is controllable and time-reversible, then p and q' are both palindromic.

A.5 Theorem 20

Let the degree of r be equal to n. Let $r = r'q$, where r' the maximal palindromic factor of r, and let n' denote the degree of r'. Assume that $\mathscr{B}$ has a conserved quantity whose two-variable polynomial representation is $\phi(\zeta,\eta)$. Then

$$\phi(\zeta,\eta) = \frac{r(\zeta) f'(\zeta,\eta) + r(\eta) f'(\eta,\zeta)}{\zeta\eta - 1},$$

for some $f' \in \mathbb{R}[\zeta,\eta]$. It is easy to see that since ϕ is $\mathscr{B}$-canonical, f' is independent of ζ and is of degree less than or equal to $n-1$ in η. Hence

$$\phi(\zeta,\eta) = \frac{r(\zeta) f(\eta) + r(\eta) f(\zeta)}{\zeta\eta - 1}, \tag{16}$$

where $f(\eta) = f'(\zeta,\eta)$. Since ϕ exists, the numerator is divisible by $\zeta\eta - 1$. Consequently by factor theorem,

$$\xi^n \left(r(\xi) f(\xi^{-1}) + r(\xi^{-1}) f(\xi) \right) = 0.$$

This implies that

$$\xi^n f(\xi^{-1}) q(\xi) = -\xi^{n-n'} q(\xi^{-1}) f(\xi).$$

Define $s(\xi) := \xi^{n-n'} f(\xi) q(\xi^{-1})$ and observe that

$$s(\xi) + \xi^{2n-n'} s(\xi^{-1}) = 0. \tag{17}$$

Two cases arise.

- Case 1: v does not have a root at 1. In this case, for equation (17) to hold, it is easy to see that f should be of the form $f(\xi) = v(\xi) f_a(\xi)$, where f_a is an antipalindromic polynomial such that

$$\xi^{n'} f_a(\xi^{-1}) + f_a(\xi) = 0.$$

  Since the degree of f is less than n, the degree of f_a is less than n'. This implies that f_a has at least one root at zero. If n' is even, then the general expression for f_a can be written as

$$f_a(\xi) = \sum_{i=1}^{n'/2-1} p_i(\xi^i - \xi^{n'-i})$$

  whereby the dimension of the space of all possible polynomials f_a is $n'/2-1$. If n' is odd, then the general expression for f_a can be written as

$$f_a(\xi) = \sum_{i=1}^{\frac{n'-1}{2}} p_i(\xi^i - \xi^{n'-i})$$

  whereby the dimension of the space of all possible polynomials f_a is $(n'-1)/2$. From (16), it can be seen that there is a one-one correspondence between ϕ and any f of degree less than or equal to $n-1$. Hence the dimension of conserved quantities for the two subcases of n' being even and odd is the same as the dimension of all possible polynomials f_a for the respective subcases, which is equal to $\lfloor (n'-1)/2 \rfloor$.

- Case 2: v has a root at 1. Let $v(\xi) = (\xi - 1)v'(\xi)$. In this case, for (17) to hold, it is easy to see that f should be of the form $f(\xi) = v'(\xi) f'(\xi)$, where f' is a palindromic polynomial, such that

$$\xi^{n'+1} f'(\xi^{-1}) - f'(\xi) = 0$$

  Since the degree of f is less than n, the degree of f' is less than or equal to n'. This implies that f' has at least one root at zero. If n' is even, the general expression for f' can be written as

$$f'(\xi) = \sum_{i=1}^{n'/2} p_i(\xi^i + \xi^{n'+1-i})$$

  whereby the dimension of the space of all possible polynomials f' is $n'/2$. If n' is odd, the general expression for f' can be written as

$$f'(\xi) = \sum_{i=1}^{\frac{n'-1}{2}} p_i(\xi^i + \xi^{n'+1-i}) + p_{\frac{n_i+1}{2}} \xi^{\frac{n_i+1}{2}}$$

  whereby the dimension of the space of all possible polynomials f' is $(n'+1)/2$. From (16), it can be seen that there is a one-one correspondence between ϕ and any f of degree less than or equal to $n-1$. Hence the dimension of conserved quantities for the two subcases of n' being even and odd is the same as the dimension of all possible polynomials f' for the respective subcases, which is equal to $\lfloor (n'+1)/2 \rfloor$.

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