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Solution of polynomial Lyapunov and Sylvester equations

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ABSTRACT

A two-variable polynomial approach to solve the one-variable polynomial Lyapunov and Sylvester equations is proposed. Lifting the problem from the one-variable to the two-variable context gives rise to associated lifted equations which live on finite-dimensional vector spaces. This allows for the design of an iterative solution method which is inspired by the method of Faddeev for the computation of matrix resolvents. The resulting algorithms are especially suitable for applications requiring symbolic or exact computation.

1. INTRODUCTION

In various areas in mathematical systems and control theory Lyapunov and Sylvester equations play an important role. For instance, they occur in the computation of certain performance criteria in control (see [1,17,18]), in stability theory (see [12,22]), and in relation to statistical quantities such as state covariance matrices and Fisher information (see [13]). In their classical form, their derivation and interpretation is usually most natural within the context of linear time-invariant state-space systems (A, B, C, D), both in the continuous-time case and in the discrete-time case.

In the behavioral approach to systems theory ([21,16]), advocating the use of models derived from first principles which are typically described by systems of high order differential equations, a convenient generalization of the classical

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Lyapunov equation attains the form of a structured polynomial matrix equation in a single variable, constituting the so-called *polynomial Lyapunov equation* (PLE):

$$(1)\qquad R(-\xi)^{\mathrm{T}}X(\xi)+X(-\xi)^{\mathrm{T}}R(\xi)=Z(\xi).$$

Here $R(\xi)$, $X(\xi)$ and $Z(\xi)$ are $q\times q$ real polynomial matrices in the indeterminate ξ, with $R(\xi)$ nonsingular (i.e., $\det(R(\xi))\neq 0$) and with $X(\xi)$ denoting the polynomial matrix to solve for. From the symmetric structure of the left-hand side of this equation it directly follows that solutions to the PLE may exist only if $Z(\xi)$ is a so-called para-Hermitian matrix, which means that $Z(\xi)=Z(-\xi)^{\mathrm{T}}$.

In many practical situations the PLE happens to attain the special form

$$(2)\qquad R(-\xi)^{\mathrm{T}}X(\xi)+X(-\xi)^{\mathrm{T}}R(\xi)=Q(-\xi)^{\mathrm{T}}\Sigma Q(\xi),$$

where $R(\xi)$ is nonsingular, Σ is a $p\times p$ signature matrix (i.e., a diagonal matrix with entries ± 1 on its main diagonal) and $Q(\xi)$ is a $p\times q$ real polynomial matrix which moreover has the property of being R-canonical (i.e., $Q(\xi)R(\xi)^{-1}$ is a strictly proper rational matrix in ξ). In this case one also restricts the search for a solution to the finite-dimensional subspace of R-canonical polynomial matrices $X(\xi)$, which can be done without affecting solvability properties of the equation as will be shown in Section 4. We shall refer to Eq. (2) as the 'PLE in canonical form'. As it turns out, the problem of solving a PLE of the form (1) can always be reduced to that of solving a PLE in canonical form (2); see Section 5 for details. Therefore it is natural to focus attention exclusively on Eq. (2).

A new solution method for this PLE in canonical form, based on lifting the problem to a two-variable polynomial setting and exploiting an algorithm inspired by the method of Faddeev for computing matrix resolvents, has recently been developed in [15]. Here, the results of that paper are briefly reviewed and then extended to deal with a more general type of polynomial matrix equation, which we propose to call the *polynomial Sylvester equation* (PSE). In its general form the PSE is defined as

$$(3)\qquad R_1(-\xi)^{\mathrm{T}}X_{12}(\xi)+X_{21}(-\xi)^{\mathrm{T}}R_2(\xi)=Z(\xi),$$

with $R_1(\xi)$ and $R_2(\xi)$ nonsingular real polynomial matrices in ξ of size $q_1\times q_1$ and $q_2\times q_2$, respectively, and $Z(\xi)$ a real polynomial matrix of size $q_1\times q_2$. Here the polynomial matrices $X_{21}(\xi)$ of size $q_2\times q_1$ and $X_{12}(\xi)$ of size $q_1\times q_2$ constitute the *pair* of unknown quantities to solve for.

As in the Lyapunov case, in many practical situations the PSE attains the following special form which shall be referred to as the 'PSE in canonical form':

$$(4)\qquad R_1(-\xi)^{\mathrm{T}}X_{12}(\xi)+X_{21}(-\xi)^{\mathrm{T}}R_2(\xi)=Q_1(-\xi)^{\mathrm{T}}\Sigma Q_2(\xi),$$

where Σ is a $p\times p$ signature matrix, $Q_1(\xi)$ is a $p\times q_1$ real polynomial matrix which is R_1-canonical and $Q_2(\xi)$ is a $p\times q_2$ real polynomial matrix which is R_2-canonical. In addition, one now may restrict the search for a solution pair $(X_{21}(\xi),X_{12}(\xi))$

to the finite-dimensional subspace of pairs of R_1-canonical matrices $X_{21}(\xi)$ and R_2-canonical matrices $X_{12}(\xi)$. The problem of solving a PSE of the form (3) can always be reduced to that of solving a PSE in canonical form (4); see again Section 5 for details.

The definition of the PLE and PSE in canonical form is motivated primarily by their connection with the problem of computing norms and inner products of the time signals produced by linear time-invariant autonomous systems in kernel form, which is demonstrated by a worked example in Section 7. A markedly distinguishing feature of the PSE when compared to the PLE is that it requires the determination of a solution pair of polynomial matrices $(X_{21}(\xi), X_{12}(\xi))$, while the solution of the PLE consists only of a single polynomial matrix $X(\xi)$.

The solution approach towards the PSE (4) presented here is similar to the approach of [15] to the solution of the PLE (2). By lifting the problem to a two-variable polynomial setting, a new equation is introduced which is called the *lifted polynomial Sylvester equation* (LPSE). In contrast to the PSE, this LPSE requires the determination of a *single* two-variable polynomial matrix only, from which a solution *pair* of one-variable polynomial matrices for the PSE can then be constructed. The proposed algorithm to solve the LPSE is again inspired by the method of Faddeev. It applies to the regular case where the associated Sylvester operator is nonsingular. The algorithm is again designed to be particularly suited for exact and symbolic computation. In contrast to the available algorithms described in the literature (see, e.g., [7–9]), it does not require substantial preprocessing or the transformation of any of the matrices involved into some canonical form.

This chapter is organized as follows. In Section 2 we review several concepts from the literature regarding polynomial matrices and shifts in a single variable. In Section 3 these notions are extended to the case of two-variable polynomial matrices and we define the *Sylvester operator* as a two-variable shift operator on a particular finite-dimensional vector space. The development of the two-variable framework for the study of the PSE is completed in Section 4. There, the PSE is lifted to a two-variable context, giving rise to the LPSE. Next we explore the intimate relationship that exists between the PSE and the LPSE. Section 5 constitutes an intermezzo where we address details of the reduction of the PLE (1) to the PLE in canonical form (2) and of the PSE (3) to the PSE in canonical form (4). We also show how these equations relate to the classical Lyapunov and Sylvester equations for state-space systems (A, B, C, D). In Section 6 the Sylvester operator is used to formulate an iterative algorithm to compute a solution Y to the LPSE which is inspired by the method of Faddeev for computing matrix resolvents and generalizes the algorithm of [15]. From this two-variable solution matrix Y a one-variable solution pair (X_{21}, X_{12}) to the PSE (4) is constructed. In Section 7 the algorithm is demonstrated by a worked example. A section containing final remarks concludes the chapter. Because of space limitations no proofs are included. Most of these proofs can be obtained as generalizations of the proofs employed in the Lyapunov case addressed in [15]; they will be given elsewhere.

2. R-EQUIVALENCE AND THE ONE-VARIABLE SHIFT OPERATOR

In this section we briefly review a number of well-known results on polynomial matrices in a single variable which are important in the sequel. The concepts and notions introduced in this section are not new, although the terminology used elsewhere may differ. See also [2,3,20] and [15].

Let R be an element of $\mathbb{R}^{q\times q}[\xi]$, the set of $q \times q$ real polynomial matrices in the indeterminate ξ. Assume that R is nonsingular, i.e., $\det(R)$ does not vanish identically. Then R induces an equivalence relation on the set of polynomial row vectors $\mathbb{R}^{1\times q}[\xi]$ as follows.

Definition 2.1. Two polynomial vectors $D_1, D_2 \in \mathbb{R}^{1\times q}[\xi]$ are called *R-equivalent* if there exists a polynomial vector $P \in \mathbb{R}^{1\times q}[\xi]$ such that $D_1 - D_2 = PR$. A polynomial vector $D \in \mathbb{R}^{1\times q}[\xi]$ is called *R-canonical* if the rational vector DR^{-1} is strictly proper.

Every $1 \times q$ polynomial vector D admits a unique R-canonical polynomial vector D' which is R-equivalent to D. This R-canonical representative D' of the R-equivalence class of D can be computed as $D' = SR = D - PR$, where P denotes the polynomial part and S the strictly proper part of $DR^{-1} = P + S$. We alternatively denote D' by $D \bmod R$. The subset of $\mathbb{R}^{1\times q}[\xi]$ consisting of all R-canonical polynomial vectors is denoted by $\mathcal{C}_R^{1\times q}[\xi]$, for which the following proposition holds.

Proposition 2.2. *The space $\mathcal{C}_R^{1\times q}[\xi]$ is a finite-dimensional vector space over $\mathbb{R}$ of dimension $n = \deg(\det(R))$. It can be identified with the vector space of R-equivalence classes in $\mathbb{R}^{1\times q}[\xi]$ in a natural way.*

We proceed to define the polynomial shift operator σ on $\mathcal{C}_R^{1\times q}[\xi]$.

Definition 2.3. The (one-variable) *polynomial shift operator* $\sigma : \mathcal{C}_R^{1\times q}[\xi] \to \mathcal{C}_R^{1\times q}[\xi]$ is the linear operator defined by the action:

$$\sigma\big(D(\xi)\big) := \xi D(\xi) \bmod R(\xi).$$

Proposition 2.4. *The characteristic polynomial $\chi_\sigma(z)$ of the operator σ on $\mathcal{C}_R^{1\times q}[\xi]$ is given by*

$$\chi_\sigma(z) = \det\big(R(z)\big)/r_0,$$

where r_0 denotes the leading coefficient of $\det(R(z))$.

The definition of the shift σ can obviously be extended from $\mathbb{R}^{1\times q}[\xi]$ to $\mathbb{R}^{k\times q}[\xi]$ in a row-by-row manner. The concepts of R-equivalence and R-canonicity are extended likewise. The subspace of R-canonical elements of $\mathbb{R}^{k\times q}[\xi]$ is denoted by $\mathcal{C}_R^{k\times q}[\xi]$.

3. Two-variable (R_1, R_2)-equivalence and the Sylvester operator

In this section we study (R_1, R_2)-equivalence, (R_1, R_2)-canonicity and shift operators on spaces of symmetric and nonsymmetric polynomial matrices in two variables. The material of this section is in part a review of notions developed in the context of quadratic differential forms, see [20]. It extends the results of [15] on symmetric two-variable polynomial matrices and the Lyapunov operator to the nonsymmetric case, thereby introducing the Sylvester operator.

The vector space of $q_1 \times q_2$ real polynomial matrices in the two indeterminates ζ and η is denoted by $\mathbb{R}^{q_1\times q_2}[\zeta, \eta]$. A (square) polynomial matrix $Y \in \mathbb{R}^{q\times q}[\zeta, \eta]$ is called *symmetric* if $Y(\zeta, \eta) = Y(\eta, \zeta)^{\mathrm{T}}$. The subspace of all symmetric polynomial matrices in $\mathbb{R}^{q\times q}[\zeta, \eta]$ is denoted by $\mathbb{R}^{q\times q}_{\mathrm{sym}}[\zeta, \eta]$.

Let $R_1 \in \mathbb{R}^{q_1\times q_1}[\xi]$ and $R_2 \in \mathbb{R}^{q_2\times q_2}[\xi]$ be nonsingular. Then R_1 and R_2 together induce an equivalence relation on $\mathbb{R}^{q_1\times q_2}[\zeta, \eta]$ in the following way.

Definition 3.1. Two $q_1 \times q_2$ polynomial matrices $Y_1, Y_2 \in \mathbb{R}^{q_1\times q_2}[\zeta, \eta]$ are called (R_1, R_2)-*equivalent* if there exist two polynomial matrices $P_1 \in \mathbb{R}^{q_1\times q_2}[\zeta, \eta]$ and $P_2 \in \mathbb{R}^{q_2\times q_1}[\zeta, \eta]$ such that $Y_1(\zeta, \eta) - Y_2(\zeta, \eta) = R_1(\zeta)^{\mathrm{T}} P_1(\zeta, \eta) + P_2(\eta, \zeta)^{\mathrm{T}} R_2(\eta)$. A polynomial matrix $Y \in \mathbb{R}^{q_1\times q_2}[\zeta, \eta]$ is called (R_1, R_2)-*canonical* if the rational two-variable matrix $R_1(\zeta)^{-\mathrm{T}} Y(\zeta, \eta) R_2(\eta)^{-1}$ is strictly proper in ζ and in η.

Every $Y \in \mathbb{R}^{q_1\times q_2}[\zeta, \eta]$ admits a unique (R_1, R_2)-canonical two-variable polynomial matrix Y' which is (R_1, R_2)-equivalent to Y. Computation of this (R_1, R_2)-canonical representative Y' of the (R_1, R_2)-equivalence class of Y may proceed as follows. First determine a factorization of Y of the form $Y(\zeta, \eta) = M(\zeta)^{\mathrm{T}} N(\eta)$. Note that this can always be achieved with M and N not necessarily square; see also [20] and [14]. Then $Y'(\zeta, \eta) = M'(\zeta)^{\mathrm{T}} N'(\eta)$, where $M' = M \bmod R_1$ and $N' = N \bmod R_2$ (in the sense of one-variable R_1-equivalence and R_2-equivalence, respectively).

The (R_1, R_2)-canonical representative Y' of the (R_1, R_2)-equivalence class of $Y \in \mathbb{R}^{q_1\times q_2}[\zeta, \eta]$ is alternatively denoted by $Y' = Y \bmod(R_1, R_2)$. The subset of $\mathbb{R}^{q_1\times q_2}[\zeta, \eta]$ of all (R_1, R_2)-canonical two-variable polynomial matrices is denoted by $\mathcal{C}^{q_1\times q_2}_{R_1,R_2}[\zeta, \eta]$.

Proposition 3.2. *The space* $\mathcal{C}^{q_1\times q_2}_{R_1,R_2}[\zeta, \eta]$ *is a finite-dimensional vector space over* $\mathbb{R}$ *of dimension* $n_1 n_2$*, where* $n_1 = \deg(\det(R_1))$ *and* $n_2 = \deg(\det(R_2))$*. It can be identified with the vector space of* (R_1, R_2)*-equivalence classes in* $\mathbb{R}^{q_1\times q_2}[\zeta, \eta]$ *in a natural way.*

We proceed to define the two-variable shift operator $\mathcal{S}_{R_1,R_2}$ acting on the space $\mathcal{C}^{q_1\times q_2}_{R_1,R_2}[\zeta, \eta]$ of (R_1, R_2)-canonical two-variable polynomial matrices. This linear operator will be referred to as the *Sylvester operator* associated with R_1 and R_2 for reasons that will become clear in the next section.

Definition 3.3. The *Sylvester operator* $\mathcal{S}_{R_1,R_2} : \mathcal{C}^{q_1\times q_2}_{R_1,R_2}[\zeta,\eta] \to \mathcal{C}^{q_1\times q_2}_{R_1,R_2}[\zeta,\eta]$ is defined by the action

$$\mathcal{S}_{R_1,R_2}\big(Y(\zeta,\eta)\big) := (\zeta+\eta)Y(\zeta,\eta) \bmod (R_1,R_2). \tag{5}$$

Proposition 3.4. *The characteristic polynomial $\chi_{\mathcal{S}_{R_1,R_2}}(z)$ of the Sylvester operator $\mathcal{S}_{R_1,R_2}$ acting on $\mathcal{C}^{q_1\times q_2}_{R_1,R_2}[\zeta,\eta]$ is given by*

$$\chi_{\mathcal{S}_{R_1,R_2}}(z) := \prod_{i=1}^{n_1}\prod_{j=1}^{n_2}\big(z-(\lambda_i+\mu_j)\big), \tag{6}$$

where $n_1 = \deg(\det(R_1))$ and $n_2 = \deg(\det(R_2))$, and where $\lambda_1,\ldots,\lambda_{n_1}$ and $\mu_1,\ldots,\mu_{n_2}$ denote the zeros of $\det(R_1)$ *and* $\det(R_2)$ *respectively* (*including multiplicities*).

In [15] attention has been focused exclusively on the symmetric case $\mathbb{R}^{q\times q}_{\mathrm{sym}}[\zeta,\eta]$. There, the concept of two-variable R-equivalence was introduced, which can be shown to coincide on this subspace with the concept of (R_1,R_2)-equivalence introduced above when $R_1 = R_2 = R$. Also the *Lyapunov* operator $\mathcal{L}_R$ was introduced as the two-variable shift operator on the space of R-canonical two-variable symmetric polynomial matrices $\mathcal{C}^{q\times q}_{R,\mathrm{sym}}[\zeta,\eta]$ which is readily seen to be a subspace of $\mathcal{C}^{q\times q}_{R,R}[\zeta,\eta]$. On this subspace the Lyapunov operator coincides with the restriction of the Sylvester operator $\mathcal{S}_{R,R}$. Note that the subspace $\mathcal{C}^{q\times q}_{R,\mathrm{sym}}[\zeta,\eta]$ has dimension $n(n+1)/2$ instead of n^2 (with $n = \deg(\det(R))$), so that the characteristic polynomial of $\mathcal{L}_R$ is different from the characteristic polynomial of $\mathcal{S}_{R,R}$ (unrestricted) as expressed by the fact that the multiplicities of its zeros are lower. Since the degrees of the characteristic polynomials of these operators determine the number of iterations to be carried out in our solution algorithms, this shows how the implicit incorporation of symmetry in the Lyapunov case leads to a more efficient algorithm than the present nonsymmetric Sylvester approach would give on such a more structured problem.

4. The lifted polynomial Sylvester equation

In this section we complete the framework for the study of the PSE. First we lift the problem of computing a solution to the PSE in canonical form (4) from the one-variable polynomial context in which it was formulated above, to a two-variable polynomial context. To this end we now introduce the following two-variable polynomial equation associated with the matrices R_1, R_2, Q_1, Q_2 and Σ which define the PSE (4). The equation

$$(\zeta+\eta)Y(\zeta,\eta) \bmod (R_1,R_2) = Q_1(\zeta)^{\mathrm{T}}\Sigma Q_2(\eta) \tag{7}$$

in the unknown (R_1,R_2)-canonical two-variable polynomial matrix $Y \in \mathcal{C}^{q_1\times q_2}_{R_1,R_2}[\zeta,\eta]$ is called the *lifted polynomial Sylvester equation* (LPSE). As in the Lyapunov case treated in [15], solvability of the PSE is equivalent to solvability of the LPSE, as the following proposition shows.

Proposition 4.1. *Let $R_1 \in \mathbb{R}^{q_1 \times q_1}[\xi]$ and $R_2 \in \mathbb{R}^{q_2 \times q_2}[\xi]$ both be nonsingular, let $Q_1 \in \mathbb{R}^{p \times q_1}[\xi]$ be R_1-canonical, let $Q_2 \in \mathbb{R}^{p \times q_2}[\xi]$ be R_2-canonical and let Σ be a $p \times p$ signature matrix. Then the following two statements are equivalent.*

(1) *There exists a solution pair $(X_{21}, X_{12}) \in \mathbb{R}^{q_2 \times q_1}[\xi] \times \mathbb{R}^{q_1 \times q_2}[\xi]$ for the associated PSE* (4).
(2) *There exists a solution $Y \in \mathcal{C}^{q_1 \times q_2}_{R_1,R_2}[\zeta, \eta]$ for the associated LPSE* (7).

A solution pair (X_{21}, X_{12}) for the PSE is called (R_1, R_2)-canonical if X_{21} is R_1-canonical and X_{12} is R_2-canonical. The next proposition characterizes the solution space of the PSE (4) as a direct sum of (R_1, R_2)-canonical solution pairs and the solution space to the homogeneous PSE.

Proposition 4.2. *Let $R_1 \in \mathbb{R}^{q_1 \times q_1}[\xi]$ and $R_2 \in \mathbb{R}^{q_2 \times q_2}[\xi]$ both be nonsingular, let $Q_1 \in \mathbb{R}^{p \times q_1}[\xi]$ be R_1-canonical, let $Q_2 \in \mathbb{R}^{p \times q_2}[\xi]$ be R_2-canonical and let Σ be a $p \times p$ signature matrix.*

Let $\mathcal{X}_{R_1,R_2} \subset \mathcal{C}^{q_2 \times q_1}_{R_1}[\xi] \times \mathcal{C}^{q_1 \times q_2}_{R_2}[\xi]$ be the set of all (R_1, R_2)-canonical solution pairs of the PSE.

Then the space of all solutions pairs (X_{21}, X_{12}) of the PSE is given by

$$\mathcal{X}_{R_1,R_2} \oplus \left\{(SR_1, -S^{\sim}R_2) \mid S \in \mathbb{R}^{q_2 \times q_1}[\xi]\right\},$$

where $S^{\sim}(\xi) := S(-\xi)^{\mathrm{T}}$.

Observe that Proposition 4.2 implies that the PSE admits a solution pair if and only if it admits an (R_1, R_2)-canonical solution pair. Consequently, as a corollary, the search for a solution pair of the PSE can be restricted from the infinite-dimensional space $\mathbb{R}^{q_2 \times q_1}[\xi] \times \mathbb{R}^{q_1 \times q_2}[\xi]$ to the space $\mathcal{C}^{q_2 \times q_1}_{R_1}[\xi] \times \mathcal{C}^{q_1 \times q_2}_{R_2}[\xi]$ of finite dimension $q_2 n_1 \times q_1 n_2$.

From an arbitrary solution pair (X_{21}, X_{12}) for the PSE a two-variable solution Y of the LPSE can explicitly be constructed, and vice versa. Indeed, if (X_{21}, X_{12}) is a solution pair for the PSE let $\widehat{Y}$ be defined by $\widehat{Y}(\zeta, \eta) := [Q_1(\zeta)^{\mathrm{T}} \Sigma Q_2(\eta) - R_1(\zeta)^{\mathrm{T}} X_{12}(\eta) - X_{21}(\zeta)^{\mathrm{T}} R_2(\eta)]/(\zeta + \eta)$. Observe that $\widehat{Y}$ is indeed a polynomial matrix (since the numerator matrix polynomial vanishes by construction when ζ is put equal to $-\eta$), which however need not be (R_1, R_2)-canonical. Now let Y be defined as the (R_1, R_2)-canonical representative $Y := \widehat{Y} \bmod (R_1, R_2)$ of the (R_1, R_2)-equivalence class of $\widehat{Y}$. It can then be verified directly that Y solves the LPSE.

Conversely, and more important for our purposes, if Y is a solution to the LPSE then by definition of (R_1, R_2)-equivalence there exist two polynomial matrices $P_1 \in \mathbb{R}^{q_1 \times q_2}[\zeta, \eta]$ and $P_2 \in \mathbb{R}^{q_2 \times q_1}[\zeta, \eta]$ such that

$$(\zeta + \eta) Y(\zeta, \eta) + R_1(\zeta)^{\mathrm{T}} P_1(\zeta, \eta) + P_2(\eta, \zeta)^{\mathrm{T}} R_2(\eta) = Q_1(\zeta)^{\mathrm{T}} \Sigma Q_2(\eta). \quad (8)$$

A solution to the PSE is then obtained from P_1 and P_2 by substituting $\zeta = -\xi$ and $\eta = \xi$, yielding $X_{21}(\xi) := P_2(-\xi, \xi)$ and $X_{12}(\xi) := P_1(-\xi, \xi)$. This, however, is an indirect way of computing a solution pair (X_{21}, X_{12}) from Y, requiring determination of the two-variable polynomial matrices P_1 and P_2. The following proposition shows how an (R_1, R_2)-canonical solution pair (X_{21}, X_{12}) for the PSE can in fact be expressed directly in terms of a solution Y to the LPSE.

Proposition 4.3. *Let $Y \in \mathcal{C}^{q_1\times q_2}_{R_1,R_2}[\zeta, \eta]$ be a solution of the LPSE. Then an (R_1, R_2)-canonical solution pair $(X_{21}, X_{12}) \in \mathcal{C}^{q_2\times q_1}_{R_1}[\xi] \times \mathcal{C}^{q_1\times q_2}_{R_2}[\xi]$ for the PSE is given by*

$$(9)\qquad \begin{aligned} X_{21}(\xi) &:= -\lim_{|\mu|\to\infty} \mu R_2(\mu)^{-\mathrm{T}} Y(\xi, \mu)^{\mathrm{T}}, \\ X_{12}(\xi) &:= -\lim_{|\mu|\to\infty} \mu R_1(\mu)^{-\mathrm{T}} Y(\mu, \xi). \end{aligned}$$

Moreover, for such (X_{21}, X_{12}) it holds that $(\zeta + \eta)Y(\zeta, \eta) + R_1(\zeta)^{\mathrm{T}} X_{12}(\eta) + X_{21}(\zeta)^{\mathrm{T}} R_2(\eta) = Q_1(\zeta)^{\mathrm{T}} \Sigma Q_2(\eta)$.

Note that the last statement of this proposition makes clear that the two-variable polynomial matrices P_1 and P_2 required in the indirect computation of (X_{21}, X_{12}) from Y based on Eq. (8), can in fact be chosen to be one-variable polynomials in η and ζ, respectively.

Propositions 4.1–4.3 show that *to solve the PSE one can first solve the LPSE and then construct an (R_1, R_2)-canonical solution pair for the PSE from the solution of the LPSE*.

If we denote the right-hand side of the LPSE by $\Phi(\zeta, \eta) := Q_1(\zeta)^{\mathrm{T}} \Sigma Q_2(\eta)$, then the LPSE can be written compactly as $\mathcal{S}_{R_1,R_2}(Y) = \Phi$, with $\mathcal{S}_{R_1,R_2}$ the Sylvester operator. From Proposition 3.4 a necessary and sufficient condition for the existence of a unique solution to the LPSE is immediate. It is remarkable that *the same condition also characterizes the existence of a unique (R_1, R_2)-canonical solution pair for the PSE*.

Proposition 4.4. *Let $R_1 \in \mathbb{R}^{q_1\times q_1}[\xi]$ and $R_2 \in \mathbb{R}^{q_2\times q_2}[\xi]$ be nonsingular, let $Q_1 \in \mathbb{R}^{p\times q_1}[\xi]$ be R_1-canonical, let $Q_2 \in \mathbb{R}^{p\times q_2}[\xi]$ be R_2-canonical and let Σ be a $p \times p$ signature matrix. Let $n_1 = \deg(\det(R_1))$, $n_2 = \deg(\det(R_2))$ and let $\lambda_1, \ldots, \lambda_{n_1}$ and $\mu_1, \ldots, \mu_{n_2}$ be the zeros of $\det(R_1)$ and $\det(R_2)$, respectively. Then the following three statements are equivalent.*

(1) *The following condition is satisfied*:

$$(10)\qquad \lambda_i + \mu_j \neq 0 \quad \text{for all } i = 1, 2, \ldots, n_1;\ j = 1, 2, \ldots, n_2.$$

(2) *The LPSE has a unique solution (which is (R_1, R_2)-canonical).*
(3) *The PSE has a unique (R_1, R_2)-canonical solution pair.*

For obvious reasons we call condition (10) the *invertibility condition* for the operator $\mathcal{S}_{R_1,R_2}$. Observe that this condition is certainly satisfied when R_1 and R_2

are Hurwitz, i.e., when all λ_i and μ_j are in the open left half of the complex plane. The invertibility condition is similar to well-known sufficient conditions for the existence of a solution to the classical matrix Lyapunov and Sylvester equations (see, for example, [4, Section VIII.3]).

5. Reduction of the PSE to the PSE in canonical form and its relationship with the classical Sylvester equation

In this section we supply additional details on two topics. First we consider the issue of the reduction of a PLE (PSE) of the general form (1) (or (3)) to a PLE (PSE) in the canonical form (2) (or (4)). Next we investigate the relationship between the PLE (PSE) and the classical Lyapunov (Sylvester) equation which emerges as a special case associated with the conventional context of state-space systems (A, B, C, D).

To start with the first issue, let $Z(\xi)$ be the right-hand side of a PLE of the form (1). If $Z(\xi) \neq Z(-\xi)^{\mathrm{T}}$ there are no solutions to the PLE. Otherwise put $p = 2q$ and define $Q(\xi)$ and Σ as follows:

$$(11) \qquad Q(\xi) := \begin{pmatrix} (Z(\xi) + I_q)/2 \\ (Z(\xi) - I_q)/2 \end{pmatrix}, \qquad \Sigma := \begin{pmatrix} I_q & 0 \\ 0 & -I_q \end{pmatrix}.$$

It then is straightforward to verify that the associated PLE of the form (2) is equivalent to the PLE (1). In case of a PSE of the form (3) the situation is even easier because symmetry aspects do not play a role. Here one may simply put $p = q_2$ and define

$$(12) \qquad Q_1(\xi) := Z(-\xi)^{\mathrm{T}}, \qquad Q_2(\xi) := I_{q_2}, \qquad \Sigma := I_{q_2}.$$

Alternatively, depending on the dimensions q_1 and q_2, it may be preferable to put $p = q_1$ and to define

$$(13) \qquad Q_1(\xi) := I_{q_1}, \qquad Q_2(\xi) := Z(\xi), \qquad \Sigma := I_{q_1}.$$

Other solutions are obviously possible.

Once a PSE in the form (4) has been obtained according to the recipe given above, the next step is to enforce R_1-canonicity of Q_1 and R_2-canonicity of Q_2. To this end, let Q_1' be the R_1-canonical representative of the R_1-equivalence class of Q_1 with T_1 a polynomial matrix such that $Q_1 = Q_1' + T_1 R_1$. Likewise let Q_2' be the R_2-canonical representative of the R_2-equivalence class of Q_2 with T_2 a polynomial matrix such that $Q_2 = Q_2' + T_2 R_2$. Then the right-hand side of the PSE can be expanded into a sum four terms, yielding

$$(14) \qquad \begin{aligned} Q_1(-\xi)^{\mathrm{T}} \Sigma Q_2(\xi) = {} & Q_1'(-\xi)^{\mathrm{T}} \Sigma Q_2'(\xi) \\ & + Q_1'(-\xi)^{\mathrm{T}} \Sigma T_2(\xi) R_2(\xi) + R_1(-\xi)^{\mathrm{T}} T_1(-\xi)^{\mathrm{T}} \Sigma Q_2'(\xi) \\ & + R_1(-\xi)^{\mathrm{T}} T_1(-\xi)^{\mathrm{T}} \Sigma T_2(\xi) R_2(\xi). \end{aligned}$$

Because of linearity of the PSE, individual solution pairs with respect to each term may be superimposed. The first term of the expansion above corresponds to a PSE exactly in the canonical form (4) that we are reducing to, with all the required properties. A particular solution pair for the PSE corresponding to the remaining terms is easily verified to be given by

$$(15)\qquad \big(T_2(-\xi)^{\mathrm{T}}\Sigma\big(Q_1'(\xi)+T_1(\xi)R_1(\xi)/2\big),\, T_1(-\xi)^{\mathrm{T}}\Sigma\big(Q_2'(\xi)+T_2(\xi)R_2(\xi)/2\big)\big).$$

In case of a PLE in the form (2) a similar procedure can be adopted with all the indices dropped to obtain a PLE with R-canonicity holding for Q.

For the second issue, it will be natural to associate the following two linear time-invariant autonomous systems Σ_1 and Σ_2 with the polynomial matrices R_1, Q_1, R_2 and Q_2 in the PSE:

$$\Sigma_1 := \begin{cases} R_1\left(\dfrac{d}{dt}\right)w_1 = 0, \\ y_1 = Q_1\left(\dfrac{d}{dt}\right)w_1, \end{cases} \quad \text{and} \quad \Sigma_2 := \begin{cases} R_2\left(\dfrac{d}{dt}\right)w_2 = 0, \\ y_2 = Q_2\left(\dfrac{d}{dt}\right)w_2. \end{cases}$$

It is clear that the output signals y_1 and y_2 remain unaffected by replacement of Q_1 and Q_2 by arbitrary R_1-equivalent and R_2-equivalent matrices, respectively. Thus the requirement of R_1-canonicity of Q_1 and R_2-canonicity of Q_2 appears naturally in such a context.

In the classical situation of state-space systems (A, B, C, D), the quantities w_1 and w_2 serve as state vectors and the systems consist of first-order differential equations where the polynomial matrices R_1 and R_2 attain the special form

$$(16)\qquad R_1(\xi) = \xi I_{q_1} - A_1, \qquad R_2(\xi) = \xi I_{q_2} - A_2.$$

The properties of R_1-canonicity of Q_1 and R_2-canonicity of Q_2 then amount to these matrices being constant:

$$(17)\qquad Q_1(\xi) = C_1, \qquad Q_2(\xi) = C_2.$$

Thus, with $\Sigma = I_p$, the PSE attains the form

$$(18)\qquad \big(-\xi I_{q_1} - A_1^{\mathrm{T}}\big)X_{12} + X_{21}^{\mathrm{T}}(\xi I_{q_2} - A_2) = C_1^{\mathrm{T}} C_2.$$

Here, (R_1, R_2)-canonicity of the solution pair (X_{21}, X_{12}) also implies that the matrices X_{21} and X_{12} are constant. By comparing the terms that are linear in ξ it is obtained that $X_{12} = X_{21}^{\mathrm{T}} =: X$, say. Then the remaining terms yield the equation

$$(19)\qquad A_1^{\mathrm{T}} X + X A_2 = -C_1^{\mathrm{T}} C_2,$$

which is precisely the classical Sylvester equation for X. The Lyapunov case can be handled in an entirely analogous fashion by dropping the indices.

6. A RECURSIVE ALGORITHM TO SOLVE THE PSE

In this section we present a recursive procedure to solve the PSE (2) under the assumption that the invertibility condition (10) is satisfied. It generalizes the procedure of [15] for the solution of the PLE (2). The method is conceptually and computationally transparent in the sense that the matrices R_1 and R_2 need not be transformed to some desired canonical representation, and that the amount of bookkeeping is kept to a minimum. The algorithm is particularly suited for computation in an exact or symbolic context.

The method is inspired by the Faddeev algorithm for computing the resolvent $(zI_n - A)^{-1}$ of an $n \times n$ matrix A. (See, for example, [6] and [4, Section IV.4] for a more detailed exposition.) Assume that A is invertible and let $\chi_A(z) = \det(zI_n - A) = z^n + \chi_1 z^{n-1} + \cdots + \chi_{n-1}z + \chi_n$ be the characteristic polynomial of A. Then $\chi_n = (-1)^n \det(A) \neq 0$ and also $\chi_A(A) = 0$ according to the well-known theorem of Cayley and Hamilton. Note that it follows that $A(A^{n-1} + \chi_1 A^{n-2} + \cdots + \chi_{n-1}I_n) = -\chi_n I_n$, whence the inverse of A is given by $A^{-1} = -\frac{1}{\chi_n}(A^{n-1} + \chi_1 A^{n-2} + \cdots + \chi_{n-1}I_n)$. Observe that the unique solution $\hat{x} = A^{-1}b$ to the linear system of equations $Ax = b$ can therefore be computed by the following iterative procedure:

$$x_0 := b, \tag{20}$$

$$x_k := Ax_{k-1} + \chi_k b \quad (k = 1, 2, \ldots, n-1), \tag{21}$$

$$\hat{x} := -\frac{1}{\chi_n} x_{n-1}. \tag{22}$$

Prior knowledge of the coefficients χ_k of the characteristic polynomial of the matrix A is fundamental for applicability of this procedure. In case of the LPSE, we are dealing with a linear system of equations on a finite-dimensional vector space, namely $\mathcal{S}_{R_1,R_2}(Y) = \Phi$. The characteristic polynomial of the Lyapunov operator $\mathcal{S}_{R_1,R_2}$ is available and described by Eq. (6). In order to come up with a procedure to solve the PSE we therefore only need to adapt the recursion (20)–(22) to the case at hand. This yields the main result of this section.

Proposition 6.1. *Let $R_1 \in \mathbb{R}^{q_1\times q_1}[\xi]$ and $R_2 \in \mathbb{R}^{q_2\times q_2}[\xi]$ both be nonsingular, let $Q_1 \in \mathbb{R}^{p\times q_1}[\xi]$ be R_1-canonical, let $Q_2 \in \mathbb{R}^{p\times q_2}[\xi]$ be R_2-canonical and let Σ be a $p \times p$ signature matrix. Let $n_1 = \deg(\det(R_1))$, $n_2 = \deg(\det(R_2))$ and let $\lambda_1, \ldots, \lambda_{n_1}$ and $\mu_1, \ldots, \mu_{n_2}$ be the zeros of $\det(R_1)$ and $\det(R_2)$, respectively. Assume that the invertibility condition (10) holds. Let $\chi_{\mathcal{S}_{R_1,R_2}}(z) = z^d + \gamma_1 z^{d-1} + \cdots + \gamma_{d-1}z + \gamma_d$ be the characteristic polynomial of the Lyapunov operator $\mathcal{S}_{R_1,R_2}$ as given by Eq. (6) with $d = n_1 n_2$. Denote the right-hand side of the LPSE by $\Phi(\zeta,\eta) := Q_1(\zeta)^{\mathrm{T}}\Sigma Q_2(\eta)$. Consider the recursion*:

$$Y_0 := \Phi, \tag{23}$$

$$Y_k := \mathcal{S}_{R_1,R_2}(Y_{k-1}) + \gamma_k \Phi, \tag{24}$$

for $k = 1, 2, \ldots, d-1$. Then the two-variable polynomial matrix

$$Y := -\frac{1}{\gamma_d} Y_{d-1} \tag{25}$$

yields the unique solution of the LPSE. From Y the unique (R_1, R_2)-canonical solution pair (X_{21}, X_{12}) for the PSE is computed as:

$$\begin{aligned} X_{21}(\xi) &:= -\lim_{|\mu|\to\infty} \mu R_2(\mu)^{-\mathrm{T}} Y(\xi,\mu)^{\mathrm{T}}, \\ X_{12}(\xi) &:= -\lim_{|\mu|\to\infty} \mu R_1(\mu)^{-\mathrm{T}} Y(\mu,\xi). \end{aligned} \tag{26}$$

As stated above, knowledge of the characteristic polynomial of $\mathcal{S}_{R_1,R_2}$ is fundamental for applicability of the algorithm above. Observe that in the context of symbolic or exact computation it is not advisable to compute the characteristic polynomial of $\mathcal{S}_{R_1,R_2}$ from the zeros λ_i and μ_j of $\det(R_1)$ and $\det(R_2)$ as might be suggested by Eq. (6). An efficient rational algorithm to compute the coefficients of $\chi_{\mathcal{S}_{R_1,R_2}}$ directly from the coefficients of the polynomials $\det(R_1)$ and $\det(R_2)$ can be designed using Faddeev-type recursions analogous to those of [6, Section 5]. This generalizes the corresponding algorithm of [15] for the Lyapunov case:

Proposition 6.2. *Let $R_1 \in \mathbb{R}^{q_1\times q_1}[\xi]$ and $R_2 \in \mathbb{R}^{q_2\times q_2}[\xi]$ both be nonsingular. Let $n_1 = \deg(\det(R_1))$, $n_2 = \deg(\det(R_2))$ and let $\lambda_1, \ldots, \lambda_{n_1}$ and $\mu_1, \ldots, \mu_{n_2}$ be the zeros of $\det(R_1)$ and $\det(R_2)$, respectively. Define $\alpha(z) = z^{n_1} + \alpha_1 z^{n_1-1} + \cdots + \alpha_{n_1-1}z + \alpha_{n_1} := \prod_{i=1}^{n_1}(z-\lambda_i)$ and put $\alpha_k := 0$ for all $k > n_1$. Likewise, define $\beta(z) = z^{n_2} + \beta_1 z^{n_2-1} + \cdots + \beta_{n_2-1}z + \beta_{n_2} := \prod_{j=1}^{n_2}(z-\mu_j)$ and define $\beta_k := 0$ for all $k > n_2$. Let $d = n_1 n_2$ and consider the following four recursions that define the quantities t_k, s_k, u_k and γ_k for $k = 1, 2, \ldots, d$.*

$$t_k := -\left(k\alpha_k + \sum_{\ell=1}^{k-1} t_\ell \alpha_{k-\ell}\right), \quad \text{with } t_1 := -\alpha_1, \tag{27}$$

$$s_k := -\left(k\beta_k + \sum_{\ell=1}^{k-1} s_\ell \beta_{k-\ell}\right), \quad \text{with } s_1 := -\beta_1, \tag{28}$$

$$u_k := n_1 s_k + n_2 t_k + \sum_{\ell=1}^{k-1} \binom{k}{\ell} t_\ell s_{k-\ell}, \quad \text{with } u_1 := -n_1\beta_1 - n_2\alpha_1, \tag{29}$$

$$\gamma_k := -\left(u_k + \sum_{\ell=1}^{k-1} u_\ell \gamma_{k-\ell}\right)\Big/ k, \quad \text{with } \gamma_1 := n_1\beta_1 + n_2\alpha_1. \tag{30}$$

Then the characteristic polynomial $\chi_{\mathcal{S}_{R_1,R_2}}(z)$ of the Sylvester operator $\mathcal{S}_{R_1,R_2}$ is given by

$$\chi_{\mathcal{S}_{R_1,R_2}}(z) = z^d + \gamma_1 z^{d-1} + \cdots + \gamma_{d-1}z + \gamma_d. \tag{31}$$

Note that the above result shows that the exact computation of the coefficients of the characteristic polynomial of the Sylvester operator is possible even in cases where the computation of the zeros of $\det(R_1)$ or of $\det(R_2)$ is infeasible, such as when these depend on symbolic, unspecified parameters.

Remark 1. The algorithm (23)–(26) involves the computation of the (R_1, R_2)-canonical representatives of $(\zeta + \eta)Y_{k-1}(\zeta, \eta)$ for $k = 1, 2, \ldots, d-1$. It is easy to see that if one defines the matrices $Y'_{1,k}(\xi) := -\lim_{|\mu|\to\infty} \mu R_1(\mu)^{-\mathrm{T}} Y_k(\mu, \xi)$ and $Y'_{2,k}(\xi) := -\lim_{|\mu|\to\infty} \mu R_2(\mu)^{-\mathrm{T}} Y_k(\xi, \mu)^{\mathrm{T}}$ it holds that $(\zeta + \eta)Y_{k-1}(\zeta, \eta) \bmod (R_1, R_2) = (\zeta + \eta)Y_{k-1}(\zeta, \eta) + R_1(\zeta)^{\mathrm{T}} Y'_{2,k-1}(\eta) + Y'_{1,k-1}(\zeta)^{\mathrm{T}} R_2(\eta)$. The authors have devised a Faddeev-type recursion that enables the computation of $Y'_{1,k-1}$ and $Y'_{2,k-1}$ with polynomial operations only and which only requires division between the highest-power coefficients of certain univariate polynomials. Such implementation details will be discussed elsewhere; see also [14] for similar considerations in the Lyapunov case.

Remark 2. In many cases the matrices $R_1(\xi)$ and $R_2(\xi)$ have the property that their leading coefficient matrices are nonsingular. For example, this always happens for the scalar PSE: $r_1(-\xi)x_{12}(\xi) + x_{21}(-\xi)r_2(\xi) = q_1(-\xi)q_2(\xi)$, where r_1, r_2, q_1, q_2, x_{12} and $x_{21} \in \mathbb{R}[\xi]$. An algorithm can then be developed that takes advantage of this property. Full details will again be presented elsewhere; see also [14].

7. Example

In this section we demonstrate the algorithm of this chapter by means of a worked example. We also present an interpretation of the PSE by addressing the case of a PSE derived from a PLE in canonical form (2) with a block-diagonal matrix R.

Let the polynomial matrices R and Q be defined by

$$R(\xi) = \begin{pmatrix} R_1(\xi) & 0 \\ 0 & R_2(\xi) \end{pmatrix}, \qquad Q(\xi) = \begin{pmatrix} Q_1(\xi) & Q_2(\xi) \end{pmatrix},$$

with R_1, R_2, Q_1 and Q_2 given by

$$R_1(\xi) = \begin{pmatrix} 2\xi+1 & \xi^2-1 & 1 \\ 1 & \xi^2+2\xi+3 & 1 \\ \xi & -\xi+1 & \xi+1 \end{pmatrix}, \qquad R_2(\xi) = \begin{pmatrix} \xi-2 & \xi^2+4 \\ -\xi-4 & 4 \end{pmatrix},$$

$$Q_1(\xi) = \begin{pmatrix} 1 & 3\xi-2 & -1 \\ 2 & \xi+1 & 1 \end{pmatrix}, \qquad Q_2(\xi) = \begin{pmatrix} -1 & \xi \\ 2 & -6 \end{pmatrix}.$$

Then Q is easily verified to be R-canonical, which due to the block-diagonal structure of R is equivalent to Q_1 being R_1-canonical and Q_2 being R_2-canonical. The PLE associated with R, Q and $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is given by

$$R(-\xi)^{\mathrm{T}} X(\xi) + X(-\xi)^{\mathrm{T}} R(\xi) = Q(-\xi)^{\mathrm{T}} \Sigma Q(\xi),$$

which is to be solved for the R-canonical matrix $X(\xi)$. If $X(\xi)$ is block-partitioned as

$$X(\xi) = \begin{pmatrix} X_{11}(\xi) & X_{12}(\xi) \\ X_{21}(\xi) & X_{22}(\xi) \end{pmatrix},$$

then the PLE gives rise to an equivalent set of three matrix equations of reduced size:

$$\begin{aligned} R_1(-\xi)^{\mathrm{T}} X_{11}(\xi) + X_{11}(-\xi)^{\mathrm{T}} R_1(\xi) &= Q_1(-\xi)^{\mathrm{T}} \Sigma Q_1(\xi), \\ R_1(-\xi)^{\mathrm{T}} X_{12}(\xi) + X_{21}(-\xi)^{\mathrm{T}} R_2(\xi) &= Q_1(-\xi)^{\mathrm{T}} \Sigma Q_2(\xi), \\ R_2(-\xi)^{\mathrm{T}} X_{22}(\xi) + X_{22}(-\xi)^{\mathrm{T}} R_2(\xi) &= Q_2(-\xi)^{\mathrm{T}} \Sigma Q_2(\xi). \end{aligned}$$

Note that the first and third equations are both PLEs while the second equation is a PSE. The R-canonicity property of X is equivalent to X_{11} and X_{21} being R_1-canonical and X_{12} and X_{22} being R_2-canonical. Thus, these three equations are all in canonical form.

The autonomous system associated with R and Q is described by a set of equations

$$\begin{cases} R\left(\dfrac{d}{dt}\right) w = 0, \\ y = Q\left(\dfrac{d}{dt}\right) w, \end{cases}$$

which represents a parallel connection of the two autonomous subsystems associated with R_1, Q_1, R_2 and Q_2, where $w = \binom{w_1}{w_2}$ and $y = y_1 + y_2$.

The PLE (see, e.g., [1]) can be associated with a quadratic cost integral:

$$J = \int_0^\infty \|y(t)\|^2 \, dt.$$

Since $\|y\|^2 = \|y_1\|^2 + 2\langle y_1, y_2\rangle + \|y_2\|^2$, this may be decomposed into a sum of three cost integrals involving the two individual subsystems: $J = J_1 + 2J_{12} + J_2$ with

$$J_1 = \int_0^\infty \|y_1(t)\|^2 \, dt, \qquad J_{12} = \int_0^\infty y_1(t)^{\mathrm{T}} y_2(t) \, dt, \qquad J_2 = \int_0^\infty \|y_2(t)\|^2 \, dt.$$

Here the cost integrals J_1 and J_2 are associated with the two reduced size PLEs while the cost integral J_{12} relates to the PSE.

The polynomials $\det(R_1(\xi))$ and $\det(R_2(\xi))$ are easily computed as

$$\begin{aligned} \det\bigl(R_1(\xi)\bigr) &= -2\bigl(\xi^4 + 3\xi^3 + 6\xi^2 + 3\xi + 2\bigr), \\ \det\bigl(R_2(\xi)\bigr) &= \xi^3 + 4\xi^2 + 8\xi + 8, \end{aligned}$$

having degrees 4 and 3 respectively. They are both easily verified to be Hurwitz, whence the cost integrals all converge regardless of the specific initial conditions. The algorithm of Proposition 6.2 then produces the following characteristic polynomial $\chi_{S_{R_1,R_2}}(\xi)$ of degree 12 for the Sylvester operator associated with the LPSE:

$$\xi^{12} + 25\xi^{11} + 305\xi^{10} + 2376\xi^9 + 13066\xi^8 + 53157\xi^7 + 163553\xi^6 \\ + 382761\xi^5 + 675150\xi^4 + 874127\xi^3 + 788370\xi^2 + 445740\xi + 120096.$$

Using this polynomial, the algorithm (23)–(25) produces the solution $Y(\zeta,\eta)$ to the LPSE in 12 iteration steps:

$$Y(\zeta,\eta) = \begin{pmatrix} 365/139 & (1924 + 2179\eta)/834 \\ 2(109 + 151\zeta)/139 & (-2784 + 1336\zeta + 1167\eta + 1090\zeta\eta)/834 \\ -437/417 & -2(-185 + 7\eta)/1251 \end{pmatrix}.$$

According to Eq. (26) this solution Y gives rise to the following unique (R_1, R_2)-canonical solution pair $(X_{21}(\xi), X_{12}(\xi))$ for the PSE:

$$X_{21}(\xi) = \begin{pmatrix} -2179/834 & (-1167 - 1090\xi)/834 & 14/1251 \\ 11/834 & (141 + 722\xi)/834 & -1297/1251 \end{pmatrix},$$

$$X_{12}(\xi) = \begin{pmatrix} -766/417 & (-5032 - 6565\xi)/5004 \\ -140/417 & (-2984 + 25\xi)/5004 \\ 437/417 & 2(-185 + 7\xi)/1251 \end{pmatrix}.$$

8. Conclusions

In this chapter we have introduced and studied the polynomial Sylvester equation by exploring analogies with the polynomial Lyapunov equation and generalizing the results of [15]. The algorithm for solving the PSE presented here is an extension of the algorithm developed for the PLE in [15] and works directly with the polynomial matrices that constitute the PSE. No preprocessing or transformations to canonical forms are required. The amount of bookkeeping necessary to perform the computations is kept to a minimum and the procedure is straightforward to implement. Moreover, the methods employed make the algorithm especially suitable for exact and symbolic computation purposes, and has been illustrated by a worked example. An implementation of the algorithm as a *Mathematica* Notebook is available upon request from the authors.

The application of the two-variable polynomial framework proposed in this chapter to the solution of other polynomial equations relevant for systems and control applications is currently being studied. Another issue under investigation concerns the case of singular Lyapunov and Sylvester operators.

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