Dissipativity preserving model reduction by retention of trajectories of minimal dissipation

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October 9, 2007

Abstract

We present a method for model reduction based on ideas from the behavioral theory of dissipative systems, in which the reduced-order model is required to reproduce a subset of the set of trajectories of minimal dissipation of the original system. The passivity-preserving model reduction method of Antoulas and Sorensen proposed in [2, 16] is shown to be a particular case of this more general class of model reduction procedures.

Keywords: model reduction, strictly dissipative systems, behaviors, minimal dissipation, driving variable representation, output nulling representations, Nevanlinna interpolation problem

1 Introduction

Model reduction aims at finding a system that approximates a given one and has lower complexity than the original, with the complexity being measured by its McMillan degree, i.e. the minimal dimension of the state space of the model. In the linear setting, classical model reduction methods are balancing (see [12]), Padé approximation (see [4]), moment-matching (see [21, 9]), and H_∞-approximation (see [8]). An up-to-date and exhaustive source on the problem of model reduction and approximation is the book [1].

Usually, besides the reduction in complexity of the original model, preservation of certain properties of the original model is required. An example of this is preservation of stability. However, often it is also demanded that the reduced model retains other characteristics of the original system, passivity being one of them. Several methods for model reduction with stability and passivity preservation have been introduced in the past, see for example [6, 7, 22, 5, 13].

Recently, Antoulas (see [2]) and Sorensen (see [16]) have presented a new technique and efficient numerical algorithms to perform model reduction with passivity- and stability preservation. The novel approach pioneered by Antoulas in [2] is based on the idea of combining Krylov projection methods with positive-real interpolation techniques; the reduced-order model is obtained by interpolating a subset of the spectral zeros of the

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original system. In the closely related paper [16], Sorensen shows that for all practical purposes there is no need for explicit interpolation in the implementation: rather, the reduced-order model can be found by computing a suitable basis for the stable invariant subspace of a Hamiltonian matrix associated with the system. This idea renders Antoulas' model reduction method applicable also to systems with large McMillan degree.

The purpose of the present paper is to present a different point of view on the method of [2] using ideas from the behavioral theory of dissipative systems. We show that the model reduction approach of Antoulas can be interpreted as special case of a general method for model reduction applicable to dissipative systems. For a given dissipative behavior we introduce the subbehavior of trajectories that are in a sense *local minima of dissipation*. Next, for the reduced order approximation we require that a particular part of its subbehavior of minimal dissipation is contained in the subbehavior of minimal dissipation of the original system: the approximating behavior 'inherits' this part of the subbehavior of minimal dissipation from the original system. We will call this technique *model reduction by retention of trajectories of minimal dissipation*.

In our setting, the original system will be given as the behavior of a linear, time-invariant differential system. We assume that the behavior is dissipative with respect to a given supply rate. The complexity of the behavior is measured by its McMillan degree. The problem that we will study in this paper is to find a (approximating) behavior: (1) whose McMillan degree is strictly less than that of the original behavior, (2) that has the same number of inputs as the original behavior, (3) that is again dissipative with respect to the given supply rate, and (4) that retains (or: inherits) a maximal number of a priori given antistable trajectories of minimal dissipation of the original behavior. Interpreted in this sense, the method of passivity preserving model reduction as initiated by Antoulas and Sorensen has the same heuristic flavour as the method of positive real balancing (see [5]), where it can be argued that the reduced order model is obtained by deleting typically that part of the system along which a relatively large amount of dissipation takes place.

We will establish algorithmic procedures to compute, for a given behavior represented in driving variable representation or output nulling representation, a reduced order behavior that solves the problem stated above. Subsequently, we will show that certain transfer matrices associated with our reduced order behavior are in fact solutions of a Nevanlinna type tangential interpolation problem (see also [10]). In fact, both for driving variable as well as output nulling representations, the transfer matrix of the reduced order behavior will turn out to interpolate the transfer matrix of the original behavior in certain directions, with interpolation points at some of the antistable spectral zeroes of the original behavior.

The outline of this paper is as follows. In section 2 we review the basic material on behaviors that we need in this paper. Section 3 reviews the concepts of dissipativity, storage function, and dissipation function. Also, in this section the notion of subbehavior of minimal dissipation is introduced and elaborated. In section 4 we state the exact problem that this paper deals with: the problem of dissipativity preserving model reduction by retention of trajectories of minimal dissipation. In section 5 we turn to behaviors in driving variable representation, and characterize strict dissipativity in terms of the representation. We also establish a representation of the subbehavior of minimal dissipation in terms of the matrices of the driving variable representation. Using these results, in section 6 we give an algorithm to solve our main problem (the problem introduced in section 4) for the case that the behavior to be reduced is in driving variable representation. We also show that our reduced order behavior solves a Nevanlinna tangential interpolation problem. Sections 7 and 8 deal with behaviors in output nulling representation. In section 9 we give concluding remarks. Finally, section 10 contains an Appendix in which we review the necessary material on driving variable and output nulling representations and the way they interact.

In this paper we will use the following notation:

The space of $\mathtt{n}$ dimensional real, respectively complex, vectors is denoted by $\mathbb{R}^{\mathtt{n}}$, respectively $\mathbb{C}^{\mathtt{n}}$, and the space of $\mathtt{m} \times \mathtt{n}$ real, respectively complex, matrices, by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$, respectively $\mathbb{C}^{\mathtt{m}\times\mathtt{n}}$. Whenever one of the two dimensions is not specified, a bullet $\bullet$ is used. Given two column vectors x and y, we denote with $\mathrm{col}(x, y)$ the vector obtained by stacking x over y; a similar convention holds for the stacking of matrices with the same number of columns. Given a Hermitian matrix $S \in \mathbb{C}^{\mathtt{w}\times\mathtt{w}}$, we define its *inertia* as the triple $\sigma(S) := (\sigma_-, \sigma_0, \sigma_+)$ where σ_+ is the number of positive eigenvalues of S, σ_- is the number of negative eigenvalues of S, and σ_0 is the multiplicity of 0 as an eigenvalue of S.

The ring of polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$; the ring of two-variable polynomials with real coefficients in the indeterminates ζ and η is denoted by $\mathbb{R}[\zeta, \eta]$. The space of all $\mathtt{m} \times \mathtt{n}$ polynomial matrices in the indeterminate ξ is denoted by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}[\xi]$, and that consisting of all $\mathtt{m} \times \mathtt{n}$ polynomial matrices in the indeterminates ζ and η by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}[\zeta, \eta]$. Given a matrix $R \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}[\xi]$, we define $R^{\sim}(\xi) := R(-\xi)^T \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$.

We denote with $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ the set of infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$, with $\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ the subspace of $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ consisting of all compactly supported functions, with $\mathfrak{L}_2^{loc}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ the set of all Lebesgue measurable functions w from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$ for which the integral $\int_\Omega \|w\|^2 dt$ is finite for all compact sets $\Omega \subset \mathbb{R}$. Sometimes, when the domain and co-domain are obvious from the context, we simply write $\mathfrak{C}^\infty$, $\mathfrak{D}$ and $\mathfrak{L}_2^{loc}$. If $F(t)$ is a real $\mathtt{p} \times \mathtt{m}$ matrix valued function, then the space of all functions formed as real linear combinations of the columns of $F(t)$ is denoted by $\mathrm{span}\{F(t)\} := \{F(t)x_0 \mid x_0 \in \mathbb{R}^{\mathtt{m}}\}$,

2 Behaviors and quadratic differential forms

A subset $\mathfrak{B} \subset \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ is called a linear time-invariant differential system (briefly, a *behavior*) if there exists a polynomial matrix $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ such that $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0\}$. By $\mathfrak{L}^{\mathtt{w}}$ we denote the set of all linear time-invariant differential systems with $\mathtt{w}$ variables. We note that while we *define* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ as the kernel of a differential operator, $\mathfrak{B}$ is often *not specified* in this way. We speak about a *kernel representation* when $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is represented by $R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0$, i.e., $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0\}$. Another representation is a *latent variable representation*, defined through polynomial matrices R and M by $R(\frac{\mathrm{d}}{\mathrm{d}t})w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell$, with $\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists\ \ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\ell})$ such that $R(\frac{\mathrm{d}}{\mathrm{d}t})w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell\}$. This type of model is the kind of model that usually results from first principles modeling, with the w's the vector of variables that the model aims at, and the ℓ's the vector of auxiliary variables introduced in the modeling process (for example state variables). The behavior $\mathfrak{B}$ is then called the *external* behavior, and $\mathfrak{B}_{\text{full}} = \{(w, \ell) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}+\ell}) \mid R(\frac{\mathrm{d}}{\mathrm{d}t})w = M(\frac{\mathrm{d}}{\mathrm{d}t})\ell\}$, the *full behavior.* If $\mathfrak{B}$ is the external behavior of $\mathfrak{B}_{\text{full}}$, then we often write $\mathfrak{B} = (\mathfrak{B}_{\text{full}})_{\text{ext}}$.

We also need the notion of state for a behavior. We refer to [15] for a detailed exposition, with only a brief review here. A latent variable representation of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is called a state representation if the latent variable (denoted here by x) has the *property of state*, i.e.: if $(w_1, x_1), (w_2, x_2) \in \mathfrak{B}_{\text{full}}$ are such that $x_1(0) = x_2(0)$ then $(w_1, x_1) \wedge (w_2, x_2)$, the concatenation (at $t = 0$, here), belongs to the $\mathfrak{L}_1^{\text{loc}}$-closure of $\mathfrak{B}_{\text{full}}$. We call such an x a state for $\mathfrak{B}$.

A latent variable representation is a state representation of its manifest behavior if and only if its full behavior can be represented by a differential equation that is zero-th order in w and first order in x, i.e., by $R_0 w = M_0 x + M_1 \frac{\mathrm{d}}{\mathrm{d}t} x$, with R_0, M_0, M_1 constant matrices. There are many, more structured, state representations as, for instance, a *driving variable representation* $\frac{\mathrm{d}}{\mathrm{d}t} x = Ax + Bv, \quad w = Cx + Dv$, with v an, obviously free, additional latent variable; an *output nulling representation* $\frac{\mathrm{d}}{\mathrm{d}t} x = Ax + Bw, \quad 0 = Cx + Dw$; or an *input/state/output representations* $\frac{\mathrm{d}}{\mathrm{d}t} x = Ax + Bu, \quad y = Cx + Du, \quad w = (u, y)$, the most popular of them all. Every system $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ admits such a representation after a suitable permutation of the components of w and a suitable choice of the state. In this paper, an important role is played by driving variable representations and output nulling representations. We have collected the basic material on these representations in Appendix A.

In this paper, we restrict ourselves to controllable behaviors. Roughly speaking, controllable behaviors are defined as behaviors in which for any two of its elements there exists a third element which coincides with the first one on the past and the second one on the future (for details, see [14]). $\mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ (a subset of $\mathfrak{L}^{\mathtt{w}}$) denotes the set of controllable behaviors.

Given a behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, it is possible to choose some components of w as any function in $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R})$. The maximal number of such components that can be chosen arbitrarily is called the *input cardinality* of $\mathfrak{B}$ and is denoted as $\mathtt{m}(\mathfrak{B})$. This number is exactly equal to the dimension of the input u in any input/state/output representation of $\mathfrak{B}$. The complementary number $\mathtt{w} - \mathtt{m}(\mathfrak{B})$ is called the *output cardinality* of $\mathfrak{B}$.

This paper also uses the formalism of *quadratic differential form* (*QDF*) developed in [27]. We now review the basic elements of the theory of QDF's. A two-variable polynomial matrix $\Phi \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ can be written as $\Phi(\zeta, \eta) = \sum_{h,k=0}^{N} \Phi_{h,k} \zeta^h \eta^k$, where $\Phi_{h,k} \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$ for all h, k, and N is a nonnegative integer. The two-variable polynomial matrix $\Phi(\zeta, \eta)$ induces a quadratic functional acting on $\mathtt{w}$-dimensional infinitely differentiable trajectories, defined as $Q_\Phi(w) = \sum_{h,k=0}^{N} (\frac{d^h w}{dt^h})^T \Phi_{h,k} \frac{d^k w}{dt^k}$ Such a functional is called a *quadratic differential form* (QDF). It is easy to see that without loss of generality we may restrict our attention to *symmetric* two-variable polynomial matrices $\Phi(\zeta, \eta)$, i.e. $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^\top$. In this paper we always assume that this is the case. By $\mathbb{R}_s^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ we will denote the subset of $\mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\zeta, \eta]$ of all symmetric two-variable polynomial matrices.

3 Dissipativity and the subbehavior of minimal dissipation

For an extensive treatment of dissipative systems in a behavioral context we refer to [24, 27, 28, 17]. Here we review the basic material. Let $\Sigma \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$ and $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$. Write $Q_\Sigma(w) := w^T \Sigma w$. $\mathfrak{B}$ is said to be *dissipative* with respect to Q_Σ (or briefly, Σ-dissipative) if $\int_{-\infty}^{+\infty} Q_\Sigma(w)\, dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$. Further, it is said to be dissipative on $\mathbb{R}_-$ with respect to Q_Σ (or briefly, Σ-dissipative on $\mathbb{R}_-$) if $\int_{-\infty}^{0} Q_\Sigma(w)\, dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$. We also use the analogous definition of dissipativity on $\mathbb{R}_+$. It is easily seen that if $\mathfrak{B}$ is

Σ-dissipative on $\mathbb{R}_-$ or $\mathbb{R}_+$, then it is Σ-dissipative. A controllable behavior $\mathfrak{B}$ is said to be *strictly dissipative* with respect to Q_Σ (or briefly, strictly Σ-dissipative) if there exists an $\epsilon > 0$ such that $\mathfrak{B}$ is dissipative with respect to $Q_{\Sigma-\epsilon I}$. We have the obvious definitions for strict dissipativity on $\mathbb{R}_-$ and on $\mathbb{R}_+$. If $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$ or $\mathbb{R}_+$, then it is strictly Σ-dissipative.

The QDF Q_Ψ induced by $\Psi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is called a *storage function* for $(\mathfrak{B}, Q_\Sigma)$ if

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Q_\Psi(w) \le Q_\Sigma(w) \text{ for all } w \in \mathfrak{B} \cap \mathfrak{C}^\infty. \tag{1}$$

The QDF Q_Δ induced by $\Delta \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$ is called a *dissipation function* for $(\mathfrak{B}, Q_\Sigma)$ if $Q_\Delta(w) \ge 0$ for all $w \in \mathfrak{B} \cap \mathfrak{C}^\infty$ and

$$\int_{-\infty}^{\infty} Q_\Sigma(w)dt = \int_{-\infty}^{\infty} Q_\Delta(w)dt \text{ for all } w \in \mathfrak{B} \cap \mathfrak{D}.$$

If the supply rate Q_Σ, the dissipation function Q_Δ, and the storage function Q_Ψ satisfy

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Q_\Psi(w) = Q_\Sigma(w) - Q_\Delta(w) \text{ for all } w \in \mathfrak{B} \cap \mathfrak{C}^\infty \tag{2}$$

then we call the triple $(Q_\Sigma, Q_\Psi, Q_\Delta)$ *matched on* $\mathfrak{B}$. Equation (2) expresses that, along $w \in \mathfrak{B}$, the increase in internal storage is equal to the rate at which supply is delivered minus the rate at which supply is dissipated. The following is well-known, see e.g. [17].

Proposition 3.1 : *The following conditions are equivalent*

1. $(\mathfrak{B}, Q_\Sigma)$ *is dissipative,*
2. $(\mathfrak{B}, Q_\Sigma)$ *admits a storage function,*
3. $(\mathfrak{B}, Q_\Sigma)$ *admits a dissipation function.*

Furthermore, for any dissipation function Q_Δ there exists a unique storage function Q_Ψ, and for any storage function Q_Ψ there exists a unique dissipation function Q_Δ such that $(Q_\Sigma, Q_\Psi, Q_\Delta)$ is matched on $\mathfrak{B}$.

We now introduce the notion of *subbehavior of minimal dissipation*. For a given Σ-dissipative system $\mathfrak{B}$, let Q_Ψ be a storage function, and Q_Δ be a dissipation function such that $(Q_\Sigma, Q_\Psi, Q_\Delta)$ is matched on $\mathfrak{B}$. Let $w \in \mathfrak{B}$. Then the integral $\int_{t_0}^{t_1} Q_\Delta(w)dt$ is equal to the dissipated supply over the interval $[t_0, t_1]$ when $\mathfrak{B}$ is taken through the trajectory w. We will now look at those w's in $\mathfrak{B}$ that are, in a sense, local minima for the dissipated supply. Fix $w \in \mathfrak{B}$, and for $\delta \in \mathfrak{B} \cap \mathfrak{D}$ define

$$J_w(\delta) := \int_{-\infty}^{+\infty} Q_\Delta(w+\delta) - Q_\Delta(w)dt.$$

Then w is called *a trajectory of minimal dissipation* if $J_w(\delta) \ge 0$ for all $\delta \in \mathfrak{B} \cap \mathfrak{D}$. Define

$$\mathfrak{B}^* := \{w \in \mathfrak{B} \mid \ w \text{ is a trajectory of minimal dissipation}\}.$$

It turns out that the subset $\mathfrak{B}^*$ of $\mathfrak{B}$ of trajectories of minimal dissipation is independent of the chosen dissipation function Q_Δ, forms a behavior again, and admits an easy characterization in terms of $\mathfrak{B}$ and Σ. For this, define the Σ-*orthogonal complement* $\mathfrak{B}^{\perp_\Sigma}$ of $\mathfrak{B}$ as

$$\mathfrak{B}^{\perp_\Sigma} := \{w \in \mathfrak{C}^\infty \mid \int_{-\infty}^{+\infty} w^\top \Sigma \delta \ \mathrm{d}t = 0 \text{ for all } \delta \in \mathfrak{B} \cap \mathfrak{D}\}.$$

It can be proven that $\mathfrak{B}^{\perp_\Sigma}$ is also a controllable behavior, see section 10 of [27]. If $\Sigma = I$, we simply write $\mathfrak{B}^\perp$, called the *orthogonal complement* of $\mathfrak{B}$. We then have:

Theorem 3.2 : *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$ and $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. Assume that $\mathfrak{B}$ is Σ-dissipative. Then $\mathfrak{B}^* \in \mathfrak{L}^{\mathtt{w}}$ and*

$$\mathfrak{B}^* = \mathfrak{B} \cap \mathfrak{B}^{\perp_\Sigma} = \mathfrak{B} \cap (\Sigma\mathfrak{B})^\perp.$$

Proof : Let Q_Δ be a dissipation function. Let Q_Ψ a storage function such that $\frac{\mathrm{d}}{\mathrm{d}t}Q_\Psi = Q_\Sigma - Q_\Delta$. It is then easily seen that for all $w \in \mathfrak{B}$ and for all $\delta \in \mathfrak{B} \cap \mathfrak{D}$ we have

$$J_w(\delta) = \int_{-\infty}^{+\infty} Q_\Sigma(w+\delta) - Q_\Sigma(w)dt.$$

Clearly, $J_w(\delta) = \int_{-\infty}^{+\infty} \delta^\top\Sigma\delta \;\mathrm{d}t + 2\int_{-\infty}^{+\infty} w^\top\Sigma\delta \;\mathrm{d}t$, and it can be seen that $J_w(\delta) \geq 0$ for all $\delta \in \mathfrak{B} \cap \mathfrak{D}$ if and only if the linear term is equal to zero for all $\delta \in \mathfrak{B} \cap \mathfrak{D}$. Consequently, the subset of trajectories $\mathfrak{B}$ of minimal dissipation is equal to

$$\mathfrak{B}^* = \{w \in \mathfrak{B} \mid \int_{-\infty}^{+\infty} w^\top\Sigma\delta \;\mathrm{d}t = 0 \text{ for all } \delta \in \mathfrak{B} \cap \mathfrak{D}\} = \mathfrak{B} \cap (\mathfrak{B})^{\perp_\Sigma}.$$

□

In the sequel, we will refer to $\mathfrak{B}^*$ as *the subbehavior of minimal dissipation.* It turns out that if $\mathfrak{B}$ is *strictly* dissipative, then the subbehavior $\mathfrak{B}^*$ of minimal dissipation is *autonomous.* In fact, we have

Theorem 3.3 : *Assume that $\mathfrak{B}$ is strictly Σ-dissipative. Then*

1. *$\mathfrak{B}^*$ is autonomous,*
2. *$\mathfrak{B}^* = \mathfrak{B}^*_{\text{stab}} \oplus \mathfrak{B}^*_{\text{antistab}}$, where we define $\mathfrak{B}^*_{\text{stab}} := \{w \in \mathfrak{B}^* \mid \lim_{t\to\infty} w(t) = 0\}$ and $\mathfrak{B}^*_{\text{antistab}} = \{w \in \mathfrak{B}^* \mid \lim_{t\to-\infty} w(t) = 0\}$,*
3. *$\mathtt{n}(\mathfrak{B}^*) = 2\mathtt{n}(\mathfrak{B})$ and $\mathtt{n}(\mathfrak{B}^*_{\text{stab}}) = \mathtt{n}(\mathfrak{B}^*_{\text{antistab}}) = \mathtt{n}(\mathfrak{B})$.*

Proof : A proof of this follows immediately from lemma 5.3 (and the remarks following it) in section 5 of this paper. □

4 Problem statement

In this section we will formulate the problem of model reduction by retention of trajectories of minimal dissipation.

Main Problem. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$. Let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. Assume that $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}^-$. Let $(\mathfrak{B}^*)_{\text{antistable}}$ be the antistable part of the subbehavior of minimal dissipation $\mathfrak{B}^*$. Let $\mathtt{k} < \mathtt{n}(\mathfrak{B})$ be given together with a subbehavior $\mathfrak{B}' \subset (\mathfrak{B}^*)_{\text{antistable}}$ such that $\mathtt{n}(\mathfrak{B}') = \mathtt{k}$. Find $\hat{\mathfrak{B}} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$ such that

1. $\mathtt{n}(\hat{\mathfrak{B}}) \leq \mathtt{k}$,
2. $\mathtt{m}(\hat{\mathfrak{B}}) = \mathtt{m}(\mathfrak{B})$,

3. $\hat{\mathfrak{B}}$ is strictly dissipative on $\mathbb{R}^-$ with respect to Q_Σ,

4. The antistable part $(\hat{\mathfrak{B}}^*)_{\text{antistable}}$ of $\hat{\mathfrak{B}}^*$ is a subbehavior of $\mathfrak{B}'$.

Any behavior $\hat{\mathfrak{B}}$ as above has the property that the $\hat{\mathtt{n}}$-dimensional antistable part of the subbehavior of minimal dissipation (with $\hat{\mathtt{n}} = \mathtt{n}(\hat{\mathfrak{B}})$) is contained in the antistable part of the subbehavior of minimal dissipation of the original system $\mathfrak{B}$. Thus $\hat{\mathfrak{B}}$ inherits from $\mathfrak{B}$ a $\hat{\mathtt{n}}$-dimensional subbehavior of its subbehavior of minimal dissipation. By virtue of this property, $\hat{\mathfrak{B}}$ is considered as an *approximation* of $\mathfrak{B}$. Note that there are many choices for the $\mathtt{k}$-dimensional subbehavior $\mathfrak{B}'$ of $(\mathfrak{B}^*)_{\text{antistable}}$. Different choices of $\mathfrak{B}'$ will of course result in different approximations $\hat{\mathfrak{B}}$.

In the sequel we will prove that the the subbehavior of minimal dissipation $\mathfrak{B}^*$ is associated with the so called *spectral zeroes* of the original system. It will turn out that any of the approximants $\hat{\mathfrak{B}}$ that we will obtain as solution to our problem allows an interpretation as a solution of a *rational interpolation problem*, with as interpolation points some of the antistable spectral zeroes, see also [2] and [10].

Of course, it is also possible to formulate a version of the above problem with strict dissipativity on $\mathbb{R}^+$, and with $\mathfrak{B}'$ a subbehavior of the stable part of $\mathfrak{B}^*$, in which the problem is to find a reduced order behavior $\hat{\mathfrak{B}}$ such that $\hat{\mathfrak{B}}^*$ is a subbehavior of the stable part of $\hat{\mathfrak{B}}^*$. The details are left to the reader.

In formulating the problem of model reduction by retention of trajectories of minimal dissipation, we have kept with one of the tenets of behavioral systems theory, that of articulating concepts at the most intrinsic possible level, that of trajectories. In practice, though, the to-be-approximated behavior $\mathfrak{B}$ is represented in some form, be it kernel, image, latent variable, state space, etc., and the issue arises of *how to pass from the original representation to a representation of a reduced-order approximation*, for example for the purposes of simulation, of control, etc. In the remainder of this paper we consider this topic for two types of models, namely *driving-variable* (in the following abbreviated with DV) and *output-nulling* (in the following abbreviated with ON), and delay the discussion of other types of representations to the conclusion section, where we outline some of the lines of research currently pursued. The definition of DV and ON representation, and some of the essential notions necessary in order to understand the material presented in this paper, are gathered in the Appendix in section 10.

5 Dissipativity and minimal dissipation for DV representations

In this section we examine strict dissipativity and the the subbehavior of minimal dissipation for the case that our system is represented by a DV-representation.

The connection between dissipativity, the algebraic Riccati equation (ARE), and the Hamiltonian matrix of the system is well-known, see [24, 25, 26, 29]. In the following, we will review this connection for the case of *half-line* dissipativity. First note the following:

Lemma 5.1 : *Let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$ be strictly Σ-dissipative. Then there exists a minimal driving variable representation $\mathfrak{B}_{DV}(A,B,C,D)$ of $\mathfrak{B}$ with $D^\top\Sigma D = I$ and $D^\top\Sigma C = 0$.*

Proof : Let $\hat{A}, \hat{B}, \hat{C}, \hat{D}$ be such that $\mathfrak{B}_{DV}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is a minimal DV representation of $\mathfrak{B}$. Then $\hat{D}$ has full column rank (see Appendix, Proposition 10.1). Then, using an argument analogous to that used in the proof of Th. 5.3.4 of [11], it can be proven that $\hat{D}^\top \Sigma \hat{D} > 0$. Let W be a nonsingular matrix such that $\hat{D}^\top \Sigma \hat{D} = W^\top W$. By applying the state feedback transformation $\hat{v} = -(\hat{D}^\top \Sigma \hat{D})^{-1} \hat{D}^\top \Sigma \hat{C} x + W^{-1} v$ to $\mathfrak{B}_{DV}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ we obtain a new driving variable representation $\mathfrak{B}_{DV}(A, B, C, D)$ of $\mathfrak{B}$, with

$$\begin{aligned} A &= \hat{A} - \hat{B}(\hat{D}^\top \Sigma \hat{D})^{-1} \hat{D}^\top \Sigma \hat{C} \\ B &= \hat{B} W^{-1} \\ C &= \hat{C} - \hat{D}(\hat{D}^\top \Sigma \hat{D})^{-1} \hat{D}^\top \Sigma \hat{C} \\ D &= \hat{D} W^{-1}. \end{aligned}$$

Observe that D is injective, and that from the minimality of $\mathfrak{B}_{DV}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ and statement (2) of Proposition 10.1 it follows that $\mathfrak{B}_{DV}(A, B, C, D)$ is also a minimal representation of $\mathfrak{B}$. It is easy to see that $D^\top \Sigma D = I$, and moreover

$$D^\top \Sigma C = W^{-\top} \hat{D}^\top \Sigma (\hat{C} - \hat{D}(\hat{D}^\top \Sigma \hat{D})^{-1} \hat{D}^\top \Sigma \hat{C}) = 0.$$

This concludes the proof. □

We then have the following:

Proposition 5.2 : *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{contr}}$, and let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$. Let $\mathfrak{B}_{DV}(A, B, C, D)$ be a minimal driving variable representation of $\mathfrak{B}$ such that $D^\top \Sigma D = I$ and $D^\top \Sigma C = 0$. Then the following statements are equivalent:*

1. *$\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}^-$ ($\mathbb{R}^+$),*
2. *the ARE*
$$A^\top K + KA - C^\top \Sigma C + KBB^\top K = 0 \tag{3}$$
has a real symmetric solution K with $K > 0$ ($K < 0$) and $A + BB^\top K$ is antistable (stable),
3. *The Hamiltonian matrix $H = \begin{bmatrix} A & BB^\top \\ C^\top \Sigma C & -A^\top \end{bmatrix}$ has no eigenvalues on the imaginary axis, and there exists $X_1, Y_1 \in \mathbb{R}^{\mathtt{n} \times \mathtt{n}}$, with X_1 nonsingular, and $M \in \mathbb{R}^{\mathtt{n} \times \mathtt{n}}$ antistable (stable) such that*
$$H \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} M,$$
with $X_1^\top Y_1 > 0$ ($X_1^\top Y_1 < 0$).

If K satisfies the conditions in (2.) above then it is unique, and it is the largest (smallest) real symmetric solution of (3). We denote it by K^+ (K^-). If X_1, Y_1 satisfy the conditions in (3.) above, then $Y_1 X_1^{-1}$ is equal to this largest (smallest) real symmetric solution K^+ (K^-) of the ARE (3).

Proof : A proof of this was given in [11], theorem 5.3.4. The equivalence of (2) and (3) follows from standard results on the relation between the algebraic Riccati equation and Hamiltonian matrices, see e.g. [29]. □

In the remainder of this section, for systems represented in DV form we will obtain a representation of the subbehavior of minimal dissipation, and of its antistable and stable part.

From Proposition 10.12 in the Appendix it follows that if $\mathfrak{B}_{DV}(A,B,C,D)$ is a minimal driving variable representation of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$, then $\mathfrak{B}_{ON}(-A^\top, C^\top\Sigma, B^\top, -D^\top\Sigma)$ is a minimal output nulling representation of $\mathfrak{B}^{\perp_\Sigma}$. Using theorem 3.2 we then find that if $\mathfrak{B}$ is Σ-dissipative, then the subbehavior of minimal dissipation of $\mathfrak{B}$ is given by

$$\mathfrak{B}^* = \mathfrak{B}_{DV}(A,B,C,D)_{\text{ext}} \cap \mathfrak{B}_{ON}(-A^\top, C^\top\Sigma, B^\top, -D^\top\Sigma)_{\text{ext}}. \tag{4}$$

For strictly Σ-dissipative systems this yields the following state space representation of $\mathfrak{B}^*$:

Lemma 5.3 : *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$ be strictly Σ-dissipative, and let $\mathfrak{B}_{DV}(A,B,C,D)$ be a minimal DV representation of $\mathfrak{B}$ such that $D^\top\Sigma D = I$ and $D^\top\Sigma C = 0$. Then $\mathfrak{B}^*$ is equal to the external behavior of*

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} &= \begin{bmatrix} A & BB^\top \\ C^\top\Sigma C & -A^\top \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \\ w &= \begin{bmatrix} C & DB^\top \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}, \end{aligned} \tag{5}$$

i.e., $\mathfrak{B}^* = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid$ there exist $x, z \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}})$ such that (5) holds$\}$.

Proof : By (4), $w \in \mathfrak{B}^*$ if and only if there exist x, z, v such that

$$\begin{aligned} \dot{x} &= Ax + Bv, \\ \dot{z} &= -A^\top z + C^\top\Sigma w, \\ w &= Cx + Dv, && (6) \\ 0 &= B^\top z - D^\top\Sigma w. && (7) \end{aligned}$$

Since $D^\top\Sigma D = I_{\mathtt{m}}$ and $D^\top\Sigma C = 0$ from, (6) and (7) it follows that $B^\top z = D^\top\Sigma w = D^\top\Sigma(Cx + Dv) = v$. Also, $\dot{z} = -A^\top z + C^\top\Sigma(Cx + Dv) = -A^\top z + C^\top\Sigma Cx$. This proves the claim of the lemma. □

The full behavior represented by the equations (5) will be called the *Hamiltonian behavior of $\mathfrak{B}_{DV}(A,B,C,D)$ with respect to* Σ, and we denote it with $B_H(A,B,C,D)$. Clearly, the antistable (stable) part of the external behavior of (5) can be obtained by considering the antistable (stable) invariant subspace $X_+(H)$ ($X_-(H)$) of the Hamiltonian matrix H. Indeed, assuming that H has no imaginary axis eigenvalues, if $X_1, Y_1 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ are such that the columns of $\text{col}(X_1, Y_1)$ form a basis of $X_+(H)$, and $M \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ is the matrix of $H\mid_{X_+(H)}$ with respect to this basis, then

$$(\mathfrak{B}^*)_{\text{antistab}} = \text{span}\{(CX_1 + DB^\top Y_1)e^{Mt}\}. \tag{8}$$

Now suppose a subbehavior $\mathfrak{B}'$ of $\mathfrak{B}^*$ is given, with $\mathtt{n}(\mathfrak{B}') = \mathtt{k}$. It can be shown that any such $\mathfrak{B}'$ corresponds to a unique $\mathtt{k}$-dimensional H-invariant subspace of $\mathbb{R}^{2\mathtt{n}}$, the state space of $\mathfrak{B}^*$. Thus we obtain the following:

Theorem 5.4 : *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$ be strictly Σ-dissipative on $\mathbb{R}^-$, and let $\mathfrak{B}_{DV}(A,B,C,D)$ be a minimal DV representation of $\mathfrak{B}$, with $D^\top \Sigma D = I$ and $D^\top \Sigma C = 0$. Let $\mathtt{k} < \mathtt{n}(\mathfrak{B})$ be a positive integer. Let $\mathfrak{B}'$ be a subbehavior of $(\mathfrak{B}^*)_{\text{antistab}}$ with $\mathtt{n}(\mathfrak{B}') = \mathtt{k}$. Then there exist $X_1^1, Y_1^1 \in \mathbb{R}^{\mathtt{n}\times\mathtt{k}}$, $X_1^2, Y_1^2 \in \mathbb{R}^{\mathtt{n}\times(\mathtt{n}-\mathtt{k})}$, and matrices M_{11}, M_{12}, M_{22} with M_{11} and M_{22} antistable and $X_1 := [X_1^1\ X_1^2]$ nonsingular, such that*

$$\begin{bmatrix} A & BB^\top \\ C^\top \Sigma C & -A^\top \end{bmatrix} \begin{bmatrix} X_1^1 & X_1^2 \\ Y_1^1 & Y_1^2 \end{bmatrix} = \begin{bmatrix} X_1^1 & X_1^2 \\ Y_1^1 & Y_1^2 \end{bmatrix} \underbrace{\begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}}_{=:M},$$

$$\mathfrak{B}' = \text{span}\{(CX_1^1 + DB^\top Y_1^1)e^{M_{11}t}\},$$

and

$$(\mathfrak{B}^*)_{\text{antistab}} = \text{span}\{(CX_1 + DB^\top Y_1)e^{Mt}\}.$$

Here, we define $Y_1 := [Y_1^1\ Y_1^2]$.

Proof : A proof follows immediately from Lemma 5.3 and the remarks above. □

A similar theorem of course holds holds for the stable part $(\mathfrak{B}^*)_{\text{stable}}$ of $\mathfrak{B}^*$ under the assumption of strict Σ-dissipativity on $\mathbb{R}^+$.

To conclude this section we formulate a result that will be of importance in our reduction procedure. The result deals with a general, possibly non-controllable behavior, represented by a minimal DV-representation. It states that the subbehavior of minimal dissipation of the controllable part of $\mathfrak{B}$ is *contained* in the external behavior of the Hamiltonian system (5):

Lemma 5.5 : *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. Let $\mathfrak{B}_{DV}(A,B,C,D)$ be a minimal driving variable representation of $\mathfrak{B}$ such that $D^\top \Sigma D = I$ and $D^\top \Sigma C = 0$. Assume that $\mathfrak{B}_{\text{contr}}$ is strictly Σ-dissipative. Then $(\mathfrak{B}_{\text{contr}})^* \subseteq \mathfrak{B}_H(A,B,C,D)_{\text{ext}}$. Consequently, if the Hamiltonian matrix H has no imaginary axis eigenvalues, then $((\mathfrak{B}_{\text{contr}})^*)_{\text{antistable}}$ is contained in the antistable part of the external behavior of the Hamiltonian system, as given by (5)*

Proof : We prove the first part of the lemma. The second part follows easily from it. We proceed by computing a minimal driving variable representation of the controllable part of $\mathfrak{B}$. In order to do this, we first compute a driving variable representation $\mathfrak{B}_{DV}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})$ of $\mathfrak{B}_{\text{contr}}$ following proposition 10.4. Observe that this is in general not a minimal representation of $\mathfrak{B}_{\text{contr}}$. Now apply a feedback transformation $v = Fx + v'$ as in Proposition 10.2, in order to obtain a minimal representation $\mathfrak{B}_{DV}(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1, \tilde{D})$ of $\mathfrak{B}_{\text{contr}}$. Then we have

$$\begin{aligned}
(\mathfrak{B}_{\text{contr}})^* &= \mathfrak{B}_{DV}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})_{\text{ext}} \cap (\mathfrak{B}_{DV}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})_{\text{ext}})^{\perp_\Sigma} \\
&= \mathfrak{B}_{DV}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})_{\text{ext}} \cap (\mathfrak{B}_{DV}(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1, \tilde{D})_{\text{ext}})^{\perp_\Sigma} \\
&= \mathfrak{B}_{DV}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})_{\text{ext}} \cap\ \mathfrak{B}_{ON}(-\tilde{A}_{11}^\top, \tilde{C}_1^\top \Sigma, \tilde{B}_1^\top, -\tilde{D}^\top \Sigma)_{\text{ext}}.
\end{aligned}$$

Now observe that

$$\begin{aligned}
\mathfrak{B}_{ON}(-\tilde{A}_{11}^\top, \tilde{C}_1^\top \Sigma, \tilde{B}_1^\top, -\tilde{D}^\top \Sigma)_{\text{ext}} &\subseteq \mathfrak{B}_{ON}(-\bar{A}_{11}^\top, \bar{C}_1^\top \Sigma, \bar{B}_1^\top, -\bar{D}^\top \Sigma)_{\text{ext}} \\
&\subseteq \mathfrak{B}_{ON}(-A^\top, C^\top \Sigma, B^\top, -D^\top \Sigma)_{\text{ext}}.
\end{aligned}$$

Consequently we have

$$\begin{aligned}(\mathfrak{B}_{\text{contr}})^* &\subseteq \mathfrak{B}_{DV}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})_{\text{ext}} \cap \mathfrak{B}_{ON}(-A^\top, C^\top\Sigma, B^\top, -D^\top\Sigma)_{\text{ext}} \\ &\subseteq \mathfrak{B}_{DV}(A, B, C, D)_{\text{ext}} \cap \mathfrak{B}_{ON}(-A^\top, C^\top\Sigma, B^\top, -D^\top\Sigma)_{\text{ext}} \\ &= B_H(A, B, C, D), \end{aligned} \tag{9}$$

which proves the claim of the lemma. □

6 A reduction algorithm for DV-representations

In this section we give an algorithmic procedure to compute for a given controllable behavior $\mathfrak{B}$, strictly Σ-dissipative on $\mathbb{R}^-$, a given integer $\mathtt{k} \leq \mathtt{n}(\mathfrak{B})$, and a given McMillan degree $\mathtt{k}$ subbehavior of the antistable part of the subbehavior of minimal dissipation, a DV-representation of a solution to our Main Problem as stated in section 4. Subsequently, we will show that the transfer matrix from driving variable to manifest variable of any of our solutions is a solution to a rational interpolation problem associated with the data of the model reduction problem.

ALGORITHM 1. (from DVR to DVR)

Input: $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$ strictly Σ-dissipative on $\mathbb{R}^-$, an integer $0 \leq \mathtt{k} \leq \mathtt{n}(\mathfrak{B})$ and a subbehavior $\mathfrak{B}'$ of $(\mathfrak{B}^*)_{\text{antistable}}$ of McMillan degree $\mathtt{k}$.
Output: A minimal DV-representation of $\hat{\mathfrak{B}} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$ satisfying the requirements of the Main Problem.

Step 1. Represent $\mathfrak{B}$ by a minimal DV-representation $\mathfrak{B}_{DV}(A, B, C, D)$ such that $D^\top\Sigma D = I$ and $D^\top\Sigma C = 0$.

Step 2. Compute $X_1 = [X_1^1 \ X_1^2] \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ nonsingular, $Y_1 = [Y_1^1 \ Y_1^2] \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ such that

$$\begin{bmatrix} A & BB^\top \\ C^\top\Sigma C & -A^\top \end{bmatrix} \begin{bmatrix} X_1^1 & X_1^2 \\ Y_1^1 & Y_1^2 \end{bmatrix} = \begin{bmatrix} X_1^1 & X_1^2 \\ Y_1^1 & Y_1^2 \end{bmatrix} \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix},$$

where M_{11} and M_{22} are antistable and $\mathfrak{B}' = \text{span}\{(CX_1^1 + DB^\top Y_1^1)e^{M_{11}t}\}$.

Step 3. Compute a Cholesky factorization $P^\top P = X_1^\top Y_1$, (P is a nonsingular upper triangular matrix).
Comment: Such factorization exists, since $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$ is strictly Σ-dissipative on $\mathbb{R}^-$, so $X_1^\top Y_1$ is symmetric and positive definite, see proposition 5.2. This also implies that Y_1 is nonsingular.

Step 4. Define $S = X_1 P^{-1} = Y_1^{-\top} P^\top$.

Step 5. Compute $(\bar{A}, \bar{B}, \bar{C}, \bar{D}) = (S^{-1}AS, S^{-1}B, CS, D)$.

Step 6. Denote the truncation of $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ to the first $\mathtt{k}$ components of the state vector by $(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})$.

Step 7. Perform a Kalman controllability decomposition:

$$T^{-1}\bar{A}_{11}T = \begin{bmatrix} \hat{A} & * \\ 0 & * \end{bmatrix}, T^{-1}\bar{B}_1 = \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix}, \bar{C}_1 T = \begin{bmatrix} \hat{C} & * \end{bmatrix}, \bar{D} = \hat{D}.$$

Step 8 Output

$$\hat{\mathfrak{B}} := \mathfrak{B}_{DV}(\hat{A}, \hat{B}, \hat{C}, \hat{D})_{\text{ext}}.$$

Proposition 6.1 : *The behavior $\hat{\mathfrak{B}}$ computed by Algorithm 1 is a solution to the Main Problem as formulated in section 4.*

Proof : By construction $\mathtt{n}(\hat{\mathfrak{B}}) \leq \mathtt{k}$. Also, $D = \hat{D}$ has full column rank, so the number of driving variable components in the original and new DV-representation are equal. Since the number of driving variable components of a minimal DV-representation is equal to the input cardinality of its external behavior, we obtain $\mathtt{m}(\hat{\mathfrak{B}}) = \mathtt{m}(\mathfrak{B})$.

We now prove that $\hat{\mathfrak{B}}$ is strictly Σ-dissipative on $\mathbb{R}^-$. It is easily verified that for $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ as computed in Step 5 above we have

$$\begin{bmatrix} \bar{A} & \bar{B}\bar{B}^\top \\ \bar{C}^\top \Sigma \bar{C} & -\bar{A}^\top \end{bmatrix} \begin{bmatrix} P \\ P \end{bmatrix} = \begin{bmatrix} P \\ P \end{bmatrix} M, \text{ with } M = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}$$

as in Step 2. Denote the $(1,1)$-block of the upper triangular matrix P by P_{11}. Then the truncated system $(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})$ computed in Step 6 satisfies

$$\begin{bmatrix} \bar{A}_{11} & \bar{B}_1\bar{B}_1^\top \\ \bar{C}_1^\top \Sigma \bar{C}_1 & -\bar{A}_{11}^\top \end{bmatrix} \begin{bmatrix} P_{11} \\ P_{11} \end{bmatrix} = \begin{bmatrix} P_{11} \\ P_{11} \end{bmatrix} M_{11}. \tag{10}$$

From (10) it then follows that the maximal solution of the ARE

$$\bar{A}_{11}^\top \bar{K} + \bar{K}\bar{A}_{11} - \bar{C}_1^\top \Sigma \bar{C}_1 + \bar{K}\bar{B}_1\bar{B}_1^\top \bar{K} = 0 \tag{11}$$

is given by $\bar{K}^+ = P_{11}P_{11}^{-1} = I$. Moreover, from (10) we also obtain $(\bar{A}_{11} + \bar{B}_1\bar{B}_1^\top)P_{11} = P_{11}M_{11}$, which implies that $\bar{A}_{11} + \bar{B}_1\bar{B}_1^\top$ is similar to M_{11} and therefore antistable.

Now consider the ARE corresponding to the DV-representation of the reduced order (controllable) behavior $\hat{\mathfrak{B}}$ computed in Step 8:

$$\hat{A}^\top \hat{K} + \hat{K}\hat{A} - \hat{C}^\top \Sigma \hat{C} + \hat{K}\hat{B}\hat{B}^\top \hat{K} = 0 \tag{12}$$

and observe that any solution of (12) is the $(1,1)$-block of a solution of (11). In particular, I is a solution of (12). Moreover, we have

$$\bar{A}_{11} + \bar{B}_1\bar{B}_1^\top I = \begin{bmatrix} \hat{A} + \hat{B}\hat{B}^\top I & * \\ 0 & * \end{bmatrix}$$

which implies that $\hat{A} + \hat{B}\hat{B}^\top I$ is antistable. By proposition 5.2 we conclude that $\hat{\mathfrak{B}}$ is strictly Σ-dissipative on $\mathbb{R}^-$.

We finally prove that the antistable part of the subbehavior of minimal dissipation of the reduced order behavior $\hat{\mathfrak{B}}$ is contained in $\mathfrak{B}'$. In order to do so, first observe that

$$\begin{aligned}
\mathfrak{B}' &= \text{span}\{(CX_1^1 + DB^\top Y_1^1)e^{M_{11}t}\} = \text{span}\{((CS)(S^{-1}X_1^1) + (DB^\top S^{-\top})(S^\top Y_1^1))e^{M_{11}t}\} \\
&= \text{span}\{(\bar{C}(S^{-1}X_1^1) + \bar{D}\bar{B}^\top (S^\top Y_1^1))e^{M_{11}t}\} \\
&= \text{span}\{\begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix} \begin{bmatrix} P_{11} \\ 0 \end{bmatrix} e^{M_{11}t} + \begin{bmatrix} \bar{D}\bar{B}_1^\top & \bar{D}\bar{B}_2^\top \end{bmatrix} \begin{bmatrix} P_{11} \\ 0 \end{bmatrix} e^{M_{11}t}\} \\
&= \text{span}\{(\bar{C}_1 + \bar{D}\bar{B}_1^\top)P_{11}e^{M_{11}t}\}.
\end{aligned}$$

Note that the external behavior $\mathfrak{B}_{\text{trunc}} := \mathfrak{B}_{DV}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})_{\text{ext}}$ may not be controllable, but that we do have $\bar{D}^\top \Sigma \bar{D} = I$ and $\bar{D}^\top \Sigma \bar{C}_1 = 0$. By applying proposition 5.5 we have

$$\hat{\mathfrak{B}}^* = ((\mathfrak{B}_{\text{trunc}})_{\text{contr}})^* \subseteq \mathfrak{B}_H(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D}).$$

We then conclude that

$$(\hat{\mathfrak{B}}^*)_{\text{antistable}} \subseteq (\mathfrak{B}_H(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D}))_{\text{antistable}} = \text{span}\{(\bar{C}_1 + \bar{D}\bar{B}_1^\top)P_{11}e^{M_{11}t}\} = \mathfrak{B}'.$$

This concludes the proof. □

Of course, a similar algorithmic procedure can be given for the alternative problem in which the original system $\mathfrak{B}$ is strictly dissipative on $\mathbb{R}^+$, and with $\mathfrak{B}'$ a subbehavior of the stable part of $\mathfrak{B}^*$, and where it is required to find a reduced order behavior $\hat{\mathfrak{B}}$ such that $\hat{\mathfrak{B}}^*$ is a subbehavior of the stable part of $\hat{\mathfrak{B}}^*$. Again, the details are left to the reader.

6.1 Rational interpolation at the spectral zeroes

In this subsection we will show that the transfer matrix associated with any reduced order system $\hat{\mathfrak{B}}$ obtained in Algorithm 1 is in fact a solution of a tangential Nevanlinna rational interpolation problem. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$ be represented by the minimal DV-representation $\mathfrak{B}_{DV}(A, B, C, D)$, and let $G(s) := D + C(sI - A)^{-1}B$ be its transfer matrix from driving variable to manifest variable. Let $\mathfrak{B}'$ be a given subbehavior of $(\mathfrak{B}^*)_{\text{antistable}}$, and let $\hat{\mathfrak{B}} := \mathfrak{B}_{DV}(\hat{A}, \hat{B}, \hat{C}, \hat{D})_{\text{ext}}$ be any reduced order system obtained from Algorithm 1. Let $\hat{G}(s) := \hat{D} + \hat{C}(sI - \hat{A})^{-1}\hat{B}$.

As noted before, $\mathfrak{B}'$ is associated with a unique $\mathtt{k}$-dimensional H-invariant subspace $\mathcal{V}$ of the antistable subspace $X_+(H)$ of H. In the remainder of this section, for simplicity we assume that the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{\mathtt{k}}$ of the restriction $H|_{\mathcal{V}}$ are *distinct*. In that case, the matrix M_{11} in Step 2 of Algorithm 1 (being a matrix representation of this restriction) can be diagonalized: there exists a nonsingular complex $\mathtt{k} \times \mathtt{k}$ matrix U such that $M_{11} = U^{-1}\Lambda U$, with $\Lambda := \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{\mathtt{k}})$. Let P be a nonsingular upper triangular matrix from Step 3 of Algorithm 1, say

$$P = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix}.$$

Consider the complex $\mathtt{n} \times \mathtt{k}$ matrix $\begin{bmatrix} P_{11} \\ 0 \end{bmatrix} U^{-1}$ and let $p_1, p_2, \ldots, p_{\mathtt{k}} \in \mathbb{C}^{\mathtt{n}}$ be its $\mathtt{k}$ columns. Finally, let $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ be the system matrices obtained after applying the similarity transformation S in Step 5, and let $\bar{H}$ denote the corresponding Hamiltonian matrix. We will now show that the reduced order transfer matrix $\hat{G}(s)$ is a solution of a rational tangential interpolation problem at the interpolation points $\lambda_1, \lambda_2, \ldots, \lambda_{\mathtt{k}}$, with data given by the values $G(\lambda_i)$ and the vectors p_i:

Theorem 6.2 : *For $i = 1, 2, \ldots, \mathtt{k}$, assume λ_i is not an eigenvalue of A and not an eigenvalue of $\bar{A}_{11}$. Define*

$$v_i := \bar{B}^\top p_i, \quad w_i := G(\lambda_i)v_i.$$

Then $\hat{G}(s)$ satisfies $w_i = \hat{G}(\lambda_i)v_i$ ($i = 1, 2, \ldots, \mathtt{k}$).

Note that in the case that the driving variable is one-dimensional, equivalently, the input cardinality of the systems $\mathfrak{B}$ and $\hat{\mathfrak{B}}$ is equal to one, then $\hat{G}(\lambda_i) = G(\lambda_i)$ for $i = 1, 2, \ldots, \mathtt{k}$, so the transfer matrix $\hat{G}$ of the reduced order system actually interpolates the values $G(\lambda_i)$ at the interpolation points $\lambda_1, \lambda_2, \ldots, \lambda_{\mathtt{k}}$.

Proof : First note that each p_i is of the form $(p_{i1}^\top, 0)^\top$, with $p_{i1} \in \mathbb{R}^{\mathtt{k}}$ the *ith* column of $P_{11}U^{-1}$. Also note that $(p_i^\top, p_i^\top)^\top \in \mathbb{R}^{2\mathtt{n}}$ is an eigenvector of $\bar{H}$ with eigenvalue λ_i $(i = 1, 2, \ldots, \mathtt{k})$. This implies $(\bar{A} + \bar{B}\bar{B}^\top)p_i = \lambda_i p_i$, so $(\lambda_i I - \bar{A})^{-1}\bar{B}\bar{B}^\top p_i = p_i$. This immediately implies

$$G(\lambda_i)\bar{B}^\top p_i = (\bar{D}\bar{B}^\top + \bar{C})p_i. \tag{13}$$

On the other hand, with $(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})$ the truncated system obtained in Step 6., we have $(A_{11} + B_1 B_1^\top)p_{i1} = \lambda_i p_{i1}$, so $(\lambda_i I - A_{11})^{-1}B_1 B_1^\top p_{i1} = p_{i1}$, which implies that

$$G_1(\lambda_i)B_1^\top p_{i1} = (\bar{D}B_1^\top + C_1)p_{i1}, \tag{14}$$

where $G_1(s) := D + C_1(Is - A_{11})^{-1}B_1$ is the transfer matrix associated with the truncated system. Combining (13) and (14), upon noting that $\bar{B}^\top p_i = B_1^\top p_{i1}$ and $\bar{C}p_i = C_1 p_{i1}$ we obtain that $G(\lambda_i)\bar{B}^\top p_i = G_1(\lambda_i)\bar{B}^\top p_i$. The proof is then completed by noting that $\hat{G} = G_1$. □

The above shows that Algorithm 1 in fact computes, for the given transfer matrix $G(s) = D + C(sI - A)^{-1}B$, a transfer matrix $\hat{G}(s)$ representing a reduced order behavior which is strictly Σ-dissipative on $\mathbb{R}^-$, and which interpolates $G(\lambda_i)$ in the sense that $\hat{G}(\lambda_i)v_i = G(\lambda_i)v_i$ with $v_i := \bar{B}^\top p_i$ $(i = 1, 2, \ldots, \mathtt{k})$. Thus Algorithm 1 solves a Nevanlinna type tangential interpolation problem, with interpolation point at $\mathtt{k}$ (antistable) spectral zeroes of the original system.

Remark 6.3 : It is well known that this tangential interpolation problem admits a solution if and only if the associated *Pick matrix* is positive definite. The Pick matrix for the problem at hand is the Hermitian $\mathtt{k} \times \mathtt{k}$ matrix $T := (T_{ij})$, with

$$T_{ij} = \frac{1}{\bar{\lambda}_i + \lambda_j} p_i^\top \bar{B} G(\bar{\lambda}_i)\Sigma G(\lambda_i)\bar{B}^\top p_j.$$

We will show now that the Pick matrix T is indeed positive definite, since it is congruent to $(X_1^1)^\top Y_1^1$, the $\mathtt{k} \times \mathtt{k}$ left upper block of the positive definite symmetric matrix $X_1^\top Y_1$. Indeed, by (13),

$$T_{ij} = \frac{1}{\bar{\lambda}_i + \lambda_j} p_i^\top (\bar{D}\bar{B}^\top + \bar{C})^\top \Sigma (\bar{D}\bar{B}^\top + \bar{C})p_j = \frac{1}{\bar{\lambda}_i + \lambda_j} p_i^\top (\bar{C}^\top \Sigma \bar{C} + \bar{B}\bar{B}^\top)p_j$$

Since $(p_i^*, p_i^*)^*$ is an eigenvector of $\bar{H}$ with eigenvalue λ_i, we obtain $p_i^*(\bar{A} + \bar{B}\bar{B}^\top)p_j = \lambda_j p_i^* p_j$ and $p_j^*(\bar{C}^\top \Sigma \bar{C} - \bar{A}^\top)p_i = \lambda_i p_j^* p_i$. Using this we get $p_i^\top(\bar{C}^\top \Sigma \bar{C} + \bar{B}\bar{B}^\top)p_j = (\bar{\lambda}_i + \lambda_j)p_i^* p_j$. We conclude that $T_{ij} = p_i^* p_j = p_{i1}^* p_{j1}$, so $T = (U^{-1})^* P_{11}^\top P_{11} U^{-1} = (U^{-1})^*(X_1^1)^\top Y_1^1 U^{-1}$.

7 Dissipativity and minimal dissipation for ON representations

In this section we study the subbehavior of minimal dissipation for the case that our system is represented by an ON-representation, and examine conditions under which a system in output nulling representation is strictly Σ-dissipative on $\mathbb{R}_-$ or $\mathbb{R}_+$.

Proposition 7.1 : *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$. Let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular. Assume that $\mathfrak{B}$ is strictly Σ-dissipative. Then there exists a minimal output nulling representation $\mathfrak{B}_{ON}(A, B, C, D)$ of $\mathfrak{B}$ such that*

$$D\Sigma^{-1}D^\top = J, \text{ with } J := \text{blockdiag}(I_{\text{rowdim}(D)-\mathtt{q}}, -I_{\mathtt{q}}) \text{ and } \mathtt{q} = \sigma_-(\Sigma), \tag{15}$$

$$B\Sigma^{-1}D^\top = 0. \tag{16}$$

Proof : The proof follows easily by combining Lemma 5.1 and Proposition 10.9 given in Appendix A. □

The following result is analogous to that of Proposition 5.2.

Proposition 7.2 : *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$. Let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular, and let $\mathfrak{B}_{ON}(A, B, C, D)$ be a minimal ON representation of $\mathfrak{B}$ such that (15) and (16) hold. Then the following conditions are equivalent:*

1. *$\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}^-$ ($\mathbb{R}^+$),*
2. *The ARE*

$$AK + KA^\top + B\Sigma^{-1}B^\top - KC^\top JCK = 0 \tag{17}$$

has a real symmetric solution $K > 0$ such that $A - KC^\top JC$ is stable (antistable),

3. *the Hamiltonian matrix $H' = \begin{bmatrix} A & B\Sigma^{-1}B^\top \\ C^\top JC & -A^\top \end{bmatrix}$ has no eigenvalues on the imaginary axis, and there exist $X_1, Y_1 \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, with Y_1 nonsingular, and $M \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ antistable (stable) such that*

$$H' \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} = \begin{bmatrix} X_1 \\ Y_1 \end{bmatrix} M$$

with $Y_1^\top X_1 > 0$ ($Y_1^\top X_1 < 0$).

If K satisfies the conditions in (2.) above then it is unique, and it is the smallest (largest) real symmetric solution of (17). We denote it by K^- (K^+). If X_1, Y_1 satisfy the conditions in (3.) above, then $X_1Y_1^{-1}$ is equal to this smallest (largest) real symmetric solution K^- ($K+$) of the ARE (17).

Proof : A proof of this can be given as follows: using Proposition 10.9 from Appendix A, associate with the given minimal output nulling representation a minimal driving variable representation satisfying the condtions of Proposition 5.2. Then apply proposition 5.2 to this driving variable representation. Finally, restate the conditions obtained in terms of the original output nulling representation (see also Theorem 5.3.5 in [11]). The equivalence of (2) and (3) again follows from standard results on the relation between the algebraic Riccati equation and the Hamiltonian matrix, see e.g. [29]. □

We now consider the problem of computing a representation for the subbehavior of minimal dissipation of a strictly Σ-dissipative behavior represented in ON form. Recalling the results on the representation of the orthogonal behavior of a behavior given in ON form, we find that if $\mathfrak{B}_{ON}(A,B,C,D)$ is a minimal representation of the Σ-dissipative behavior $\mathfrak{B}$, then the subbehavior of minimal dissipation $\mathfrak{B}^*$ is given by

$$\begin{aligned} \mathfrak{B}^* &= \mathfrak{B}_{ON}(A,B,C,D)_{\text{ext}} \\ &\quad \cap \mathfrak{B}_{DV}(-A^\top, C^\top, \Sigma^{-1}B^\top, -\Sigma^{-1}D^\top)_{\text{ext}}. \end{aligned} \tag{18}$$

This observation immediately leads to the following:

Proposition 7.3 : *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$. Let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular. Assume that $\mathfrak{B}$ is strictly Σ-dissipative, and let $\mathfrak{B}_{ON}(A,B,C,D)$ be a minimal ON representation such that (15) and (16) hold. Then $\mathfrak{B}^*$ is equal to the external behavior of the state space system*

$$\begin{aligned} \begin{bmatrix} \dot{x} \\ \dot{z} \end{bmatrix} &= \begin{bmatrix} A & B\Sigma^{-1}B^\top \\ C^\top JC & -A^\top \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} \\ w &= \begin{bmatrix} -\Sigma^{-1}D^\top JC & \Sigma^{-1}B^\top \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix}. \end{aligned} \tag{19}$$

Proof : From (18), it follows that $w \in \mathfrak{B}^*$ if and only if there exist x, z, v such that $\dot{x} = Ax + Bw$, $\dot{z} = -A^\top z + C^\top v$, $0 = Cx + Dw$, $w = \Sigma^{-1}B^\top z - \Sigma^{-1}D^\top v$. Since $D\Sigma^{-1}D^\top = J$ and $B\Sigma^{-1}D^\top = 0$, from $w = \Sigma^{-1}B^\top z - \Sigma^{-1}D^\top v$ it follows that $Dw = D\Sigma^{-1}B^\top z - D\Sigma^{-1}D^\top \Sigma v = -Jx$. Subtituting in $0 = Cx + Dw$, we get $v = JCx$. Consequently, $\dot{x} = Ax + B(\Sigma^{-1}B^\top z - \Sigma^{-1}D^\top \Sigma v) = Ax + B\Sigma^{-1}B^\top z$, $\dot{z} = -A^\top z + C^\top JCx$. This yields the claim. □

Again, we call the full behavior represented by the equations (19) the *Hamiltonian behavior of $\mathfrak{B}_{ON}(A,B,C,D)$ with respect to* Σ, and we denote it with $B_{H'}(A,B,C,D)$. The following result is the analogue of Theorem 5.4, and follows immediately from Proposition 7.3. The result shows how we can use the Hamiltonian behavior of $\mathfrak{B}_{ON}(A,B,C,D)$ in order to represent subbehaviors of the antistable part of the subbehavior of minimal dissipation.

Theorem 7.4 : *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$. Let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular. Assume $\mathfrak{B}$ strictly Σ-dissipative on $\mathbb{R}^-$, and let $\mathfrak{B}_{ON}(A,B,C,D)$ be a minimal ON-representation of $\mathfrak{B}$ such that (15) and (16) hold. Let $\mathtt{k} < \mathtt{n}(\mathfrak{B})$ be a positive integer. Let $\mathfrak{B}'$ be a subbehavior $(\mathfrak{B}^*)_{\text{antistab}}$ with $\mathtt{n}(\mathfrak{B}') = \mathtt{k}$. Then there exist $X_1^1, Y_1^1 \in \mathbb{R}^{\mathtt{n}\times\mathtt{k}}$, $X_1^2, Y_1^2 \in \mathbb{R}^{\mathtt{n}\times(\mathtt{n}-\mathtt{k})}$, and matrices M_{11}, M_{12}, M_{22} with M_{11} and M_{22} antistable and $Y_1 := [Y_1^1 \; Y_1^2]$ nonsingular, such that*

$$\begin{bmatrix} A & B\Sigma^{-1}B^\top \\ C^\top J\Sigma C & -A^\top \end{bmatrix} \begin{bmatrix} X_1^1 & X_1^2 \\ Y_1^1 & Y_1^2 \end{bmatrix} = \begin{bmatrix} X_1^1 & X_1^2 \\ Y_1^1 & Y_1^2 \end{bmatrix} \underbrace{\begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}}_{=:M},$$

$$\mathfrak{B}' = \text{span}\{(-\Sigma^{-1}D^\top JCX_1^1 + \Sigma^{-1}B^\top Y_1^1)e^{M_{11}t}\},$$

and

$$(\mathfrak{B}^*)_{\text{antistable}} = \text{span}\{(-\Sigma^{-1}D^\top JCX_1 + \Sigma^{-1}B^\top Y_1)e^{Mt}\}.$$

Here, we define $X_1 := [X_1^1 \; X_1^2]$.

Again, a similar theorem of course holds holds for the stable part $\mathfrak{B}^*_{\text{stable}}$ of $\mathfrak{B}^*$ under the assumption of strict Σ-dissipativity on $\mathbb{R}^+$.

Finally, in a result parallel to that of Lemma 5.5, we show that the subbehavior of minimal dissipation of the controllable part of $\mathfrak{B}$ is *contained* in the external behavior of the Hamiltonian system (19):

Lemma 7.5 : *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$. Let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular. Let $\mathfrak{B}_{ON}(A,B,C,D)$ be a minimal ON-representation of $\mathfrak{B}$ such that (15) and (16) hold. Assume that $\mathfrak{B}_{\text{contr}}$ is strictly Σ-dissipative. Then $(\mathfrak{B}_{\text{contr}})^* \subseteq \mathfrak{B}_{H'}(A,B,C,D)_{\text{ext}}$. Consequently, if the Hamiltonian matrix H' has no imaginary axis eigenvalues, then $(\mathfrak{B}_{\text{contr}})^*_{\text{antistable}}$ is contained in the antistable part of the external behavior of the Hamiltonian system, as given by (19)*

Proof : We prove only the first part of the claim, since the second one follows easily. We first compute an ON representation of the controllable part of $\mathfrak{B}$ using Proposition 10.8, thus obtaining a not necessarily minimal ON representation $\mathfrak{B}_{ON}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})$ of $\mathfrak{B}_{\text{cont}}$. Observe that

$$(\mathfrak{B}_{\text{contr}})^* = \mathfrak{B}_{ON}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})_{\text{ext}} \cap [\mathfrak{B}_{ON}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})_{\text{ext}}]^{\perp_\Sigma}$$

In order to compute a minimal ON representation of $\mathfrak{B}$, we apply Proposition 10.6, obtaining $\mathfrak{B}_{ON}(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1, \tilde{D})$. Conclude that

$$\begin{aligned}(\mathfrak{B}_{\text{contr}})^* &= \mathfrak{B}_{ON}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})_{\text{ext}} \cap [\mathfrak{B}_{ON}(\tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1, \tilde{D})_{\text{ext}}]^{\perp_\Sigma} \\ &= \mathfrak{B}_{ON}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})_{\text{ext}} \cap \mathfrak{B}_{DV}(-\tilde{A}_{11}^\top, \tilde{C}_1^\top, \Sigma^{-1}\tilde{B}_1^\top, -\Sigma^{-1}\tilde{D}^\top)_{\text{ext}}.\end{aligned}$$

It is not difficult to see that

$$\begin{aligned}\mathfrak{B}_{DV}(-\tilde{A}_{11}^\top, \tilde{C}_1^\top, \Sigma^{-1}\tilde{B}_1^\top, -\Sigma^{-1}\tilde{D}^\top)_{\text{ext}} &\subseteq \mathfrak{B}_{DV}(-\bar{A}_{11}^\top, \bar{C}_1^\top, \Sigma^{-1}\bar{B}_1^\top, -\Sigma^{-1}\bar{D}^\top)_{\text{ext}} \\ &\subseteq \mathfrak{B}_{DV}(-A^\top, C^\top, \Sigma^{-1}B^\top, -\Sigma^{-1}D^\top)_{\text{ext}}.\end{aligned}$$

Consequently

$$\begin{aligned}(\mathfrak{B}_{\text{contr}})^* &\subseteq \mathfrak{B}_{ON}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})_{\text{ext}} \cap \mathfrak{B}_{DV}(-A^\top, C^\top, \Sigma^{-1}B^\top, -\Sigma^{-1}D^\top)_{\text{ext}} \\ &\subseteq \mathfrak{B}_{ON}(A,B,C,D)_{\text{ext}} \cap \mathfrak{B}_{DV}(-A^\top, C^\top, \Sigma^{-1}B^\top, -\Sigma^{-1}D^\top)_{\text{ext}} \\ &= B_{H'}(A,B,C,D),\end{aligned}$$

where $B_{H'}(A,B,C,D)$ is the Hamiltonian behavior defined by (19). □

8 A reduction algorithm for ON-representations

In this section we give an algorithmic procedure to compute for a given controllable behavior $\mathfrak{B}$, strictly Σ-dissipative on $\mathbb{R}^-$, a given integer $\mathtt{k} \leq \mathtt{n}(\mathfrak{B})$, and a given subbehavior of the antistable part of the subbehavior of minimal dissipation, an ON-representation of a solution to our Main Problem as stated in section 4. Again, we will also show that the transfer matrix associated with the ON-representation of any of our solutions is a solution to a rational interpolation problem

ALGORITHM 2. (from ONR to ONR)

Input: $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ nonsingular, and $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$, strictly Σ-dissipative on $\mathbb{R}^-$, an integer $0 \leq \mathtt{k} \leq \mathtt{n}(\mathfrak{B})$ and a subbehavior $\mathfrak{B}'$ of $(\mathfrak{B}^*)_{\text{antistable}}$ such that $\mathtt{n}(\mathfrak{B}') = \mathtt{k}$.

Output: a minimal ON-representation of $\hat{\mathfrak{B}} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$ solving the Main Problem.

Step 1. Compute a minimal ON-representation $\mathfrak{B}_{ON}(A, B, C, D)$ of $\mathfrak{B}$ such that (15) and (16) hold.

Step 2. Compute $X_1 = [X_1^1 \; X_1^2] \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, and $Y_1 = [Y_1^1 \; Y_1^2] \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ nonsingular, such that

$$\begin{bmatrix} A & B\Sigma^{-1}B^\top \\ C^\top JC & -A^\top \end{bmatrix} \begin{bmatrix} X_1^1 & X_1^2 \\ Y_1^1 & Y_1^2 \end{bmatrix} = \begin{bmatrix} X_1^1 & X_1^2 \\ Y_1^1 & Y_1^2 \end{bmatrix} \underbrace{\begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}}_{=:M},$$

where M_{11} and M_{22} are antistable and $\mathfrak{B}' = \text{span}\{(-\Sigma^{-1}D^\top JCX_1^1 + \Sigma^{-1}B^\top Y_1^1)e^{M_{11}t}\}$,

Step 3. Compute a Cholesky factorization $P^\top P = Y_1^\top X_1$, with P a nonsingular upper triangular matrix.
Comment: Such factorization exists, since $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$ is strictly Σ-dissipative on $\mathbb{R}^-$ and consequently $Y_1^\top X_1 > 0$ (see Proposition 5.2).

Step 4. Compute $S = X_1 P^{-1} = Y_1^{-\top} P^\top$.

Step 5. Compute $(\bar{A}, \bar{B}, \bar{C}, \bar{D}) := (S^{-1}AS, S^{-1}B, CS, D)$.

Step 6. Let $(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})$ denote the truncation of $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ to the first $\mathtt{k}$ components of the state, and let $\mathfrak{B}_{\text{trunc}} := \mathfrak{B}_{ON}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})_{\text{ext}}$

Step 7. Perform a Kalman controllability decomposition to compute the controllable part of $\mathfrak{B}_{\text{trunc}}$:

$$\bar{A}_{11} = \begin{bmatrix} \hat{A} & * \\ 0 & * \end{bmatrix}, \bar{B}_1 = \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix}, \bar{C}_1 = \begin{bmatrix} \hat{C} & * \end{bmatrix}, \bar{D} = \hat{D},$$

Step 8. Output

$$\hat{\mathfrak{B}} := (\mathfrak{B}_{\text{trunc}})_{\text{contr}} = \mathfrak{B}_{ON}(\hat{A}, \hat{B}, \hat{C}, \hat{D})_{\text{ext}}.$$

Theorem 8.1 : *The behavior $\hat{\mathfrak{B}}$ computed in Algorithm 2 is a solution to the Main Problem.*

Proof : By construction $\mathtt{n}(\hat{\mathfrak{B}}) \leq \mathtt{k}$. Also, $D = \hat{D}$ has full row rank, so the number of algebraic equations in the original and new ON-representation are equal. From the fact that the number of algebraic equations in a minimal ON-representation is equal to the output cardinality of its external behavior, we deduce $\mathtt{m}(\hat{\mathfrak{B}}) = \mathtt{m}(\mathfrak{B})$.

We now prove that $\hat{\mathfrak{B}}$ is strictly Σ-dissipative on $\mathbb{R}^-$. For $(\bar{A}, \bar{B}, \bar{C}, \bar{D})$ as computed in Step 5 in Algorithm 2 we have

$$\begin{bmatrix} \bar{A} & \bar{B}\Sigma^{-1}\bar{B}^\top \\ \bar{C}^\top J\bar{C} & -\bar{A}^\top \end{bmatrix} \begin{bmatrix} P \\ P \end{bmatrix} = \begin{bmatrix} P \\ P \end{bmatrix} M, \text{ with } M = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix} \tag{20}$$

as in Step 2. Denote the $(1,1)$-block of the upper triangular matrix P by P_{11}. Then the truncated system $(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})$ computed in Step 6 satisfies

$$\begin{bmatrix} \bar{A}_{11} & \bar{B}_1\Sigma^{-1}\bar{B}_1^\top \\ \bar{C}_1^\top J\bar{C}_1 & -\bar{A}_{11}^\top \end{bmatrix} \begin{bmatrix} P_{11} \\ P_{11} \end{bmatrix} = \begin{bmatrix} P_{11} \\ P_{11} \end{bmatrix} M_{11}. \tag{21}$$

From (21) it follows that the maximal solution of the ARE

$$\bar{A}_{11}\bar{K} + \bar{K}\bar{A}_{11}^\top - \bar{K}\bar{C}_1^\top J\bar{C}_1\bar{K} + \bar{B}_1\Sigma^{-1}\bar{B}_1^\top = 0 \tag{22}$$

is $\bar{K} = P_{11}P_{11}^{-1} = I$. Since $(\bar{A}_{11}^\top - \bar{C}_1^\top J\bar{C}_1)P_{11} = -P_{11}M_{11}$ and M_{11} is antistable, $\bar{A}_{11}^\top - \bar{C}_1^\top J\bar{C}_1$ is stable. Now consider the ARE corresponding to the ON-representation of the reduced order (controllable) behavior $\hat{\mathfrak{B}}$ computed in Step 8:

$$\hat{A}\hat{K} + \hat{K}\hat{A}^\top - \hat{K}\hat{C}^\top J\hat{C}\hat{K} + \hat{B}^\top\Sigma^{-1}\hat{B} = 0. \tag{23}$$

Observe that any solution of (23) is the $(1,1)$-block of a solution of (22). In particular, I is a solution of (23). Moreover, we have

$$\bar{A}_{11}^\top - \bar{C}^\top J\bar{C} = \begin{bmatrix} \hat{A}^\top - \hat{C}^\top J\hat{C} & * \\ 0 & * \end{bmatrix},$$

which implies that $\hat{A}^\top - \hat{C}^\top J\hat{C}$ is stable. By Proposition 7.2 we conclude that $\hat{\mathfrak{B}}$ is strictly Σ-dissipative on $\mathbb{R}^-$.

Finally, we prove that the antistable part of the subbehavior of minimal dissipation of the reduced order behavior $\hat{\mathfrak{B}}$ is contained in $\mathfrak{B}'$. From the definition of $\mathfrak{B}'$ conclude that

$$\begin{aligned}
\mathfrak{B}' &= \operatorname{span}\{((-\Sigma^{-1}D^\top JCS)(S^{-1}X_1^1) + (\Sigma^{-1}B^\top S^{-\top})(S^\top Y_1^1))e^{M_{11}t}\} \\
&= \operatorname{span}\{(-\Sigma^{-1}\bar{D}^\top J\bar{C}(S^{-1}X_1^1) + \Sigma^{-1}\bar{B}^\top(S^\top Y_1^1))e^{M_{11}t}\} \\
&= \operatorname{span}\{-\Sigma^{-1}\bar{D}^\top J\begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix}\begin{bmatrix} P_{11} \\ 0 \end{bmatrix}e^{M_{11}t} + \Sigma^{-1}\begin{bmatrix} \bar{B}_1^\top & \bar{B}_2^\top \end{bmatrix}\begin{bmatrix} P_{11} \\ 0 \end{bmatrix}e^{M_{11}t}\} \\
&= \operatorname{span}\{(-\Sigma^{-1}\bar{D}^\top J\bar{C}_1 + \Sigma^{-1}\bar{B}_1^\top)P_{11}e^{M_{11}t}\}.
\end{aligned}$$

It follows from Lemma 7.5 that $\hat{\mathfrak{B}}^* = ((\mathfrak{B}_{\text{trunc}})_{\text{contr}})^* \subseteq \mathfrak{B}_{H'}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D})$. Thus $(\hat{\mathfrak{B}}^*)_{\text{antistable}} \subseteq (\mathfrak{B}_{H'}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, \bar{D}))_{\text{antistable}}$. The latter is equal to $\operatorname{span}\{(-\Sigma^{-1}\bar{D}^\top J\bar{C}_1 + \Sigma^{-1}\bar{B}_1^\top)P_{11}e^{M_{11}t}\}$, which is equal to $\mathfrak{B}'$. □

Again, a similar algorithmic procedure can be given for the problem where the original system $\mathfrak{B}$ is strictly dissipative on $\mathbb{R}^+$, and with $\mathfrak{B}'$ a subbehavior of the stable part of $\mathfrak{B}^*$, and where it is required to find a reduced order behavior $\hat{\mathfrak{B}}$ such that $\hat{\mathfrak{B}}^*$ is a subbehavior of the stable part of $\hat{\mathfrak{B}}^*$. The details are left to the reader.

8.1 Rational interpolation at the spectral zeroes

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{contr}}$ be represented by the minimal ON-representation $\mathfrak{B}_{ON}(A, B, C, D)$, and let $G(s) := D + C(sI - A)^{-1}B$. Let $\mathfrak{B}'$ be a given subbehavior of $(\mathfrak{B}^*)_{\text{antistable}}$, and let $\hat{\mathfrak{B}} := \mathfrak{B}_{DV}(\hat{A}, \hat{B}, \hat{C}, \hat{D})_{\text{ext}}$ be any reduced order system obtained from Algorithm 2. Let $\hat{G}(s) := \hat{D} + \hat{C}_1(sI - \hat{A}_{11})^{-1}\hat{B}_1$.

As before, the given subbehavior $\mathfrak{B}'$ is associated with a unique H'-invariant subspace $\mathcal{W}$ of the antistable subspace $X_+(H')$ of the Hamiltonian matrix H'. For simplicity, assume that the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_{\mathtt{k}}$ of the restriction $H' \mid_{\mathcal{W}}$ are distinct. Again, in that case the matrix M_{11} from step 2 of Algorithm 2 can be diagonalized: there exists a nonsingular complex $\mathtt{k} \times \mathtt{k}$ matrix T such that $M_{11} = U^{-1}\Lambda U$, with

$\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_{\mathtt{k}})$. Let P be a nonsingular upper triangular matrix obtained in step 3, say

$$P = \begin{bmatrix} P_{11} & P_{12} \\ 0 & P_{22} \end{bmatrix}.$$

Now consider the complex $\mathtt{n} \times \mathtt{k}$ matrix $\begin{bmatrix} P_{11} \\ 0 \end{bmatrix} U^{-1}$ and let $p_1, p_2, \ldots, p_{\mathtt{k}} \in \mathbb{C}^{\mathtt{n}}$ be its $\mathtt{k}$ columns. Then we have the following result:

Theorem 8.2 : *For $i = 1, 2, \ldots, \mathtt{k}$, assume that λ_i is not an eigenvalue of A and not an eigenvalue of $\bar{A}_{11}$. Define*

$$w_i := \Sigma^{-1} \bar{B}^\top p_i, \quad z_i := G(\lambda_i) w_i.$$

Then $\hat{G}(s)$ satisfies $z_i = \hat{G}(\lambda_i) w_i$ $(i = 1, 2, \ldots, \mathtt{k})$.

Proof : The proof is analoguous to that of theorem 6.2. It uses the facts that p_i is an eigenvector of $\bar{H}'$ with eigenvalue λ_i, and that p_i is of the form $(p_{i1}^\top, 0)^\top$. The details are left to the reader. □

9 Conclusions

In this paper we have introduced and resolved the problem of dissipativity preserving model reduction by retention of trajectories of minimal dissipation. The problem is to find, for a given dissipative behavior $\mathfrak{B}$ of McMillan degree $\mathtt{n}$, and a degree $\mathtt{k}$ subbehavior $\mathfrak{B}'$ of the subbehavior of minimal dissipation, a dissipative approximative behavior $\hat{\mathfrak{B}}$ of McMillan degree $\mathtt{k}$ whose subbehavior of minimal dissipation is contained in $\mathfrak{B}'$. This means that the approximative behavior $\hat{\mathfrak{B}}$ "inherits" trajectories of minimal dissipation from $\mathfrak{B}$. We have given algorithmic procedures to compute $\hat{\mathfrak{B}}$ from $\mathfrak{B}$ in two cases, the case that $\mathfrak{B}$ is given in driving variable representation, and the case that $\mathfrak{B}$ is given in output nulling representation. In both cases the algorithms are based on analysis of invariant subspaces of a Hamiltonian matrix, and on truncation of a state space model obtained after suitable state space transformation. The use of the Hamiltonian matrix for computing an approximative system is reminescent to the work of Sorensen in [16], where a Hamiltonian matrix is used to compute a passive approximation of a given input/state/output system. Indeed, the work in the present paper can be seen as a behavioral formulation and interpretation of the ideas of Antoulas [2] and Sorensen [16] on passivity preserving model reduction using rational interpolation. Of course, the results in our paper are valid for general supply rates. In our paper we show, a fortiori, that the transfer matrices of our reduced order behaviors are solutions of certain tangential Nevanlinna interpolation problems, with interpolation points at the spectral zeroes of the original behavior (see also [10]).

10 Appendix: Basics of driving-variable and output-nulling representations

As already noted in section 2, linear differential systems often result as external behavior of systems with latent variables. Two particular instances of such latent variable representations are systems with driving variables, and output nulling systems. In these latent variable systems, the latent variable in fact satisfies the axiom of state. In this appendix we have collected the basic material on driving variable and output nulling representations.

10.1 Driving-variable representations

Let $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{v}}$, $C \in \mathbb{R}^{\mathtt{w}\times\mathtt{n}}$, $D \in \mathbb{R}^{\mathtt{w}\times\mathtt{v}}$, and consider the equations

$$\begin{aligned} \dot{x} &= Ax + Bv \\ w &= Cx + Dv. \end{aligned} \tag{24}$$

These equations represent the *full behavior*

$$\mathfrak{B}_{DV}(A,B,C,D) := \{(w,x,v) \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \times \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{n}}) \times \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{v}}) \mid (24) \text{ hold}\}.$$

In we interpret w as manifest variable and (x,v) as latent variable, then $\mathfrak{B}_{DV}(A,B,C,D)$ is a latent variable representation of its external behavior

$$\mathfrak{B}_{DV}(A,B,C,D)_{\text{ext}} = \{w \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid \exists\, x \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{n}}) \text{ and } v \in \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{v}}) \text{ such that } (w,x,v) \in \mathfrak{B}_{DV}(A,B,C,D)\}.$$

The variable x is in fact a state variable, the variable v is free, and is called the *driving variable.*

If $\mathfrak{B} = \mathfrak{B}_{DV}(A,B,C,D)_{\text{ext}}$ then we call $\mathfrak{B}_{DV}(A,B,C,D)$ a *driving variable representation* of $\mathfrak{B}$. A driving variable representation $\mathfrak{B}_{DV}(A,B,C,D)$ of $\mathfrak{B}$ is called *minimal* if the state dimension $\mathtt{n}$ and the driving variable dimension $\mathtt{v}$ are minimal over all such driving variable representations. In the following, let $\mathtt{n}(\mathfrak{B})$ and $\mathtt{m}(\mathfrak{B})$ denote the McMillan degree of $\mathfrak{B}$, and the input cardinality of $\mathfrak{B}$, respectively. The following result is well known:

Proposition 10.1 *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be given. Denote $\mathtt{n} = \mathtt{n}(\mathfrak{B})$ and $\mathtt{m} = \mathtt{m}(\mathfrak{B})$. Then*

1. *there exists matrices $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$, $C \in \mathbb{R}^{\mathtt{w}\times\mathtt{n}}$, $D \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}$ such that $\mathfrak{B}_{DV}(A,B,C,D)$ is a minimal driving variable representation of $\mathfrak{B}$,*

2. *if $\mathfrak{B}_{DV}(A,B,C,D)$ represents $\mathfrak{B}$, then it is a minimal representation if and only if D is injective and the pair $(C+DF, A+BF)$ is observable for all F,*

3. *if $\mathfrak{B}_{DV}(A,B,C,D)$ is a minimal representation of $\mathfrak{B}$, then $\mathfrak{B}_{DV}(A',B',C',D')$ is a minimal representation of $\mathfrak{B}$ if and only if there exist invertible matrices S and R and a matrix F such that*

$$(A',B',C',D') = (S^{-1}(A+BF)S, S^{-1}BR, (C+DF)S, DR).$$

Proof : See Theorem 3.10 in [20]. □

The next proposition states that in order to compute a minimal driving variable representation from a given one, we can use state feedback.

Proposition 10.2 *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and let $\mathfrak{B}_{DV}(A,B,C,D)$ be a driving variable representation of $\mathfrak{B}$, with D injective. Define $F := -(D^\top D)^{-1}D^\top C$. Then there is a nonsingular matrix S such that $S^{-1}(A+BF)S = \begin{bmatrix} A'_{11} & 0 \\ A'_{21} & A'_{22} \end{bmatrix}$, $S^{-1}B = \begin{bmatrix} B'_1 \\ B'_2 \end{bmatrix}$, $(C+DF)S = \begin{bmatrix} C'_1 & 0 \end{bmatrix}$ such that*

1. *the pair* $(C_1' + DF', A_{11}' + B_1'F')$ *is observable for all* F',

2. $B_{DV}(A_{11}', B_1', C_1', D)_{\text{ext}} = \mathfrak{B}_{DV}(A, B, C, D)_{\text{ext}}$.

Consequently, $\mathfrak{B}_{DV}(A_{11}', B_1', C_1', D)$ *is a minimal driving variable representation of* $\mathfrak{B}$.

Proof : Let $\mathcal{V}^*$ be the weakly unobservable subspace of (A, B, C, D) (see [19], section 7.3). By [19], Exercise 7.5, $\mathcal{V}^*$ is equal to the unobservable subspace of the pair $(C + DF, A+BF)$, with $F = -(D^\top D)^{-1}D^\top C$. With respect to a basis adapted to $\mathcal{V}^*$, $A+BF$, $C + DF$ and B have matrices partitioned as claimed above. By construction, the weakly unobservable subspace of (A_{11}', B_1', C_1', D) is zero and therefore, by [19] Theorem 7.16, statement (1) of the proposition holds.

In order to prove that $B_{DV}(A_{11}', B_1', C_1', D)_{\text{ext}} = \mathfrak{B}_{DV}(A, B, C, D)_{\text{ext}}$, observe that since coordinate transformations and state feedback do not change the external behavior, we have $\mathfrak{B}_{DV}(S^{-1}(A + BF)S, S^{-1}B, (C + DF)S, D)_{\text{ext}} = \mathfrak{B}_{DV}(A, B, C, D)_{\text{ext}}$. We now prove that $\mathfrak{B}_{DV}(S^{-1}(A+BF)S, S^{-1}B, (C+DF)S, D)_{\text{ext}} = B_{DV}(A_{11}', B_1', C_1', D)_{\text{ext}}$. The inclusion $\subseteq$ follows immediately. In order to prove the converse inclusion, let $w \in \mathfrak{B}_{DV}(A_{11}', B_1', C_1', D)_{\text{ext}}$. Then there exist x_1, v such that

$$\begin{aligned} \dot{x}_1 &= A_{11}'x_1 + B_1'v \\ w &= C_1'x_1 + Dv. \end{aligned}$$

Then, let x_2 be any solution of $\dot{x}_2 = A_{21}'x_1 + A_{22}'x_2 + B_2'v$. This proves that $w \in \mathfrak{B}_{DV}(S^{-1}(A+BF)S, S^{-1}B, (C+DF)S, D)_{\text{ext}}$, so statement (2) of the proposition holds. Finally, the minimality of (A_{11}', B_1', C_1', D) as a representation of $\mathfrak{B}$ follows from the fact that D is injective and from statement (1). □

In this paper, in the context of dissipative systems, we mostly work with controllable behaviors, and with the controllable part of a behavior. We now examine under what conditions a behavior represented in driving variable form is controllable.

Proposition 10.3 *Let* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ *be given. Then the following statements are equivalent*

1. $\mathfrak{B}$ *is controllable,*

2. *there exist matrices* A, B, C *and* D *such that* $\mathfrak{B} = \mathfrak{B}_{DV}(A, B, C, D)_{\text{ext}}$ *with* (A, B) *controllable,*

3. *for every minimal representations* $\mathfrak{B} = B_{DV}(A, B, C, D)_{\text{ext}}$, *the pair* (A, B) *is controllable.*

Proof : See Theorem 3.11 [20]. □

Now let $\mathfrak{B}$ be possibly non-controllable, and let $\mathfrak{B}_{DV}(A, B, C, D)$ be a driving variable representation. The following result shows how to compute a driving variable representation of the controllable part of $\mathfrak{B}$.

Proposition 10.4 *Let* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ *and let* $\mathfrak{B}_{DV}(A, B, C, D)$ *be a driving variable representation of* $\mathfrak{B}$. *Then there exists a nonsingular matrix* S *such that*

1. $S^{-1}AS = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix}$, $S^{-1}B = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}$, $CS = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix}$,

2. $(\bar{A}_{11}, \bar{B}_1)$ *is controllable.*

Then $\mathfrak{B}_{DV}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, D)$ *is a driving variable representation of the controllable part* $\mathfrak{B}_{\text{cont}}$ *of* $\mathfrak{B}$.

Proof : First, clearly the *full* behavior $\mathfrak{B}_{DV}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, D)$ is controllable. Define $\mathfrak{B}_0 := \{(w, (x_1, 0), v) \mid (w, x_1, v) \in \mathfrak{B}_{DV}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, D)\}$. Then $\mathfrak{B}_0$ is controllable. Also we have $\mathfrak{B}_0 \subseteq \mathfrak{B}_{DV}(S^{-1}AS, S^{-1}B, CS, D)$, and the input cardinalities of these two behaviors coincide. By [3], Lemma 2.10.3, their controllable parts then coincide, so we have $\mathfrak{B}_0 = \mathfrak{B}_{DV}(S^{-1}AS, S^{-1}B, CS, D)_{\text{cont}}$. Finally, the two operations of taking the controllable part and taking external behavior commute (see [3], Lemma 2.10.4). Thus we obtain $\mathfrak{B}_{DV}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, D)_{\text{ext}} = (\mathfrak{B}_0)_{\text{ext}} = (\mathfrak{B}_{DV}(S^{-1}AS, S^{-1}B, CS, D)_{\text{cont}})_{\text{ext}} = (\mathfrak{B}_{DV}(S^{-1}AS, S^{-1}B, CS, D)_{\text{ext}})_{\text{cont}} = \mathfrak{B}_{\text{cont}}$. □

Output-nulling representations

Output-nulling representations are defined as follows. Let $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{w}}$, $C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$, $D \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}$, and consider the equations

$$\begin{aligned} \dot{x} &= Ax + Bw \\ 0 &= Cx + Dw \end{aligned} \tag{25}$$

These equations represent the *full behavior*

$$\mathfrak{B}_{ON}(A, B, C, D) := \{(w, x) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}}) \mid (25) \text{ hold}\}.$$

Again, if we interpret w as manifest variable and x as latent variable, then $\mathfrak{B}_{ON}(A, B, C, D)$ is a latent variable representation of its external behavior

$$\mathfrak{B}_{ON}(A, B, C, D)_{\text{ext}} = \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid \exists x \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{n}}) \text{ such that } (w, x) \in \mathfrak{B}_{ON}(A, B, C, D)\}.$$

Also here, the variable x is a state variable. If $\mathfrak{B} = \mathfrak{B}_{ON}(A, B, C, D)_{\text{ext}}$ then we call $\mathfrak{B}_{ON}(A, B, C, D)$ an output nulling representation of $\mathfrak{B}$. $\mathfrak{B}_{ON}(A, B, C, D)$ is called a *minimal output nulling representation* if $\mathtt{n}$ and $\mathtt{p}$ are minimal over all output nulling representations of $\mathfrak{B}$. In the following, let $\mathtt{n}(\mathfrak{B})$ and $\mathtt{p}(\mathfrak{B})$ denote the McMillan degree of $\mathfrak{B}$, and the output cardinality of $\mathfrak{B}$, respectively. Again, the following is well-known:

Proposition 10.5 *Let* $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ *be given. Denote* $\mathtt{n} = \mathtt{n}(\mathfrak{B})$ *and* $\mathtt{p} = \mathtt{p}(\mathfrak{B})$. *Then*

1. *there exist matrices* $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{w}}$, $C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$, $D \in \mathbb{R}^{\mathtt{p}\times\mathtt{w}}$ *such that* $\mathfrak{B}_{ON}(A, B, C, D)$ *is a minimal output nulling representation of* $\mathfrak{B}$,

2. *if* $\mathfrak{B}_{ON}(A, B, C, D)$ *represents* $\mathfrak{B}$, *then it is a minimal representation if and only if* D *is surjective and* (C, A) *is observable,*

3. *if* $\mathfrak{B}_{ON}(A, B, C, D)$ *is a minimal representation of* $\mathfrak{B}$, *then* $\mathfrak{B}_{ON}(A', B', C', D')$ *is a minimal representation of* $\mathfrak{B}$ *if and only if there exist invertible matrices* S *and* R *and a matrix* J *such that*

$$(A', B', C', D') = (S^{-1}(A + JC)S, S^{-1}(B + JD), RCS, RD).$$

Proof : See Theorem 3.20 in [20]. □

The next proposition shows how to compute a minimal output nulling representation of $\mathfrak{B}$ from a given one.

Proposition 10.6 *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ and let $\mathfrak{B}_{ON}(A, B, C, D)$ be an output nulling representation of $\mathfrak{B}$ with D surjective. Then there exist a nonsingular matrix S such that*

1. $S^{-1}AS = \begin{bmatrix} A'_{11} & 0 \\ A'_{21} & A'_{22} \end{bmatrix}, S^{-1}B = \begin{bmatrix} B'_1 \\ B'_2 \end{bmatrix}, CS = \begin{bmatrix} C'_1 & 0 \end{bmatrix},$
2. *the pair (C'_1, A'_{11}) is observable.*
3. $B_{ON}(A'_{11}, B'_1, C'_1, D)_{\text{ext}} = \mathfrak{B}_{ON}(A, B, C, D)_{\text{ext}}$.

Consequently, $B_{ON}(A'_{11}, B'_1, C'_1, D)$ is a minimal output nulling representation of $\mathfrak{B}$.

Proof : The existence of a nonsingular transformation matrix S such that the conditions (1) and (2) hold follows from a standard argument. In order to prove that $B_{ON}(A'_{11}, B'_1, C'_1, D)_{\text{ext}} = \mathfrak{B}_{ON}(A, B, C, D)_{\text{ext}}$, observe first that the transformation S does not change the external behavior. We now prove that $\mathfrak{B}_{ON}(S^{-1}AS, S^{-1}B, CS, D)_{\text{ext}} = B_{ON}(A'_{11}, B'_1, C'_1, D)_{\text{ext}}$. The inclusion $\subseteq$ follows immediately from the equations. In order to prove the converse inclusion, let $w \in \mathfrak{B}_{ON}(A'_{11}, B'_1, C'_1, D)_{\text{ext}}$. Then there exist x_1 such that

$$\begin{aligned} \dot{x}_1 &= A'_{11}x_1 + B'_1 w \\ 0 &= C'_1 x_1 + D'w. \end{aligned}$$

With these x_1 and v, let x_2 be any solution of $\dot{x}_2 = A'_{21}x_1 + A'_{22}x_2 + B'_2 w$. The claim follows. □

As noted before, in the context of dissipative systems we work with controllable behaviors, and with the controllable part of a behavior. We now examine under what conditions a behavior represented in output-nulling form is controllable.

Proposition 10.7 *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ be given. Then the following statements are equivalent*

1. *$\mathfrak{B}$ is controllable,*
2. *there exist matrices A, B, C and D such that $\mathfrak{B} = \mathfrak{B}_{ON}(A, B, C, D)_{\text{ext}}$ with $(A + JC, B + JD)$ controllable for all real matrices J,*
3. *for every minimal representation $\mathfrak{B} = \mathfrak{B}_{ON}(A, B, C, D)_{\text{ext}}$ we have: the pair $(A + JC, B + JD)$ is controllable for all real matrices J.*

Proof : See Theorem 3.11 [20]. □

Next, we show how to compute an output nulling representation of the controllable part of $\mathfrak{B}$ from a given output nulling representation.

Proposition 10.8 *Let $\mathfrak{B} = \mathfrak{B}_{ON}(A, B, C, D)_{\text{ext}}$, with D surjective. Define an output injection by $G := -BD^{\top}(DD^{\top})^{-1}$. Then there exists a nonsingular matrix S such that*

1. $S^{-1}(A+GC)S = \begin{bmatrix} \bar{A}_{11} & \bar{A}_{12} \\ 0 & \bar{A}_{22} \end{bmatrix}, S^{-1}(B+GD) = \begin{bmatrix} \bar{B}_1 \\ 0 \end{bmatrix}, CS = \begin{bmatrix} \bar{C}_1 & \bar{C}_2 \end{bmatrix}$,
2. *$(\bar{A}_{11} + G'\bar{C}_1, \bar{B}_1 + G'D)$ is controllable for all real matrices G'.*

Furthermore, $\mathfrak{B}_{ON}(\bar{A}_{11}, \bar{B}_1, \bar{C}_1, D)$ is an output nulling representation of the controllable part $\mathfrak{B}_{\text{cont}}$ of $\mathfrak{B}$.

Proof : A proof of this can be given using the the notion of strongly reachable subspace (see [19], section 8.3), combined with similar ideas as in the proof of Proposition 10.4. The details are left to the reader. □

Relations between DV and ON representations

Driving variable and output nulling representations of the same behavior enjoy certain duality properties. These will be examined in this section. The first result we prove explaines how an output nulling representation can be obtained from a driving-variable representation of the same behavior, and the other way around. In order to state it, we need to introduce the "annihilator" of a matrix, defined as follows. Let D be a $\mathtt{p} \times \mathtt{m}$ matrix of full column rank; then $D_\perp$ denotes any full row rank $(\mathtt{p} - \mathtt{m}) \times \mathtt{p}$ matrix such that $D_\perp D = 0$. If D is $\mathtt{p} \times \mathtt{m}$ matrix of full row rank, then $D_\perp$ denotes any $\mathtt{m} \times (\mathtt{m} - \mathtt{p})$ full column rank matrix such that $DD_\perp = 0$.

Proposition 10.9 *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ and let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}$ be nonsingular.*

1. *Let $\mathfrak{B}_{DV}(A, B, C, D)$ be a minimal driving variable representation of $\mathfrak{B}$ such that*

   a. $D^\top \Sigma D = I$,

   b. $D^\top \Sigma C = 0$.

   Define $\hat{A} := A$, $\hat{B} := BD^\top\Sigma$, $\hat{C} := -D_\perp C$, $\hat{D} := D_\perp$. Then $\mathfrak{B}_{ON}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is a minimal output nulling representation of $\mathfrak{B}$ and

   c. $\hat{D}\Sigma^{-1}\hat{D}^\top = J$, *where* $J := \text{block diag}(I_{\mathtt{row}(\hat{\mathtt{D}})-\mathtt{q}}, -I_{\mathtt{q}})$ *and* $\mathtt{q} = \sigma_-(\Sigma)$,

   d. $\hat{B}\Sigma^{-1}\hat{D}^\top = 0$.

2. *Assume that $\mathfrak{B}_{ON}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is a minimal output nulling representation of $\mathfrak{B}$ such that the conditions c and d of statement 1 hold. Define $A := \hat{A}$, $B := \hat{B}\hat{D}_\perp$, $C := -\Sigma^{-1}\hat{D}^\top J\hat{C}$, $D := \hat{D}_\perp$. Then $\mathfrak{B}_{DV}(A, B, C, D)$ is a minimal driving variable representation of $\mathfrak{B}$ satisfying the conditions a and b of statement 1.*

Before giving a proof of Proposition 10.9 we need two lemmas. In the following, D^{-R} denotes a right inverse of a full row rank matrix D, i.e. $DD^{-R} = I$, and D^{-L} denotes a left inverse of a full row rank matrix D, i.e. $D^{-L}D = I$. The following result appears as Lemma 5.1.5 in [11].

Lemma 10.10 *Let $\mathfrak{B}_{DV}(A, B, C, D)$ define a minimal driving variable representation of $\mathfrak{B}$. Then $\mathfrak{B}_{ON}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ given by $\hat{A} := A - BD^{-L}C$, $\hat{B} := BD^{-L}$, $\hat{C} = -D_\perp C$, $\hat{D} := D_\perp$ defines a minimal output nulling representation of $\mathfrak{B}$.*

Let $\mathfrak{B}_{ON}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ define a minimal output nulling representation of $\mathfrak{B}$. Then $\mathfrak{B}_{DV}(A, B, C, D)$ given by $A := \hat{A} - \hat{B}\hat{D}^{-R}\hat{C}$, $B := \hat{B}\hat{D}_\perp$, $C := -\hat{D}^{-R}\hat{C}$, $D := \hat{D}_\perp$ defines a minimal driving variable representation of $\mathfrak{B}$.

We now prove the following "inertia Lemma".

Lemma 10.11 *Let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular.*

1. *Let D be such that $D^\top \Sigma D = I$. Then there exists a full row rank constant matrix $\hat{D}$ such that $\hat{D} = D_\perp$ and $\hat{D}\Sigma^{-1}\hat{D}^\top = \text{block diag}(I_{\mathtt{row}(\hat{\mathtt{D}})-\mathtt{q}}, -I_{\mathtt{q}}) =: J$, where $\mathtt{q}$ is number of negative eigenvalues of Σ.*

2. *Let $\hat{D}$ be such that $\hat{D}\Sigma^{-1}\hat{D}^\top = \text{block diag}(I_{\mathtt{row}(\hat{\mathtt{D}})-\mathtt{q}}, -I_{\mathtt{q}}) =: J$, where $\mathtt{q}$ is number of negative eigenvalues of Σ. Then there exists a full column rank constant matrix D such that $D = \hat{D}_\perp$ and $D^\top \Sigma D = I$.*

Proof : (1). Let $\bar{D}$ be a full row rank matrix such that $\bar{D} = D_\perp$. It follows from

$$\begin{bmatrix} \bar{D} \\ D^\top \end{bmatrix} \begin{bmatrix} \bar{D}^\top & \Sigma D \end{bmatrix} = \begin{bmatrix} \bar{D}\bar{D}^\top & \bar{D}\Sigma D \\ 0 & D^\top \Sigma D \end{bmatrix} = \begin{bmatrix} \bar{D}\bar{D}^\top & \bar{D}\Sigma D \\ 0 & I \end{bmatrix}$$

that $\begin{bmatrix} \bar{D}^\top & \Sigma D \end{bmatrix}$ is nonsingular. Moreover,

$$\begin{bmatrix} \bar{D} \\ D^\top \Sigma \end{bmatrix} \Sigma^{-1} \begin{bmatrix} \bar{D}^\top & \Sigma D \end{bmatrix} = \begin{bmatrix} \bar{D}\Sigma^{-1}\bar{D}^\top & 0 \\ 0 & D^\top \Sigma D \end{bmatrix} = \begin{bmatrix} \bar{D}\Sigma^{-1}\bar{D}^\top & 0 \\ 0 & I \end{bmatrix} \tag{26}$$

is then also nonsingular, and this implies that $\bar{D}\Sigma^{-1}\bar{D}^\top$ is nonsingular as well. By Sylvester's inertia law, the identity (26) implies that $\sigma_-\left(\bar{D}\Sigma^{-1}\bar{D}^\top\right) = \sigma_-\left(\Sigma^{-1}\right) = \mathtt{q}$. Consequently, there exists a nonsingular matrix W such that $\bar{D}\Sigma^{-1}\bar{D}^\top = WJW^\top$. Set $\hat{D} := W^{-1}\bar{D}$. It follows that $\hat{D} = D_\perp$ and $\hat{D}\Sigma^{-1}\hat{D}^\top = J$.

(2). Let $\bar{D}$ be a full column rank matrix such that $\bar{D} = \hat{D}_\perp$. It follows from

$$\begin{bmatrix} \hat{D} \\ \bar{D}^\top \end{bmatrix} \begin{bmatrix} \Sigma^{-1}\hat{D}^\top & \bar{D} \end{bmatrix} = \begin{bmatrix} \hat{D}\Sigma^{-1}\hat{D}^\top & 0 \\ \bar{D}\Sigma^{-1}\hat{D}^\top & \bar{D}^\top\bar{D} \end{bmatrix} = \begin{bmatrix} J & 0 \\ \bar{D}\Sigma^{-1}\hat{D}^\top & \bar{D}^\top\bar{D} \end{bmatrix}$$

that $\begin{bmatrix} \Sigma^{-1}\hat{D}^\top & \bar{D} \end{bmatrix}$ is nonsingular. This implies that

$$\begin{bmatrix} \hat{D}\Sigma^{-1} \\ \bar{D}^\top \end{bmatrix} \Sigma \begin{bmatrix} \Sigma^{-1}\hat{D}^\top & \bar{D} \end{bmatrix} = \begin{bmatrix} \hat{D}\Sigma^{-1}\hat{D}^\top & 0 \\ 0 & \bar{D}^\top\Sigma\bar{D} \end{bmatrix} = \begin{bmatrix} J & 0 \\ 0 & \bar{D}^\top\Sigma\bar{D} \end{bmatrix} \tag{27}$$

is nonsingular, and consequently also $\bar{D}^\top\Sigma\bar{D}$. Observe that $\mathtt{q} = \sigma_-(\Sigma)$ equals the number of negative eigenvalues of the right hand side of (27). It follows that $\bar{D}^\top\Sigma\bar{D} > 0$, and that there exists a nonsingular matrix W such that $\bar{D}\Sigma^{-1}\bar{D}^\top = WW^\top$. Set $D := W^{-1}\bar{D}$. Then $D = \hat{D}_\perp$ and $D^\top\Sigma D = I$. □

We now give a proof of Proposition 10.9.

Proof : (1). Use Lemma 10.11 to conclude that there exists $\hat{D}$ such that $\hat{D} = D_\perp$ and $\hat{D}\Sigma^{-1}\hat{D}^\top = J$. Since $D^\top\Sigma D = I$, we can choose $D^{-L} := D^\top\Sigma$. It follows from Lemma 10.10 that $\mathfrak{B}_{ON}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ given by $\hat{A} := A - BD^{-L}C = A - BD^\top\Sigma C = A$, $\hat{B} := BD^{-L} = BD^\top\Sigma$, $\hat{C} := -D_\perp C$, $\hat{D} := D_\perp$ is a minimal output nulling representation of $\mathfrak{B}$. It is straightforward to see that $\mathfrak{B}_{ON}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ satisfies conditions (c) and (d).

(2). Lemma 10.11 implies that there exists D such that $D = \hat{D}_\perp$ and $D\Sigma D^\top = I$. The fact that $\hat{D}\Sigma^{-1}\hat{D}^\top = J$ implies that we can choose $D^{-R} := \Sigma^{-1}\hat{D}^\top J$. It follows from

Lemma 10.10 that $\mathfrak{B}_{DV}(A,B,C,D)$ given by $A := \hat{A} - \hat{B}\hat{D}^{-R}\hat{C} = \hat{A} - \hat{B}\Sigma^{-1}\hat{D}^\top J\hat{C} = \hat{A}$, $B := \hat{B}\hat{D}_\perp$, $C := -\hat{D}^{-R}\hat{C} = -\Sigma^{-1}\hat{D}^\top J\hat{C}$, $D := \hat{D}_\perp$ is a minimal driving variable representation of $\mathfrak{B}$. It is straightforward to check that $\mathfrak{B}_{DV}(A,B,C,D)$ satisfies conditions (a) and (b). □

To conclude this Appendix, we recall how driving variable and output nulling representations of a behavior can be used in order to obtain representations for the orthogonal behavior.

Proposition 10.12 *Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{contr}}$ and let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular. Then*

1. *If $\mathfrak{B}_{DV}(A,B,C,D)$ is a minimal driving variable representation of $\mathfrak{B}$, then $\mathfrak{B}_{ON}(-A^\top, C^\top\Sigma, B^\top, -D^\top\Sigma)$ is a minimal output nulling representation of $\mathfrak{B}^{\perp_\Sigma}$.*
2. *If $\mathfrak{B}_{ON}(A,B,C,D)$ is a minimal output nulling representation of $\mathfrak{B}$, then $\mathfrak{B}_{DV}(-A^\top, C^\top, \Sigma^{-1}B^\top, -\Sigma^{-1}D^\top))$ is a minimal driving variable representation of $\mathfrak{B}^{\perp_\Sigma}$.*

Proof : See section VI.A of [28]. □

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