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Takagi Interpolation Problem As Time Series Modeling

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***Abstract*— In this paper, we consider Takagi interpolation probelm, in which a rational function interpolating given complex pairs with norm constraint have unstable poles, as time series modeling in a behavioral framework. Here we provide a new equivalent condition for the solvability of this problem with respect to the most powerful unfalsified model with special structure.**

***Keywords*— Takagi interpolation problem, most powerful unfalsified model, Pick matrix, behavioral approach**

I. INTRODUCTION

In this paper we consider the following problem:

Problem 1: Let N distinct points λ_i in the open right-half plane be given, together with N vectors v_i, and let

$$J := \begin{bmatrix} I_m & 0 \\ 0 & -I_p \end{bmatrix} \tag{1}$$

Find the smallest $k \in \mathbf{N}$ and $Y \in \mathbf{R}^{p\times p}[\xi]$, $U \in \mathbf{R}^{p\times m}[\xi]$ such that
(a) U, Y are left coprime;
(b) $\begin{bmatrix} U(\lambda_i) & -Y(\lambda_i) \end{bmatrix} v_i = 0$, $1 \leq i \leq N$;
(c) $\|Y^{-1}U\|_\infty < 1$;
(d) Y has k singularities in the right half-plane.

This problem is the vector version of the *Takagi interpolation problem* (in the following abbreviated with *TIP*), first studied in [17] as a generalization of the Nevanlinna-Pick interpolation problem (see [4], [12], [14], [16]), in which the denominator (i.e., Y) interpolating the given data should be a Hurwitz polynomial [1].

The TIP and the closely related Nudel'man problem have been posed and solved in several different ways in the course of time: in the discrete-time case as in the original version [17] (see also [13]), in the context of interpolation with rational matrix functions as in the book [4], with the generalized Beurling-Lax approach introduced in [5]. This paper proposed a proof based on completely different foundations than the one already known. Here, we consider this problem in the framework of exact modeling of time-series pioneered in [19], [20] and further developed in [1], [9], [3], [16]. We show that this problem can be considered as that of computing a special representation of the Most Powerful Unfalsified Model (see [20]), and that the constraints on the location of the roots of the determinant of Y and on the contractivity of $Y^{-1}U$ in the infinity-norm can be satified if one models, besides the data $\{(\lambda_i, \mathcal{V}_i),\ 1 \leq i \leq N\}$, also their "dualized version", a technique introduced in [2] and applied successfully in the context of exact identification in [9], [16].

The paper assumes that the reader is familiar with the behavioral approach to systems and control (see [15] for a thorough introduction) and, at least for some detail of the proofs, with quadratic differential forms (for more information on this subject, see [22]). In order to make the paper as self-contained as possible, the basics of exact modeling and the notion of Most Powerful Unfalsified Model (MPUM) are introduced in section II. The main result of this paper is in section III, where a new proof of the Takagi result is given, and a characterization of all solution to the TIP is established. Some examples are given in section IV. Finally, in section V we discuss some further research topics stemming from the work presented here.

Notation. In this paper we denote the sets of real numbers with $\mathbb{R}$, and the set of complex numbers with $\mathbb{C}$. The space of n dimensional real vectors is denoted by $\mathbb{R}^n$, and the space of $m \times n$ real matrices, by $\mathbb{R}^{m\times n}$. If $A \in \mathbb{R}^{m\times n}$, then $A^T \in \mathbb{R}^{n\times m}$ denotes its transpose. Whenever one of the two dimensions is not specified, a bullet $\bullet$ is used; so that for example, $\mathbb{C}^{\bullet\times \mathtt{n}}$ denotes the set of complex matrices with $\mathtt{n}$ columns and an unspecified number of rows. In order to enhance readability, when dealing with a vector space $\mathbb{R}^\bullet$ whose elements are commonly denoted with w, we use the notation $\mathbb{R}^{\mathtt{w}}$ (note the typewriter font type!); similar considerations hold for matrices representing linear operators on

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[1]In this paper, we refer a polynomial whose determinant has no roots in the closed right half-plane to as "Hurwitz"

such spaces. If $A_i \in \mathbf{R}^{\bullet\times\bullet}$, $i = 1, \dots, r$ have the same number of columns, $\text{col}(A_i)_{i=1,\dots,r}$ denotes the matrix

$$\begin{bmatrix} A_1 \\ \vdots \\ A_r \end{bmatrix}$$

If $H \in \mathbf{C}^{w\times w}$ is an Hermitian matrix, i.e. $H^* := \bar{H}^T = H$,then we define its signature to be the ordered triple $\text{sign}(H) = (\nu_-(H), \nu_0(H), \nu_+(H))$, where $\nu_-(H)$ is the number (counting multiplicities) of negative eigenvalues of H, $\nu_0(H)$ is the multiplicity of the zero eigenvalue of H, and $\nu_+(H)$ is the number (counting multiplicities) of positive eigenvalues of H.

The ring of polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$; the set of two-variable polynomials with real coefficients in the indeterminates ζ and η is denoted by $\mathbb{R}[\zeta, \eta]$. The space of all $\mathtt{n}\times\mathtt{m}$ polynomial matrices in the indeterminate ξ is denoted by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$, and that consisting of all $\mathtt{n}\times\mathtt{m}$ polynomial matrices in the indeterminates ζ and η by $\mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\zeta, \eta]$. Given a matrix $R \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}[\xi]$, we define $R^*(\xi) := R^T(-\xi) \in \mathbb{R}^{\mathtt{m}\times\mathtt{n}}[\xi]$. If $R(\xi)$ has complex coefficients, then $R^*(\xi)$ denotes the matrix obtained from R by substituting $-\xi$ in place of ξ, transposing, and conjugating.

We denote with $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{q}})$ the set of infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{q}}$.

II. The Most Powerful Unfalsified Model

Let $w_i : \mathbf{R} \to \mathbf{R}^w$, $i = 1, \dots, N$, be given functions; for the purposes of this paper, we assume that $w_i \in \mathfrak{C}^\infty(\mathbf{R}, \mathbf{R}^w)$ for all i. Let $\mathcal{M} \subseteq 2^{(\mathbf{C}^w)^{\mathbf{R}}}$ be a class of models, the choice of which reflects the assumptions that the modeler wishes to make on the structure of the phenomenon that produced the w_i's: for example linearity, time-invariance, etc.. In this paper, we choose $\mathcal{M} = \mathcal{L}^w \subseteq 2^{\mathfrak{C}^\infty(\mathbf{R},\mathbf{R}^w)}$, the class of *linear differential behaviors*, i.e. those that are the kernel of a polynomial differential operator with constant coefficients. Formally, a set $\mathfrak{B}$ of trajectories in $\mathfrak{C}^\infty(\mathbf{R}, \mathbf{R}^w)$ is a linear differential behavior if there exists $R \in \mathbf{R}^{\bullet\times w}[\xi]$ such that

$$\mathfrak{B} = \left\{ w \in \mathfrak{C}^\infty(\mathbf{R}, \mathbf{R}^w) \mid R\left(\frac{d}{dt}\right) w = 0 \right\} \tag{2}$$

where if $R(\xi) = \sum_{i=0}^{L} R_i \xi^i$, with $R_i \in \mathbf{R}^{\bullet\times w}$, then

$$R\left(\frac{d}{dt}\right) = R_0 + R_1 \frac{d}{dt} + \dots + R_L \frac{d^L}{dt^L}.$$

Equivalently, $\mathfrak{B}$ is the set of solutions of a system of linear, constant-coefficient, differential equations. The representation

$$R\left(\frac{d}{dt}\right) w = 0 \tag{3}$$

is called a *kernel representation* of the behavior (2).

$\mathfrak{B} \in \mathcal{M}$ is an *unfalsified model* for the *data set* $\{w_i\}_{i=1,\dots,N}$ if $w_i \in \mathfrak{B}$ for $i = 1, \dots, N$. We call $\mathfrak{B}^*$ the *Most Powerful Unfalsified Model* (*MPUM*) in $\mathcal{M}$ for the given data set, if it is unfalsified and moreover

$$[w_i \in \mathfrak{B}', i = 1, \dots, N, \ \mathfrak{B}' \in \mathcal{M}] \Longrightarrow [\mathfrak{B}^* \subseteq \mathfrak{B}']$$

i.e. if it is the smallest behavior in $\mathcal{M}$ containing the data.

The time series that we consider in this paper are *polynomial vector exponential functions*, i.e. they are described by

$$w(t) = \sum_{j=1}^{k} v_j \frac{t^j}{j!} exp_\lambda(t)$$

where $v_j \in \mathbf{C}^w$, $1 \leq j \leq k$, and $\lambda \in \mathbf{C}$. For the case of the model class $\mathcal{L}^w$ and vector-exponential time series, it can be shown that the MPUM always exists and that it is unique (see [3]). Indeed, let w_i, $1 \leq i \leq N$ be vector-exponential time series, and define

$$\mathfrak{B}^* := \text{span}\{w_i\}. \tag{4}$$

Observe that $\mathfrak{B}^*$ contains all trajectories w_i,$1 \leq i \leq N$. On the other hand, any other unfalsified model in $\mathcal{L}^w$ for the data must contain their linear span, and therefore it must contain $\mathfrak{B}^*$; in other words, such unfalsified model will "forbid less" than $\mathfrak{B}^*$, and consequently is "less powerful" than $\mathfrak{B}^*$.

Observe that *the MPUM of a finite set of polynomial vector exponential trajectories is always autonomous*, i.e. it is a finite dimensional subspace of $\mathcal{L}^w$. Equivalently, it can be represented as the kernel of a matrix polynomial differential operator $R(\frac{d}{dt})$, with the property that R is square and nonsingular as a polynomial matrix (see [15]).

Of course, given trajectories $w_i \in \mathfrak{C}^\infty(\mathbf{R}, \mathbf{R}^w)$, is is important to obtain a concrete representation of the MPUM $\mathfrak{B}^*$ defined in (4), for example a kernel representation taking the form of (3). The following iterative algorithm to compute a kernel representation of the MPUM $\mathfrak{B}^*$ for a given data set $\{v_i \exp_{\lambda_i}\}_{i=1,\dots,N} \subseteq \mathfrak{C}^\infty(\mathbf{R}, \mathbf{R}^w)$ can be used (see [21]). Define $R_{-1} := I_q$ and proceed iteratively as follows for $k = 0, 1, \dots, N$. At step k, define the k-th *error trajectory*

$$\varepsilon_k := R_{k-1}\left(\frac{d}{dt}\right) v_k \exp_{\lambda_k} = \underbrace{R_{k-1}(\lambda_k) v_k}_{:=e_k} \exp_{\lambda_k} = e_k \exp_{\lambda_k}$$

Now compute the polynomial matrix corresponding to a kernel representation E_k of the MPUM for ε_k, i.e. $E_k(\frac{d}{dt})\varepsilon_k = 0$; one possible choice for E_k is given in [21]:

$$E_k\left(\frac{d}{dt}\right) = \frac{d}{dt} I_w - \lambda_k \frac{e_k e_k^T}{\|e_k\|^2}$$

and define $R_k := E_k R_{k-1}$. After $N+1$ steps such algorithm produces a $w \times w$ polynomial matrix R_N such that $R_N(\frac{d}{dt}) w_i = 0$ for $1 \leq i \leq N$, and moreover

$$\mathfrak{B}^* = \ker \ R_N\left(\frac{d}{dt}\right).$$

In order to see that this is indeed the case, it is useful to remember that if $R_i \in \mathbb{R}^{g_i \times q}[\xi]$, $i = 1, 2$ are two matrices with the same number of columns, then $\ker \ R_1(\frac{d}{dt}) \subseteq \ker \ R_2\left(\frac{d}{dt}\right)$ if and only if there exists a polynomial $g_2 \times g_1$ matrix F such that $R_2 = F R_1$ (see [15]).

In the next section we will show that the algorithm illustrated above can be adapted to work in the case when the data trajectories need to be "explained" by a model having specific metric- and stability constraints.

III. Main result

We begin by showing that the Takagi interpolation problem can be cast in the framework of exact modeling developed in [19], [20].

We associate to the data $\{(\lambda_i, v_i)\}_{i=1,\dots,N}$ the set of vector-exponential trajectories $v_i \exp_{\lambda_i}$; then it is easy to see that requirement (a) is equivalent to $\ker \begin{bmatrix} U(\lambda_i) & -Y(\lambda_i) \end{bmatrix}$ being controllable. Requirement (b) in the definition of solution to the TIP is equivalent with

$$v_i \exp_{\lambda_i} \in \ker \begin{bmatrix} U(\frac{d}{dt}) & -Y(\frac{d}{dt}) \end{bmatrix}, \ i = 1, \dots, N$$

The metric- and root location aspects of the solution to the TIP (see requirements (a) and (c) above) can be accommodated in the MPUM framework, provided one constructs a special kernel representation for the MPUM associated to the "dualized data", which we now introduce.

Given the interpolation data $\{(\lambda_i, v_i)\}_{i=1,\dots,N}$, we define

$$v_i^\perp := \{v \in \mathbf{C}^{m+p} \mid v^* J v_i = 0\}$$

and the *dual* of $v_i \exp_{\lambda_i}$ as

$$v_i^\perp \exp_{-\bar{\lambda}_i} := \{v \exp_{-\bar{\lambda}_i} \mid v \in v_i^\perp\}$$

We also define the *dualized data* $\mathcal{D}$ as

$$\mathcal{D} := \cup_{i=1,\dots,N}\{v_i \exp_{\lambda_i}, v_i^\perp \exp_{-\bar{\lambda}_i}\} \tag{5}$$

Finally, we define the notion of *Pick matrix associated with the data* $\{(\lambda_i, v_i)\}_{1 \leq i \leq N}$. This is the Hermitian block-matrix

$$T_{\{(\lambda_i, v_i)\}_{1 \leq i \leq N}} := \left[\frac{v_i^* J v_j}{\bar{\lambda}_i + \lambda_j}\right]_{1 \leq i,j \leq n} \tag{6}$$

Now consider the following procedure:

Algorithm T
• Define $R_0 := I_{p+m}$;
• For $i = 1, \dots, N$
• $\quad v_i' := R_{i-1}(\lambda_i) v_i$;
• $\quad R_i(\xi) := \left[(\xi + \bar{\lambda}_i) I_{p+m} - v_i' T^{-1}_{\{(\lambda_i, v_i')\}} (v_i')^* J\right] R_{i-1}(\xi)$;
• end;

We now show that this algorithm produces a representation of the $MPUM$ for the dualized data $\mathcal{D}$ and we relate the properties of this representation to those of the Pick matrix of the data.

Theorem 2: Assume that the Pick matrix (6) is invertible. Then the following statements are equivalent:

[1.] The Pick matrix $T_{\{(\lambda_i,v_i)\}_{1\leq i\leq N}} := \left[\frac{v_i^* J v_j}{\lambda_i+\lambda_j}\right]_{i,j=1,\dots,n}$ has k negative eigenvalues;

[2.] Algorithm T produces a kernel representation of the MPUM for the dualized data set $\mathcal{D}$ defined in (5) induced by a matrix of the form

$$R := \begin{bmatrix} -D^* & N^* \\ Q & -P \end{bmatrix} \tag{7}$$

where $D \in \mathbf{R}^{m\times m}[\xi]$, $N \in \mathbf{R}^{m\times p}[\xi]$, $Q \in \mathbf{R}^{p\times m}[\xi]$, $P \in \mathbf{R}^{p\times p}[\xi]$ satisfy the following properties:

(a) D, P are nonsingular;

(b) $QD - PN = 0$;

(c) $\det(P)$ has k roots in $\mathbf{C}_+$;

(d) $RJR^* = R^*JR = pp^*J$ with $p(\xi) = \Pi_{i=1}^N(\xi + \bar{\lambda}_i)$;

(e) $\|P^{-1}Q\|_\infty < 1$;

(f) $\|N^\sim P^{-1}\|_\infty < 1$.

Proof: Let us first prove $(1) \Rightarrow (2)$. We will prove this by induction on the number N of vectors v_i.

For $N = 1$, partition v_1 as $v_1 = \text{col}(v_{11}, v_{12})$ with $v_{11} \in \mathbf{C}^m$ and $v_{12} \in \mathbf{C}^p$, and consider the model $\mathcal{B}_1$ represented in kernel form by

$$R_1(\xi) := (\xi + \bar{\lambda}_1)I_{p+m} - v_1 T_{\{v_1\}}^{-1} v_1^* J \tag{8}$$

Note that

$$\begin{aligned}(\frac{d}{dt} + \bar{\lambda}_1)v_1 exp_{\lambda_1} - v_1 T_{\{v_1\}}^{-1} v_1^* J v_1 \exp_{\lambda_1} = \\ (\lambda_1 + \bar{\lambda}_1)v_1 exp_{\lambda_1} - v_1(\frac{v_1^* J v_1}{\lambda_1 + \bar{\lambda}_1})^{-1}(v_1^* J v_1) exp_{\lambda_1} = 0.\end{aligned}$$

Note also that if $V_1^\perp$ is a $(m+p)\times(m+p-1)$ matrix such that $\text{Im}(V_1^\perp) = v_1^\perp$, there holds

$$(\frac{d}{dt} + \bar{\lambda}_1)V_1^\perp exp_{-\bar{\lambda}_1} - v_1 T_{\{v_1\}}^{-1} v_1^* J V_1^\perp exp_{-\bar{\lambda}_1} = 0.$$

Therefore, $v_1 exp_{\lambda_1} \in \mathfrak{B}_1$ and $v_1^\perp exp_{-\bar{\lambda}_1} \in \mathfrak{B}_1$. In order to prove that $\mathcal{B}_1$ is the MPUM, observe that the determinant of (8) has degree $p+m$, and therefore $\mathcal{B}_1$ contains $p+m$ independent trajectories. Since $\dim(v_1 exp_{\lambda_1} \oplus v_1^\perp exp_{-\bar{\lambda}_1}) = m+p$, the claim is proved.

In order to prove that (8) satisfies $(2a) - (2f)$, partition it according to the partition of $v_1 = \text{col}(v_{11}, v_{12})$ as

$$\begin{aligned} R_1(\xi) &:= \begin{pmatrix} -D_1^*(\xi) & N_1^*(\xi) \\ Q_1(\xi) & -P_1(\xi) \end{pmatrix} \\ &:= \begin{pmatrix} (\xi + \bar{\lambda}_1)I_m - v_{11} T_{\{v_1\}}^{-1} v_{11}^* & v_{11} T_{\{v_1\}}^{-1} v_{12}^* \\ -v_{12} T_{\{v_1\}}^{-1} v_{11}^* & (\xi + \bar{\lambda}_1)I_p + v_{12} T_{\{v_1\}}^{-1} v_{12}^* \end{pmatrix} \end{aligned} \tag{9}$$

Observe that D_1 and P_1 in (9) are row proper, and consequently nonsingular. $Q_1 D_1 - P_1 N_1 = 0$ follows from straightforward manipulations.

We now prove the claim $(2c)$ on the number of zeros of $\det(P_1)$ in the right half-plane. In order to do this, we make some preliminary remark.

Consider that without loss of generality, we can multiply $v_1 = \text{col}(v_{11}, v_{12})$ by a suitable constant in order to have $v_{11}^* v_{11} - v_{12}^* v_{12} = \pm 1$. Then

$$\begin{aligned} -P_1(\xi) &= (\xi + \bar{\lambda}_1)I_p + v_{12}(\frac{v_{11}^* v_{11} - v_{12}^* v_{12}}{\lambda_1 + \bar{\lambda}_1})^{-1} v_{12}^* \\ &= (\xi + \bar{\lambda}_1)I_p + (\lambda_1 + \bar{\lambda}_1)v_{12}(\pm 1)v_{12}^*. \end{aligned}$$

It is easy to see that given the structure of P_1, $v \in \mathbf{C}^p$ and $\lambda \in \mathbf{C}$ are such that $P_1(\lambda)v = 0$ if and only if v is an eigenvector of $v_{12}(\pm 1)v_{12}^*$. Now since $v_{12}(\pm 1)v_{12}^*$ is a dyad, it follows that $v \in \mathbf{C}^p$ and $\lambda \in \mathbf{C}$ are such that $P_1(\lambda)v = 0$ if and only if v is either orthogonal to, or proportional to v_{12}. In the first case, the $p-1$ vectors orthogonal to v_{12} are easily seen to correspond to $-\bar{\lambda}_1$ as a singularity of $P_1(\xi)$. As for the other eigenvector of $v_{12}v_{12}^*$, observe that without loss of generality it can be taken to be equal to v_{12}. Now two cases are possible. In the first one, $v_{12} = 0$, in which case the Pick matrix is positive; in this case $\det(P_1) = (\xi + \lambda_1)^p$, and the claim is proved. If $v_{12} \neq 0$, then let μ be the singularity of $\det(P_1)$ corresponding to it, and write

$$\begin{aligned} 0 = P_1(\mu)v_{12} &= (\mu + \bar{\lambda}_1)v_{12} + (\lambda_1 + \bar{\lambda}_1)v_{12}(\pm 1)v_{12}^* v_{12} \\ &= (\mu + \bar{\lambda}_1)v_{12} + (\lambda_1 + \bar{\lambda}_1)(\pm 1)\|v_{12}\|^2 v_{12}. \end{aligned}$$

Now assume that the Pick matrix equals 1; then

$$\mu = -\bar{\lambda}_1 - (\lambda_1 + \bar{\lambda}_1)\|v_{12}\|^2$$

which has negative real part, as we were to prove. If the Pick matrix equals -1, then

$$\mu = -\bar{\lambda}_1 + (\lambda_1 + \bar{\lambda}_1)\|v_{12}\|^2$$

and consequently

$$\text{Re}(\mu) = -\text{Re}(\bar{\lambda}_1) + 2\text{Re}(\bar{\lambda}_1))\|v_{12}\|^2.$$

Since $v_{11}^* v_{11} - v_{12}^* v_{12} = -1$, it follows $\|v_{12}\|^2 = \|v_{11}\|^2 + 1$, and consequently

$$\text{Re}(\mu) = \text{Re}(\bar{\lambda}_1)[-1 + 2(\|v_{11}\|^2 + 1)] = \text{Re}(\bar{\lambda}_1)(1 + 2\|v_{11}\|^2) > 0$$

as was to be proved. This concludes the proof of statement $(2c)$.

We now prove $(2d)$. Observe that

$$\begin{aligned} R_1 J R_1^* = [(\xi + \bar{\lambda}_1)I_{p+m} - v_1 T_{\{v_1\}}^{-1} v_1^* J]J[(-\xi + \lambda_1)I_{p+m} - J v_1 T_{\{v_1\}}^{-1} v_1^*] = \\ (\xi + \bar{\lambda}_1)(-\xi + \lambda_1)J - (\lambda_1 + \bar{\lambda}_1)v_1 T_{\{v_1\}}^{-1} v_1^* + (\lambda_1 + \bar{\lambda}_1)v_1 T_{\{v_1\}}^{-1} v_1^* = \\ (\xi + \bar{\lambda}_1)(-\xi + \lambda_1)J. \end{aligned} \tag{10}$$

The second equality of $(2d)$ can be proved analogously.

In order to prove $(2e)$, observe that from $(2a)$ and $(2b)$ follows that $P_1^{-1}Q_1 = N_1 D_1^{-1}$. Consequently, in order to prove $\|P_1^{-1}Q_1\|_\infty < 1$, it will suffice to prove that $D_1^*(i\omega)D_1(i\omega) - N_1^*(i\omega)N_1(i\omega) > 0$ for every $\omega \in \mathbf{R}$. Note that $D_1^* D_1 - N_1^* N_1$ is the $(1,1)$-block of $R_1 J R_1^*$ and, by property $(2e)$, on the imaginary axis it equals

$$(-i\omega + \bar{\lambda}_1)(i\omega + \lambda_1)I_m,$$

which is positive definite for every $\omega \in \mathbf{R}$. This implies $\det(D(i\omega)) \neq 0 \ \forall \ \omega \in \mathbf{R}$ and consequently $\|P_1^{-1}Q_1\|_\infty < 1$. In order to prove claim $(2f)$, note that $N_1(i\omega)N_1^*(i\omega) - P_1^*(i\omega)P_1(i\omega)$ is the $(2,2)$ block of $R_1(i\omega)^* J R_1(i\omega)$ and that by $(2e)$ this block is negative definite for all $\omega \in \mathbf{R}$.

This concludes the proof of $(2a) - (2f)$ for the representation (8) of the MPUM for $N = 1$.

Let us now assume that the claim $(1) \Rightarrow (2)$ holds for a number j of points to interpolate, $1 \leq j \leq N - 1$. In order to prove the claim for N points we proceed as follows. We have shown above that there exists a representation R_1 of the MPUM for $v_1 exp_{\lambda_1} \oplus v_1^\perp exp_{-\bar{\lambda}_1}$ that satisfies $(2a) - (2f)$. We will first find a congruence transformation on the Pick matrix of the data which will make it easier to apply the inductive assumption. Then we will apply the inductive assumption and conclude that a representation R' of the MPUM for the errors satisfying $(2a) - (2f)$ exists. Combining the representations of the two MPUMs as $R'R_1$ we obtain a representation of the MPUM for $\mathcal{D}$, and we will show that it satisfies $(2a) - (2f)$.

Assume now that a representation (9) of the MPUM for $v_1 exp_{\lambda_1} \oplus v_1^\perp exp_{-\bar{\lambda}_1}$ has been computed, satisfying $(2a) - (2f)$. The error vectors associated to this model are

$$v_i' := (\lambda_i + \bar{\lambda}_1)v_i - v_1 T_1^{-1} v_1^* J v_i, \qquad 2 \leq i \leq N.$$

We now investigate the relationship of the signature of the Pick matrix $T'_{2 \leq i \leq N} := T_{\{(\lambda_i, v_i')\}}$ associated with (λ_i, v_i'), $2 \leq i \leq N$, with the signature of the matrix $T_{1 \leq i \leq N}$. Note first that for $2 \leq i, j \leq N$, the $(i-1, j-1)$-th block element of $T'_{2 \leq i \leq N}$ is

$$\frac{v_i'^* J v_j'}{\bar{\lambda}_i + \lambda_j} = \frac{1}{\lambda_j + \bar{\lambda}_i}[(\bar{\lambda}_i + \lambda_1)v_i^* - v_i^* J v_1 T_1^{-1} v_1^*]J[(\lambda_j + \bar{\lambda}_1)v_j - v_1 T_1^{-1} v_1^* J v_j]. \tag{11}$$

Easy computations show that (11) equals

$$\frac{(\bar{\lambda}_i + \lambda_1)(\lambda_j + \bar{\lambda}_1)}{\bar{\lambda}_i + \lambda_j} v_i^* J v_j - v_i^* J v_1 T_1^{-1} v_1^* J v_j. \tag{12}$$

Partition now $T_{1 \leq i \leq N}$ as

$$\begin{pmatrix} T_1 & \bar{b}^T \\ b & T_{2 \leq i \leq N} \end{pmatrix}$$

with $b := col(\frac{v_i^* J v_1}{\bar{\lambda}_i+\lambda_1})_{2\leq i\leq N}$, and define $\Delta := diag((\bar{\lambda}_i+\lambda_1))_{2\leq i\leq N}$. Observe that

$$\begin{pmatrix} 1 & 0 \\ -\Delta b T_1^{-1} & \Delta \end{pmatrix} T_{1\leq i\leq N} \begin{pmatrix} 1 & -T_1^{-1}\bar{b}^T\bar{\Delta} \\ 0 & \bar{\Delta} \end{pmatrix} = \begin{pmatrix} T_1 & 0 \\ 0 & \Delta T_{2\leq i\leq N}\bar{\Delta} - \Delta b T_1^{-1}\bar{b}^T\bar{\Delta} \end{pmatrix} \tag{13}$$

We prove now that the $(2,2)$ block of (13) coincides with $T'_{2\leq i\leq N}$. In fact, the (i,j)-th block of $\Delta T_{2\leq i\leq N}\bar{\Delta} - \Delta b T_1^{-1}\bar{b}^T\bar{\Delta}$ equals

$$\frac{(\bar{\lambda}_i+\lambda_1)(\lambda_j+\bar{\lambda}_1)}{\bar{\lambda}_i+\lambda_j} v_i^* J v_j - v_i^* J v_1 T_1^{-1} v_1^* J v_j,$$

and, since the (i,j)-th block of $T_{2\leq i\leq N}$ is given by (12), this proves the claim.

Now observe that

$$\text{sign}(T_{1\leq i\leq N}) = \text{sign}(T'_{2\leq i\leq N}) + \text{sign}(T_1)$$

Let $\text{sign}(T_{1\leq i\leq N}) = (\nu_-, 0, \nu_+)$, and observe that

$$\begin{aligned} &\text{if sign}(T_1) = (0,0,1) \text{ then sign}(T'_{2\leq i\leq N}) = (\nu_-, 0, \nu_+ - 1); \\ &\text{if sign}(T_1) = (1,0,0) \text{ then sign}(T'_{2\leq i\leq N}) = (\nu_- - 1, 0, \nu_+) \end{aligned}$$

By inductive assumption we conclude that there exists a kernel representation $R' \in \mathbf{R}^{2\times 2}[\xi]$ of the MPUM for $\{v'_i \exp_{\lambda_i}, v_i'^{\perp} \exp_{-\bar{\lambda}_i}\}_{2\leq i\leq N}$ of the form

$$R' = \begin{pmatrix} -D'^* & N'^* \\ Q' & -P' \end{pmatrix}$$

satisfying the properties $(2a)-(2f)$ of the Theorem, and in particular $(2c)$, i.e. P' has $\nu_- - 1$ (respectively ν_-) roots in $\mathbf{C}_+$ if $\text{sign}(T_1) = (1,0,0)$ (respectively $\text{sign}(T_1) = (0,0,1)$).

It is easily verified that the MPUM for $\mathcal{D}$ is represented by

$$\begin{pmatrix} -D^* & N^* \\ Q & -P \end{pmatrix} := \begin{pmatrix} -D'^* & N'^* \\ Q' & -P' \end{pmatrix} \begin{pmatrix} -D_1^* & N_1^* \\ Q_1 & -P_1 \end{pmatrix} \tag{14}$$

We now show that (14) satisfies $(2a)-(2f)$.

In order to prove $(2a)$, we first show that P is nonsingular. From (14) it follows that

$$-P = Q'N_1^* + P'P_1 \tag{15}$$

By inductive assumption, P' and P_1 are nonsingular, and consequently

$$-(P')^{-1}PP_1^{-1} = (P')^{-1}Q' \cdot N_1^*P_1^{-1} + I_p$$

Conclude from the inductive assumption that $\|(P')^{-1}Q'\|_\infty < 1$ and that $\|N_1^*P_1^{-1}\|_\infty < 1$. It follows that $(P')^{-1}PP_1^{-1}$ is nonsingular on the imaginary axis, and consequently P is also nonsingular on the imaginary axis, and *a fortiori* nonsingular in $\mathbf{R}^{p\times p}[\xi]$.

We now show that D is nonsingular. Note from (14) that $D = D_1D' + Q_1^*N'$. Observe from the formula (9) that $Q_1 = -N_1$, and consequently $D = D_1D' - N_1^*N'$. Now use the contractivity of $N_1D_1^{-1}$ and of $N'(D')^{-1}$ to show in a manner analogous to that used for the proof of the nonsingularity of P, that D is also nonsingular. This concludes the proof of $(2a)$.

Claim $(2b)$ can be proved by a straightforward computation, using equation (14) and the inductive assumption.

We now prove $(2c)$, the claim regarding the number of roots of $\det(P)$ in $\mathbf{C}_+$. Conclude from (15) that

$$-P^{-1} = P_1^{-1}((P')^{-1}Q' \cdot N_1^*P_1^{-1} + I_p)^{-1}(P')^{-1}$$

and consequently that

$$-\frac{1}{\det(P)} = \frac{1}{\det(P_1)} \frac{1}{\det((P')^{-1}Q' \cdot N_1^*P_1^{-1} + I_p)} \frac{1}{\det(P')} \tag{16}$$

Now recall that the winding number $\text{wno}(\cdot)$ of a function f defined on the imaginary axis and admitting a meromorphic continuation in $\mathbf{C}_+$ satisfies

$$\text{wno}(f) = (\#\text{ zeros of } f \text{ in } \mathbf{C}_+) - (\#\text{ poles of } f \text{ in } \mathbf{C}_+)$$

Observe that

$$\text{wno}\left(\frac{1}{\det(\alpha(P')^{-1}Q' \cdot N_1^*P_1^{-1} + I_p)}\right)$$

for $0 \leq \alpha \leq 1$ is a continuous function of α taking integer values, and consequently its value is independent of α. This and the contractivity of $(P')^{-1}Q'$, $N_1^* P_1^{-1}$ imply that

$$\text{wno}\left(\frac{1}{\det(\alpha(P')^{-1}Q' \cdot N_1^* P_1^{-1} + I_p)}\right) = \text{wno}(\det(I_p)) = 0$$

Now apply the logarithmic property of wno$(\cdot)$ to both sides of (16) and obtain

$$\text{wno}\left(-\frac{1}{\det(P)}\right) = \text{wno}\left(\frac{1}{\det(P_1)}\right) + \text{wno}\left(\frac{1}{\det(P')}\right) \tag{17}$$

Now if sign$(T_1) = (0,0,1)$ it follows that P_1 is Hurwitz, and consequently wno $\left(\frac{1}{\det(P_1)}\right) = 0$. Moreover, in this case sign$(T'_{2\leq i\leq N}) = (\nu_-, 0, \nu_+ - 1)$ and by the inductive assumption we conclude that wno $\left(\frac{1}{\det(P')}\right) = -\nu_-$. In this case $\det(P)$ has exactly ν_- unstable roots, as was to be proved. If sign$(T_1) = (1,0,0)$ then wno $\left(\frac{1}{\det(P_1)}\right) = -1$, and by the inductive assumption wno $\left(\frac{1}{\det(P')}\right) = -\nu_- + 1$. It follows that wno $\left(-\frac{1}{\det(P)}\right) = -\nu_-$ and claim $(2c)$ is proved.

Claim $(2d)$ follows easily from (14) and the inductive assumption.

In order to prove $(2e)$, we show that $P^*P - Q^*Q > 0$ on the imaginary axis. Note that $P^*P - Q^*Q$ is the $(2,2)$ block-element of $R^{*\prime} R_1^* J R_1 R'$. Using $(2d)$, this element equals $-p(i\omega)p^*(i\omega) < 0$ for all $\omega \in \mathbf{R}$. This implies $\|P^{-1}Q\|_\infty < 1$. The proof of $(2f)$ follows a similar argument, since $NN^* - P^*P$ is the $(2,2)$ block-element of $R_1^* R'^* J R' R_1$.

This concludes the proof of $(1) \Rightarrow (2)$.

In order to prove the converse implication, we proceed by induction on the number N of points to be interpolated, showing that the model produced by the procedure is such that the number of unstable roots of $\det(P)$ equals the number of negative eigenvalues of the Pick matrix T associated with the data.

For $N = 1$, assume without loss of generality that $v_1 = \text{col}(v_{11}, v_{12})$ is such that $v_{11}^* v_{11} - v_{12}^* v_{12} = \pm 1$. Consider the model (8) and observe that

$$\begin{aligned} P_1(\xi) &= (\xi + \bar{\lambda}_1) I_p + v_{12}\left(\frac{v_{11}^* v_{11} - v_{12}^* v_{12}}{\lambda_1 + \bar{\lambda}_1}\right)^{-1} v_{12}^* \\ &= (\xi + \bar{\lambda}_1) I_p + (\lambda_1 + \bar{\lambda}_1) v_{12}(\pm 1) v_{12}^* \end{aligned}$$

is Hurwitz if and only if the Pick matrix $T_1 = \frac{\pm 1}{\lambda_1 + \bar{\lambda}_1}$ is positive definite. The claim is thus proved for $N = 1$. We now assume the claim is true for all $1 \leq j \leq N-1$ and we prove it for $j = N$.

In order to do this, consider first that the special representation R for the model for N trajectories is obtained from the model R_1 for $v_1 \exp_{\lambda_1}$ and the model R' for the error trajectories $R_1(\lambda_i) v_i \exp_{\lambda_i}$, $2 \leq i \leq N$ as $R = R' R_1$. Observe that by inductive assumption, the Pick matrix of the error trajectories has as many negative eigenvalues as the number of right half-plane singularities of the $(2,2)$ block-element of R'.

We have shown in equation (17) that the number of right half-plane singularities of P, the $(2,2)$ block-element of R, equals the number of such singularities of the corresponding block-element of R' plus the number of such singularities of P_1. Now observe that $T_{\{v_i\}_{1\leq i\leq N}}$, the Pick matrix of the data, is congruent to the matrix on the right-hand side of (13). The signature of this block-diagonal matrix equals the sum of the signature of the Pick matrix $T_{\{v_1\}}$, and that of the Pick matrix $T'_{2\leq i\leq N}$ associated to the error trajectories. This completes the proof of $(2) \Longrightarrow (1)$.

The special kernel representation of the MPUM for $\mathcal{D}$ described in Theorem 2 allows us to characterize the solutions of the TIP as follows.

Theorem 3: Assume that the Hermitian matrix $T_{\{(\lambda_i, v_i)\}_{1\leq i\leq N}} := \left[\frac{v_i^* J v_j}{\bar{\lambda}_i + \lambda_j}\right]_{i,j=1,\ldots,n}$ is invertible and has k negative eigenvalues, and let (7) be the representation of the MPUM for $\mathcal{D}$ computed with Algorithm T.

Let $U \in \mathbf{R}^{p\times m}[\xi]$, $Y \in \mathbf{R}^{p\times p}[\xi]$ be left coprime. Then $\begin{bmatrix} U & -Y \end{bmatrix} \in \mathbf{R}^{p\times(p+m)}[\xi]$ is a solution to the TIP with $\det(Y)$ having k roots in $\mathbf{C}_+$ if and only if there exist $\Pi, \Phi, F \in \mathbf{R}^{\bullet\times\bullet}[\xi]$, with Φ, F Hurwitz, and $\|\Phi^{-1}\Pi\|_\infty < 1$, such that

$$F \begin{bmatrix} U & -Y \end{bmatrix} = \begin{bmatrix} \Pi & -\Phi \end{bmatrix} \begin{bmatrix} -D^* & N^* \\ Q & -P \end{bmatrix} \tag{18}$$

Proof: We first prove sufficiency. Let $U \in \mathbf{R}^{p\times m}[\xi]$, $Y \in \mathbf{R}^{p\times p}[\xi]$ be given such that they are left coprime, and (18) holds for some $\Pi \in \mathbf{R}^{p\times m}[\xi]$, $\Phi \in \mathbf{R}^{p\times p}[\xi]$, $F \in \mathbf{R}^{p\times p}[\xi]$ such that $\|\Phi^{-1}\Pi\|_\infty < 1$ and Φ, F are Hurwitz. Consider that $F \begin{bmatrix} U & -Y \end{bmatrix}$ is a left multiple of a kernel representation of the MPUM for $\mathcal{D}$, and consequently it is unfalsified on $\mathcal{D}$. It follows that

$$F(\lambda_i) \begin{bmatrix} U(\lambda_i) & -Y(\lambda_i) \end{bmatrix} v_i = 0,$$

$1 \leq i \leq N$. Conclude from the fact that F is Hurwitz that this implies $\begin{bmatrix} U(\lambda_i) & -Y(\lambda_i) \end{bmatrix} v_i = 0$, $1 \leq i \leq N$, so that $\begin{bmatrix} U & -Y \end{bmatrix}$ is an unfalsified model for $v_i \exp_{\lambda_i}$, $1 \leq i \leq N$. The fact that $\|Y^{-1}U\|_\infty < 1$ follows from the

J-unitariness of R and from the assumption that $\|\Phi^{-1}\Pi\|_\infty < 1$. Finally, the claim on the number of roots of Y in $\mathbf{C}_+$ can be justified observing that

$$-FY = \Pi N^* + \Phi P = \Phi(\Phi^{-1}\Pi N^* P^{-1} + I_p)P$$

or equivalently

$$-Y^{-1}F^{-1} = P^{-1}(\Phi^{-1}\Pi N^* P^{-1} + I_p)^{-1}\Phi^{-1}$$

and consequently

$$-\frac{1}{\det(F)}\frac{1}{\det(Y)} = \frac{1}{\det(\Phi)}\frac{1}{\det(P)}\frac{1}{\det(\Phi^{-1}\Pi N^* P^{-1} + I_p)}$$

It follows from the fact that Φ and F are Hurwitz that $\text{wno}(\frac{1}{\det(F)}) = 0 = \text{wno}(\frac{1}{\det(\Phi)})$. It follows from the fact that $\|\Phi^{-1}\Pi\|_\infty < 1$ and that $\|P^{-1}N^*\|_\infty < 1$, that $\text{wno}(\frac{1}{\det(\Phi^{-1}\Pi N^* P^{-1}+I_p)}) = 0$. Conclude that the number of roots of $\det(Y)$ in $\mathbf{C}_+$ equals the number of roots of $\det(P)$ in $\mathbf{C}_+$; the latter is exactly k, the number of negative eigenvalues of the Pick matrix of the data. Sufficiency is thus proved.

In order to prove necessity, we proceed as follows. Let $U \in \mathbf{R}^{p\times m}[\xi]$, $Y \in \mathbf{R}^{p\times p}[\xi]$ constitute a solution of the TIP. Choose $F \in \mathbf{R}^{p\times p}[\xi]$ so that $F\begin{bmatrix} U & -Y \end{bmatrix}$ also models the trajectories $v_i^\perp \exp_{-\bar{\lambda}_i}$, $1 \leq i \leq N$ besides the trajectories $v_i \exp_{\lambda_i}$, $1 \leq i \leq N$. Observe that F can be chosen to be Hurwitz, since $\begin{bmatrix} U & -Y \end{bmatrix}$ already models $v_i \exp_{\lambda_i}$, $1 \leq i \leq N$. Conclude from the fact that $F\begin{bmatrix} U & -Y \end{bmatrix}$ models $\mathcal{D}$ and from the fact that a representation of the MPUM for $\mathcal{D}$ is given, that there exist $\Pi, \Phi \in \mathbf{R}^{\bullet\times\bullet}[\xi]$ such that (18) holds. We now prove the claim regarding the contractivity of $\Phi^{-1}\Pi$ and the Hurwitzianity of Φ.

Contractivity follows easily from the J-unitariness of R and from the contractivity of $Y^{-1}U$, since

$$\begin{aligned}&F^T(-i\omega)(U^T(-i\omega)U(i\omega) - Y^T(-i\omega)Y(i\omega))F(i\omega)\\&= (\Pi^T(-i\omega)\Pi(i\omega) - \Phi^T(-i\omega)\Phi(i\omega))\Pi_{i=1}^N(-i\omega + \bar{\lambda}_i)(i\omega - \lambda_i)\end{aligned}$$

is negative definite for all $\omega \in \mathbf{R}$ if and only if $\Pi^T(-i\omega)\Pi(i\omega) - \Phi^T(-i\omega)\Phi(i\omega) < 0$. The claim on the Hurwitzianity of Φ follows from

$$-\frac{1}{\det(F)}\frac{1}{\det(Y)} = \frac{1}{\det(\Phi)}\frac{1}{\det(P)}\frac{1}{\det(\Phi^{-1}\Pi N^* P^{-1} + I_p)}$$

and consequently

$$\begin{aligned}\underbrace{\text{wno}(\frac{1}{\det(F)})}_{=0} + \underbrace{\text{wno}(\frac{1}{\det(Y)})}_{=-k} = \text{wno}(\frac{1}{\det(\Phi)}) + \underbrace{\text{wno}(\frac{1}{\det(P)})}_{=-k}\\ +\underbrace{\text{wno}(\frac{1}{\det(\Phi^{-1}\Pi N^* P^{-1} + I_p)})}_{=0}\end{aligned}$$

The proof of the Theorem is thus complete. ■

The following conclusion can be drawn easily from the results of Theorem 2 and Theorem 3.

Corollary 4: The smallest k for which the Takagi interpolation problem has a solution is the number of negative eigenvalues of the Pick matrix $T_{\{(\lambda_i,v_i)\}_{1\leq i\leq N}}$.

IV. Examples

Example 5: Consider the (frequency, vector) pairs

$$(\lambda_1, v_1) = \left(4, \begin{bmatrix} 6 \\ -7 \end{bmatrix}\right) \quad (\lambda_2, v_2) = \left(5, \begin{bmatrix} 12 \\ -9 \end{bmatrix}\right) \quad (\lambda_3, v_3) = \left(5, \begin{bmatrix} 20 \\ -11 \end{bmatrix}\right)$$

corresponding to the Pick matrix

$$\begin{bmatrix} -\frac{13}{8} & 1 & \frac{43}{10} \\ 1 & \frac{63}{10} & \frac{141}{11} \\ \frac{43}{10} & \frac{141}{11} & \frac{93}{4} \end{bmatrix}$$

whose eigenvalues are 30.7039, -2.78503, 0.00614669. We conclude that there exists a solution of the TIP with 1 unstable pole.

The model for the first point is

$$R_1(\xi) := \begin{bmatrix} \frac{340}{13} + \xi & \frac{336}{13} \\ -\frac{336}{13} & \xi - \frac{340}{13} \end{bmatrix}$$

As was to be expected from the fact that the Pick matrix corresponding to (λ_1, v_1) is negative definite, the $(2,2)$ entry has a singularity in $\mathbf{C}_+$.

The vector corresponding to the first error trajectory is $v_2' := R_1 v_2$, with corresponding kernel representation

$$R_2'(\xi) := \begin{bmatrix} -\frac{357725}{11687} + \xi & -\frac{352920}{11687} \\ \frac{352920}{11687} & \frac{357725}{11687} + \xi \end{bmatrix}$$

Conclude that a kernel representation of the MPUM for the first two trajectories and their duals is

$$R_2(\xi) = R_2'(\xi) R_1(\xi) = \begin{bmatrix} -\frac{18020}{899} - \frac{4005}{899}\xi + \xi^2 & -\frac{24}{899}(50 + 163\xi) \\ \frac{24}{899}(-50 + 163\xi) & -\frac{18020}{899} + \frac{4005}{899}\xi + \xi^2 \end{bmatrix}$$

Observe that the $(2,2)$ entry of R_2 has a positive and a negative real root, as was to be expected from the fact that the 2×2 principal submatrix of the Pick matrix has one negative and one positive eigenvalue.

The third error trajectory is associated with the vector $v_3' := \begin{bmatrix} \frac{77672}{899} \\ \frac{23326}{899} \end{bmatrix}$. It can be shown that a kernel representation corresponding to this vector is induced by

$$R_3'(\xi) := \begin{bmatrix} -\frac{3288520930}{457403109} + \xi & \frac{1811777072}{457403109} \\ -\frac{1811777072}{457403109} & \frac{3288520930}{457403109} + \xi \end{bmatrix}$$

Conclude that a kernel representation of the MPUM for the given data is $R_3'(\xi) R_2(\xi)$, given by

$$\begin{bmatrix} \frac{70632200+14867334\xi-5924615\xi^2+508791\xi^3}{508791} & \frac{-40(887836-605418\xi+4967\xi^2)}{508791} \\ \frac{40(887836-605418\xi+4967\xi^2)}{508791} & \frac{70632200+14867334\xi-5924615\xi^2+508791\xi^3}{508791} \end{bmatrix}$$

Observe that the roots of the $(2,2)$ element of R_3 are 2.27811, $-6.9613 \pm 3.53247i$.

Example 6: We solve a problem with $m = 1$ and $p = 2$. Consider the (frequency, vector) pairs

$$(\lambda_1, v_1) = \left(4, \begin{bmatrix} 6 \\ -7 \\ -11 \end{bmatrix}\right) \quad (\lambda_2, v_2) = \left(5, \begin{bmatrix} 12 \\ -9 \\ -14 \end{bmatrix}\right) \quad (\lambda_3, v_3) = \left(5, \begin{bmatrix} 20 \\ -11 \\ -17 \end{bmatrix}\right)$$

which correspond to the Pick matrix

$$\begin{bmatrix} -\frac{67}{4} & -\frac{145}{9} & -\frac{72}{5} \\ -\frac{145}{9} & -\frac{133}{10} & -\frac{97}{11} \\ -\frac{72}{5} & -\frac{97}{11} & -\frac{5}{6} \end{bmatrix}$$

This matrix has eigenvalues -38.5789, 7.69355, 0.0020285. Consequently, we expect a representation of the MPUM with a $(1,1)$ block element having one singularity in C_+.

The kernel representation corresponding to the first trajectory and its dual is

$$\begin{bmatrix} \frac{412}{67} + \xi & \frac{168}{67} & \frac{264}{67} \\ -\frac{168}{67} & \frac{72}{67} + \xi & -\frac{308}{67} \\ -\frac{264}{67} & -\frac{308}{67} & -\frac{216}{67} + \xi \end{bmatrix}$$

Proceeding with the application of Algorithm T, we obtain as kernel representation of the MPUM a matrix whose $(2,2)$ block-element is

$$\begin{bmatrix} \frac{-1721350920+3763007528\xi+607391445\xi^2+35405647\xi^3}{35405647} & \frac{330(-31781032+4310493\xi+289807\xi^2)}{35405647} \\ \frac{330(-26518552+4320147\xi+289807\xi^2)}{35405647} & \frac{-11040998520+4339543928\xi+647295195\xi^2+35405647\xi^3}{35405647} \end{bmatrix}$$

The determinant of such matrix is

$$-58053.9 - 17244.7\xi + 6638.14\xi^2 + 3467.99\xi^3 + 535.189\xi^4 + 35.4375\xi^5 + \xi^6$$

which has roots in

$$-6, -5, -4, -11.3835 \pm 8.83631i$$

and one in 2.32962.

V. CONCLUSIONS AND FURTHER WORK

In this paper we have given a new proof of Takagi's result about metric interpolation problems associated with a non-sign-definite Pick matrix. Our approach consists essentially in the use of the interpolation as time-series modeling framework introduced in [3] and further refined in [1], [9], [16]. The work presented in this paper is being extended in several directions, most notably the following ones.

State-space formulas: The state-space case is a special case of the results presented in this paper; however, deriving explicit state-space formulas is a task deserving interest in its own right. In this respect, see also [7].

Generalizations: The most pressing generalization of the results presented in this paper is a discussion of the *subspace* version of the problem, in which the data involve subspaces $\mathcal{V}_i$, in the sense that one looks for left coprime $Y \in \mathbf{R}^{p\times p}[\xi]$, $U \in \mathbf{R}^{p\times m}[\xi]$ with $\det(Y)$ having k singularities in the right half-plane and $\|Y^{-1}U\|_\infty < 1$, such that

$$\begin{bmatrix} U(\lambda_i) & -Y(\lambda_i) \end{bmatrix} v = 0,$$

for all $v \in \mathcal{V}_i$, $1 \leq i \leq N$. See [16] for a discussion of the Nevanlinna interpolation problem from this point of view.

Applications: We are in the process of using the machinery illustrated in this paper in order to attack the problem of *stabilization with dissipative controllers*, formulated as follows. Let J be as in (1), and let $\mathfrak{B}$ be a controllable behavior. Let $\mathfrak{B}_{\text{des}}$ be a stable, autonomous subspace of $\mathfrak{B}$ representing the desired behavior after interconnection with some controller $\mathfrak{C}$. Does there exist a J-dissipative controller $\mathfrak{C}$ such that $\mathfrak{C} \cap \mathfrak{B} = \mathfrak{B}_{\text{des}}$? Assuming such a controller exists, how many unstable poles does the transfer function associated with the controllable part of $\mathfrak{C}$ have? It is expected that the particular kernel representation obtained through Algorithm T can provide significant insight in the solution of this problem. See also [6], [8], [10], [11], [18] for the use of interpolation methods in controller design.

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