The Behavioral Approach to Systems Theory

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Lecture 1: General Introduction

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Questions

- What is a **mathematical model**, really?
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- How is this specialized to dynamics?
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• How is this specialized to **dynamics**?

• How are models arrived at?
  • From basic laws: *first principles* modeling
  • Combined with interconnection: tearing, zooming, & linking
  • From measured data: **SYSID** (system identification)
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  • Combined with interconnection: tearing, zooming, & linking
  • From measured data: SYSID (system identification)

• What is the role of (differential) equations?
Questions

- What is a **mathematical model**, really?
- How is this specialized to **dynamics**?
- How are models arrived at?
  - From basic laws: ‘first principles’ modeling
  - Combined with interconnection: tearing, zooming, & linking
  - From measured data: **SYSID** (system identification)
- What is the role of (differential) **equations**?
- Importance of **latent** variables
Static models
The seminal idea

Consider a ‘phenomenon’; produces ‘outcomes’, ‘events’.

Mathematization: events belong to a set, $\mathcal{U}$.
The seminal idea

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Mathematization: events belong to a set, $\mathcal{U}$.

Modeling question: Which events can really occur?

The model specifies: Only those in the subset $\mathcal{B} \subseteq \mathcal{U}$!

$\Rightarrow\Rightarrow$ a mathematical model, with behavior $\mathcal{B}$
The seminal idea

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Modeling question: Which events can really occur?
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$\Rightarrow \Rightarrow$ a mathematical model, with behavior $\mathcal{B}$

Before modeling: events in $\mathcal{U}$ are possible
After modeling: only events in $\mathcal{B}$ are possible

Sharper model $\leadsto$ smaller $\mathcal{B}$. 
Examples

Gas law

Phenomenon: A balloon filled with a gas

¿! Model the relation between volume, quantity, pressure, & temperature ¿!
Examples

Gas law

Phenomenon: A balloon filled with a gas

Model the relation between volume, quantity, pressure, & temperature!!

Event: \( (\text{vol. } V, \text{ quant. } N, \text{ press. } P, \text{ temp. } T) \sim \mathcal{U} = \mathbb{R}_+^4 \)
Examples

Gas law

Phenomenon: A balloon filled with a gas

!! Model the relation between volume, quantity, pressure, & temperature !!


Charles Boyle and Avogadro

$\sim$ model $\frac{PV}{NT} = \text{a universal constant } =: R$

$\Rightarrow \Rightarrow \mathcal{B} = \left\{ (T, P, V, N) \in \mathbb{R}^4_+ \mid \frac{PV}{NT} = R \right\}$
An economy

Phenomenon: trading of a product

Model the relation between price, sales & production!!
An economy Phenomenon: trading of a product

Model the relation between price, sales & production

Event: \((\text{price } P, \text{ demand } D)\) \(\sim U = \mathbb{R}^2_+\)

Typical model: \(\mathcal{B} = \text{graph of a curve}\)
Examples

An economy

Phenomenon: trading of a product

\[ \text{Model the relation between price, sales & production} \]

Event: \((\text{price } P, \text{ supply } S)\) \(\sim \mathcal{U} = \mathbb{R}^2_+\)

Typical model: \(\mathcal{B} = \text{graph of a curve}\)
Examples

An economy Phenomenon: trading of a product

Model the relation between price, sales & production !!

Event: \((\text{price } P, \text{ demand } D, \text{ supply } S)\) \sim \mathcal{U} = \mathbb{R}^3_+

\mathcal{B} = \text{intersection of two graphs} \quad \sim \text{usually point(s)}
Examples

An economy Phenomenon: trading of a product

!! Model the relation between sales & production !! Price only to explain mechanism

Event: \((\text{demand } D, \text{ supply } S)\) \sim \mathcal{U} = \mathbb{R}_+^2

\mathcal{B} = \text{intersection of two graphs} : \sim \text{usually point(s)}

The price \(P\) becomes a ‘hidden’ variable. Modeling using ‘hidden’, ‘auxiliary’, ‘latent’ intermediate variables is very common.

How shall we deal with such variables?
Examples

Newton’s 2-nd law

Phenomenon: A moving mass

Model the relation between force, mass, & acceleration
Examples

Newton’s 2-nd law

Phenomenon: A moving mass

\[ \text{Model the relation between force, mass, & acceleration} \]

Event: (force \( F \), mass \( m \), acceleration \( a \))

\[ \sim \mathcal{U} = \mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R}^3 \]
Examples

Newton’s 2-nd law

Phenomenon: A moving mass

Model the relation between force, mass, & acceleration!!

Event: (force $F$, mass $m$, acceleration $a$)

$\mathcal{U} = \mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R}^3$

Model due to Newton:

$F = ma$

$\mathcal{V} = \{ (F, m, a) \in \mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R}^3 \mid F = ma \}$
Examples

Newton’s 2-nd law

Phenomenon: A moving mass

But, the aim of Newton’s law is really:

¡¡ Model the relation between force, mass, & position ¡¡
Newton’s 2-nd law

Phenomenon: A moving mass

But, the aim of Newton’s law is really:

|| Model the relation between force, mass, & position!!

Event: \((\text{force } F, \text{ mass } m, \text{ position } q)\)

\[ F = ma, \quad a = \frac{d^2 q}{dt^2} \]

not ‘instantaneous’ relation between \(F, m, q\) \(\Rightarrow\) dynamics

How shall we deal with this?
Dynamic models
Dynamical systems

Phenomenon produces ‘events’ that are functions of time.

Mathematization: It is convenient to distinguish

- **domain** (‘independent’ variables) \( T \subseteq \mathbb{R} \) ‘time-axis’
- **co-domain** (‘dependent’ variables) \( W \) ‘signal space’

A dynamical system: \( \Sigma = (T, W, B) \)
Dynamical systems

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A dynamical system :=

\[
\Sigma = (T, W, \mathcal{B}) \quad \mathcal{B} \subseteq (W)^T \quad \text{the behavior}
\]
Dynamical systems

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A dynamical system :=

\[ \Sigma = (T, W, \mathcal{B}) \]  \[ \mathcal{B} \subseteq (W)^T \]  \textit{the behavior}

- \( T = \mathbb{R}, \mathbb{R}_+, \) or interval in \( \mathbb{R} \): \textit{continuous-time} systems
- \( T = \mathbb{Z}, \mathbb{N}, \) etc.: \textit{discrete-time} systems

Later: set of independent variables = \( \mathbb{R}^n, n > 1, \) \( \text{PDE’s.} \)
Dynamical systems

Phenomenon produces ‘events’ that are functions of time

Mathematization: It is convenient to distinguish

- **domain** ('independent' variables) $T \subseteq \mathbb{R}$ ‘time-axis’
- **co-domain** ('dependent' variables) $W$ ‘signal space’

A dynamical system $\Sigma := (T, W, B)$

$B \subseteq (W)^T$ the behavior

$W = \mathbb{R}^w$, etc. lumped systems

$W$ = finite: finitary systems
$T = \mathbb{Z}$ or $\mathbb{N}$, $W$ finite: DES (discrete event systems)

$W$ = function space: DPS (distributed parameter systems)
Dynamical systems

Phenomenon produces ‘events’ that are functions of time

Mathematization: It is convenient to distinguish

*domain* (‘independent’ variables) \( T \subseteq \mathbb{R} \) ‘time-axis’

*co-domain* (‘dependent’ variables) \( W \) ‘signal space’

A dynamical system \( \Sigma \) :=

\[
\Sigma = (T, W, B)
\]

\( B \subseteq (W)^T \) the behavior

\( W \) vector space, \( B \subseteq (W)^T \) linear subspace: linear systems controllability, observability, stabilizability, dissipativity, stability, symmetry, reversibility, (equivalent) representations, etc.: to be defined in terms of the behavior \( B \)

THE BEHAVIOR IS ALL THERE IS!
Newton’s 2-nd law

Model the relation between force & position of a pointmass !!
Examples

Newton’s 2-nd law

¾ Model the relation between force & position of a pointmass !

Event: (force $F$ (a f’n of time), position $q$ (a f’n of time))

$\mapsto \mathbb{T} = \mathbb{R}, \mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3$
Examples

Newton’s 2-nd law

\[ \text{Model the relation between force & position of a pointmass}!! \]

Event:  \((\text{force } F \text{ (a f’n of time)}, \text{ position } q \text{ (a f’n of time)})\)

\[ \sim T = \mathbb{R}, W = \mathbb{R}^3 \times \mathbb{R}^3 \]

Model:

\[ F = ma, \quad a = \frac{d^2}{dt^2}q \]

\[ \sim \quad \Sigma = (\mathbb{R}, \mathbb{R}^3 \times \mathbb{R}^3, \mathcal{B}) \]

with

\[ \mathcal{B} = \left\{ (F, q) : \mathbb{R} \rightarrow \mathbb{R}^3 \times \mathbb{R}^3 \mid F = m\frac{d^2}{dt^2}q \right\} \]
Examples

RLC circuit

Phenomenon: the port voltage and current, f’ns of time

Model voltage/current histories as a f’n of time!
Examples

RLC circuit

\[ \Sigma = (\mathbb{R}, \mathbb{R}^2, \mathcal{B}) \]

behavior \( \mathcal{B} \) specified by:

Case 1: \( CR_C \neq \frac{L}{R_L} \)

\[
\left( \frac{R_C}{R_L} + \left( 1 + \frac{R_C}{R_L} \right) CR_C \frac{d}{dt} + CR_C \frac{L}{R_L} \frac{d^2}{dt^2} \right) V = \left( 1 + CR_C \frac{d}{dt} \right) \left( 1 + \frac{L}{R_L} \frac{d}{dt} \right) R_C I
\]

Case 2: \( CR_C = \frac{L}{R_L} \)

\[
\left( \frac{R_C}{R_L} + CR_C \frac{d}{dt} \right) V = (1 + CR_C) \frac{d}{dt} R_C I
\]

behavior all solutions \((V, I) : \mathbb{R} \rightarrow \mathbb{R}^2\) of this ODE
Examples

**input/output models**

\[ y(t) = f(y(t - 1), \ldots, y(t - n), u(t), u(t - 1), u(t - n)), \quad w = \begin{bmatrix} u \\ y \end{bmatrix} \]

Differential equation analogue

\[ P \left( \frac{d}{dt} \right) y = P \left( \frac{d}{dt} \right) u, \quad w = \begin{bmatrix} u \\ y \end{bmatrix}, \quad P, Q : \text{polynomial matrices} \]

or matrices of rational functions as in \( y = G(s)u \)

How shall we define the behavior with the **rational f’ns**?
Examples

input/output models

State models

\[ \frac{d}{dt} x = Ax + Bu, \quad y = Cx + Du; \quad \frac{d}{dt} x = f(x, u), \quad y = h(x, u) \]

¿¿ What is the behavior of this system ??
Examples

input/output models

State models

\[ \frac{d}{dt} x = A x + B u, \quad y = C x + D u; \quad \frac{d}{dt} x = f \circ (x, u), \quad y = h \circ (x, u) \]

What is the behavior of this system ??

In applications, we care foremost about i/o pairs \( u, y \)

\[ \sim \quad \Sigma = (\mathbb{R}, U \times Y, \mathcal{B}) \]

\[ \mathcal{B} = \{(u, y) : \mathbb{R} \rightarrow U \times Y | \exists x : \mathbb{R} \rightarrow X \text{ such that } x = f \circ (x, u), \ y = h \circ (x, u) \} \]

So, here again, we meet auxiliary variables, the state \( x \).
Latent variables
Latent variables

Auxiliary variables. We call them ‘latent’. They are ubiquitous:

- states in dynamical systems
- prices in economics
- the wave function in QM
- the basic probability space $\Omega$
- potentials in mechanics, in EM
- interconnection variables
- driving variables in linear system theory
- etc., etc.

Their importance in applications merits formalization.
Latent variable model := \((U, L, B_{\text{full}})\) with \(B_{\text{full}} \subseteq (U \times L)\)

\(U\): space of manifest variables
\(L\): space of latent variables
\(B_{\text{full}}\): ‘full behavior’
\(B = \{u \in U | \exists \ell \in L : (u, \ell) \in B_{\text{full}}\}\): ‘manifest behavior’.
Latent variables

Latent variable model := \((U, L, B_{\text{full}})\) with \(B_{\text{full}} \subseteq (U \times L)\)

\(U\): space of manifest variables
\(L\): space of latent variables

\(B_{\text{full}}\): ‘full behavior’
\(B = \{u \in U|\exists \ell \in L : (u, \ell) \in B_{\text{full}}\}\): ‘manifest behavior’.

This is readily generalized to dynamical systems.

A latent variable dynamical system :=
\((T, W, L, B_{\text{full}})\) with \(B_{\text{full}} \subseteq (W \times L)^T\)

e tc.
Example

The price in our economic example
Model voltage/current histories as a f’n of time!

How do we actually go about this modeling?

Emergence of latent variables.
Example

RLC circuit

TEARING

connector1

R\text{C} \quad 1 \quad 2

C \quad 5 \quad 6

L \quad 7 \quad 8

R\text{L} \quad 3 \quad 4

connector2
The list of the modules & the associated terminals:

<table>
<thead>
<tr>
<th>Module</th>
<th>Type</th>
<th>Terminals</th>
<th>Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_C$</td>
<td>resistor</td>
<td>(1, 2)</td>
<td>in ohms</td>
</tr>
<tr>
<td>$R_L$</td>
<td>resistor</td>
<td>(3, 4)</td>
<td>in ohms</td>
</tr>
<tr>
<td>$C$</td>
<td>capacitor</td>
<td>(5, 6)</td>
<td>in farad</td>
</tr>
<tr>
<td>$L$</td>
<td>inductor</td>
<td>(7, 8)</td>
<td>in henry</td>
</tr>
<tr>
<td>connector1</td>
<td>3-terminal connector</td>
<td>(9, 10, 11)</td>
<td></td>
</tr>
<tr>
<td>connector2</td>
<td>3-terminal connector</td>
<td>(12, 13, 14)</td>
<td></td>
</tr>
</tbody>
</table>
Example

**RLC circuit**

**TEARING**

The **interconnection architecture**:

```
<table>
<thead>
<tr>
<th>Pairing</th>
</tr>
</thead>
<tbody>
<tr>
<td>{10, 1}</td>
</tr>
<tr>
<td>{11, 7}</td>
</tr>
<tr>
<td>{2, 5}</td>
</tr>
<tr>
<td>{8, 3}</td>
</tr>
<tr>
<td>{6, 13}</td>
</tr>
<tr>
<td>{4, 14}</td>
</tr>
</tbody>
</table>
```
Example

RLC circuit

Manifest variable assignment:

the variables

\[ V_9, I_9, V_{12}, I_{12} \]

on the external terminals \{9, 12\}, i.e,

\[ V_a = V_9, I_a = I_9, V_b = V_{12}, I_b = I_{12}. \]

The internal terminals are

\{1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14\}

The variables (currents and voltages) on these terminals are our latent variables.
Example

**RLC circuit**

**Equations for the full behavior:**

<table>
<thead>
<tr>
<th>Modules</th>
<th>Constitutive equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_C$</td>
<td>$I_1 + I_2 = 0$</td>
</tr>
<tr>
<td>$R_L$</td>
<td>$I_7 + I_8 = 0$</td>
</tr>
<tr>
<td>$C$</td>
<td>$I_5 + I_6 = 0$</td>
</tr>
<tr>
<td>$L$</td>
<td>$I_7 + I_8 = 0$</td>
</tr>
<tr>
<td>connector1</td>
<td>$I_9 + I_{10} + I_{11} = 0$</td>
</tr>
<tr>
<td>connector2</td>
<td>$I_{12} + I_{13} + I_{14} = 0$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Interconnection pair</th>
<th>Interconnection equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>{10, 1}</td>
<td>$V_{10} = V_1$</td>
</tr>
<tr>
<td>{11, 7}</td>
<td>$V_{11} = V_7$</td>
</tr>
<tr>
<td>{2, 5}</td>
<td>$V_2 = V_5$</td>
</tr>
<tr>
<td>{8, 3}</td>
<td>$V_8 = V_3$</td>
</tr>
<tr>
<td>{6, 13}</td>
<td>$V_6 = V_{13}$</td>
</tr>
<tr>
<td>{4, 14}</td>
<td>$V_4 = V_{14}$</td>
</tr>
</tbody>
</table>
Example

All these eq’ns combined define a latent variable system in the manifest ‘external’ variables

\[ w = (V_a, I_a, V_b, I_b) \]

with ‘internal’ latent variables

\[ \ell = (V_1, I_1, V_2, I_2, V_3, I_3, V_4, I_4, V_5, I_5, V_6, I_6, V_7, I_7, V_8, I_8, V_{10}, I_{10}, V_{11}, I_{11}, V_{13}, I_{13}, V_{14}, I_{14}) \]

The manifest behavior \( \mathcal{B} \) is given by

\[ \mathcal{B} = \{(V_a, I_a, V_b, I_b) : \mathbb{R} \rightarrow \mathbb{R}^4 \mid \exists \ell : \mathbb{R} \rightarrow \mathbb{R}^{24} \ldots \} \]
Example

**RC circuit**

**Elimination:**

**Case 1:** \( CR_C \neq \frac{L}{R_L} \).

\[
\left( \frac{R_C}{R_L} + \left(1 + \frac{R_C}{R_L}\right)CR_C \frac{d}{dt} + CR_C \frac{L}{R_L} \frac{d^2}{dt^2} \right)(V_a - V_b) = (1 + CR_C \frac{d}{dt})(1 + \frac{L}{R_L} \frac{d}{dt})R_C I_a.
\]

\[I_a + I_b = 0\]

**Case 2:** \( CR_C = \frac{L}{R_L} \).

\[
\left( \frac{R_C}{R_L} + CR_C \frac{d}{dt} \right)(V_a - V_b) = (1 + CR_C \frac{d}{dt})R_C I_a
\]

\[I_a + I_b = 0\]

Perhaps ‘port’ variables: \( V = V_a - V_b, I = I_a = -I_b \)
Example

**RLC circuit**

Note: the eliminated equations are differential equations! Does this follow from some general principle?

Algorithms for elimination?

The modeling of this RLC circuit is an example of **tearing, zooming & linking**. It is the most prevalent way of modeling. See my website for formalization. Crucial role of latent variables.

Note: no input/output thinking; systems in nodes, connections in edges.
Controllability & Observability
In this framework, system theoretic notions like

Controllability, observability, stabilizability,...

become simpler, more general, more convincing.
System properties

In this framework, system theoretic notions like Controllability, observability, stabilizability,... become simpler, more general, more convincing.

For simplicity, we consider only time-invariant, continuous-time systems with $T = \mathbb{R}$

$$\text{time-invariant} := \left[ w \in \mathcal{B} \right] \Rightarrow \left[ w(t' + \cdot) \in \mathcal{B} \ \forall \ t' \in \mathbb{R} \right].$$
Controllability

The time-invariant system $\Sigma = (T, W, B)$ is said to be \textbf{controllable} if for all $w_1, w_2 \in B$ $\exists \ w \in B$ and $T \geq 0$ such that

$$w(t) = \begin{cases} 
  w_1(t) & t < 0 \\
  w_2(t - T) & t \geq T 
\end{cases}$$

Controllability $\Leftrightarrow$ legal trajectories must be ‘patch-able’, ‘concatenable’.
Controllability
Controllability
Examples

\[
\frac{d}{dt} x = Ax + Bu; \quad \frac{d}{dt} x = f \circ (x, u)
\]

with \( w = (x, u) \), controllable \( \iff \) ‘state point’ controllable.
Examples

\[ \frac{d}{dt} x = Ax + Bu; \quad \frac{d}{dt} x = f \circ (x, u) \]

with \( w = (x, u) \), controllable \( \iff \) ‘state point’ controllable.

likewise \( \iff \) with \( w = x \)
Examples

\[ \frac{d}{dt} x = Ax + Bu; \quad \frac{d}{dt} x = f \circ (x, u) \]

with \( w = (x, u) \), controllable \( \iff \) ‘state point’ controllable.

**RLC circuit**

**Case 2:**

\[ CR_C = \frac{L}{R_L} \]

\[ \left( \frac{R_C}{R_L} + CR_C \frac{d}{dt} \right)(V_a - V_b) = (1 + CR_C \frac{d}{dt})R_C I_a \]

\[ I_a + I_b = 0 \]

Assume also \( R_C = R_L \). Controllable?

\[ V_a - V_b = R_C I_a + \text{constant} \cdot e^{-\frac{t}{CR_C}}. \] Not controllable.
Examples

\[ \frac{d}{dt} x = Ax + Bu; \quad \frac{d}{dt} x = f \circ (x, u) \]

with \( w = (x, u) \), controllable \( \iff \) ‘state point’ controllable.

\[ p(\frac{d}{dt})y = q(\frac{d}{dt})u \]

controllable \( \iff \) \( p, q \) co-prime
Examples

\[ \frac{d}{dt} x = Ax + Bu; \quad \frac{d}{dt} x = f \circ (x, u) \]

with \( w = (x, u) \), controllable \( \iff \) ‘state point’ controllable.

\[ w = M \left( \frac{d}{dt} \right)^{\ell} \]

\( M \) a polynomial matrix, always has a controllable manifest behavior.

In fact, this characterizes the controllable linear time-invariant differentiable systems (‘image representation’).

Note emergence of latent variables, \( \ell \).
Examples

\[ w = M \left( \frac{d}{dt} \right) \ell \]

\( M \) a polynomial matrix, always has a controllable manifest behavior. Likewise,

\[ w = F \left( \frac{d}{dt} \right) \ell \]

\( F \) matrix of rat. f’ns has controllable manifest behavior. But we need to give this ‘differential equation’ a meaning.

Whence

\[ y = G \left( \frac{d}{dt} \right) u, \quad w = \begin{bmatrix} u \\ y \end{bmatrix} \]

is always controllable.
¿ Is it possible to deduce $w_2$ from $w_1$ and the model $\mathcal{B}$ ?
Consider the system $\Sigma = (T, \mathcal{W}_1 \times \mathcal{W}_2, \mathcal{B})$. Each element of $\mathcal{B}$ hence consists of a pair of trajectories $(\mathcal{w}_1, \mathcal{w}_2)$:

- $\mathcal{w}_1 :$ observed;
- $\mathcal{w}_2 :$ to-be-deduced.
Consider the system $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathbb{B})$. Each element of $\mathbb{B}$ hence consists of a pair of trajectories $(w_1, w_2)$:

\[ w_1 : \text{observed}; \]
\[ w_2 : \text{to-be-deduced}. \]

**Definition:** $w_2$ is said to be

**observable from** $w_1$

if \[ [(w_1, w_2') \in \mathbb{B}, \text{ and } (w_1, w_2'') \in \mathbb{B}] \Rightarrow [(w_2' = w_2'')] \],

i.e., if on $\mathbb{B}$, there exists a map $w_1 \mapsto w_2$.  

**Observability**
Consider the system $\Sigma = (T, W_1 \times W_2, B)$. Each element of $B$ hence consists of a pair of trajectories $(w_1, w_2)$:

\[ w_1 : \text{observed}; \]
\[ w_2 : \text{to-be-deduced.} \]

**Definition:** $w_2$ is said to be \textbf{observable from} $w_1$

\[ \text{if } [((w_1, w'_2) \in B, \text{ and } (w_1, w''_2) \in B)] \Rightarrow [w'_2 = w''_2], \]
\[ \text{i.e., if on } B, \text{ there exists a map } w_1 \mapsto w_2. \]

Very often \textit{manifest} = \text{observed, latent} = \text{to-be-deduced.}

We then speak of an \textit{observable (latent variable)} system.
Examples

\[
\frac{d}{dt} x = Ax + Bu, \quad y = Cx + Du; \quad \frac{d}{dt} x = f(x, u), \quad y = h(x, u)
\]

with \( w_1 = (u, y), \ w_2 = x \), observable \( \Leftrightarrow \) ‘state’ observable.
Controllability of this system (referring to external terminal variables) is a well-defined question.

Observability is not! **No duality** on the system’s level. Of course, there is a notion of $\mathcal{B}^\perp$, and results connecting controllability of $\mathcal{B}$ to state observability of $\mathcal{B}^\perp$. 
## Examples

### Equations for the full behavior:

<table>
<thead>
<tr>
<th>Modules</th>
<th>Constitutive equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R_C$</td>
<td>$I_1 + I_2 = 0$</td>
</tr>
<tr>
<td>$R_L$</td>
<td>$I_7 + I_8 = 0$</td>
</tr>
<tr>
<td>$C$</td>
<td>$I_5 + I_6 = 0$</td>
</tr>
<tr>
<td>$L$</td>
<td>$I_7 + I_8 = 0$</td>
</tr>
<tr>
<td>connector1</td>
<td>$I_9 + I_{10} + I_{11} = 0$</td>
</tr>
<tr>
<td>connector2</td>
<td>$I_{12} + I_{13} + I_{14} = 0$</td>
</tr>
</tbody>
</table>

### Interconnection pair

<table>
<thead>
<tr>
<th>Interconnection pair</th>
<th>Interconnection equations</th>
</tr>
</thead>
<tbody>
<tr>
<td>{10, 1}</td>
<td>$V_{10} = V_1$</td>
</tr>
<tr>
<td>{11, 7}</td>
<td>$V_{11} = V_7$</td>
</tr>
<tr>
<td>{2, 5}</td>
<td>$V_2 = V_5$</td>
</tr>
<tr>
<td>{8, 3}</td>
<td>$V_8 = V_3$</td>
</tr>
<tr>
<td>{6, 13}</td>
<td>$V_6 = V_{13}$</td>
</tr>
<tr>
<td>{4, 14}</td>
<td>$V_4 = V_{14}$</td>
</tr>
</tbody>
</table>
All these eq’ns combined define a latent variable system in the manifest variables

\[ w = (V_a, l_a, V_b, l_b) \]

with latent variables

\[ \ell = (V_1, l_1, V_2, l_2, V_3, l_3, V_4, l_4, V_5, l_5, V_6, l_6, V_7, l_7, V_8, l_8, V_{10}, l_{10}, V_{11}, l_{11}, V_{13}, l_{13}, V_{14}, l_{14}) \].

The manifest behavior \( \mathcal{B} \) is given by

\[ \mathcal{B} = \{ (V_a, l_a, V_b, l_b) : \mathbb{R} \to \mathbb{R}^4 \mid \exists \ \ell : \mathbb{R} \to \mathbb{R}^{24} \ldots \} \]

Are the latent variables observable from the manifest ones?

\[ \iff \ CR_C \neq \frac{L}{R_L} \]
Examples

\[ p\left(\frac{d}{dt}\right)y = q\left(\frac{d}{dt}\right)u \]

\( u \) is observable from \( y \) \( \Leftrightarrow \) \( q \) = non-zero constant (no zeros).

A controllable linear time-invariant differential system always has an observable ‘image’ representation

\[ w = M\left(\frac{d}{dt}\right)\ell. \]

In fact, this again characterizes the controllable linear time-invariant differentiable systems.
Kalman definitions

Special case: classical Kalman definitions for

$$\frac{d}{dt}x = f \circ (x, u), \quad y = h \circ (x, u).$$

R.E. Kalman
Kalman definitions

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**controllability:** variables = (input, state)

If a system is not (state) controllable, why is it?

**Insufficient influence of the control?**

Or **bad choice of the state?**

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**controllability:** variables = (input, state)

If a system is not (state) controllable, why is it?

- Insufficient influence of the control?
- Or bad choice of the state?

**observability:** observed = (input, output),

\[ \text{to-be-deduced} = \text{state}. \]

Why is it so interesting to try to deduce the state, of all things? The state is a derived notion, not a ‘physical’ one.
Stabilizability

The system $\Sigma = (T, \mathbb{R}^w, \mathcal{B})$ is said to be **stabilizable** if, for all $w \in \mathcal{B}$, there exists $w' \in \mathcal{B}$ such that

$$w(t) = w'(t) \text{ for } t < 0 \text{ and } w'(t) \xrightarrow{t \to \infty} 0.$$
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Stabilizability $\iff$ legal trajectories can be steered to a desired point.
Is it possible to deduce $w_2$ asymptotically from $w_1$?
 Detectability

¿ Is it possible to deduce $w_2$ asymptotically from $w_1$?

Definition: $w_2$ is said to be detectable from $w_1$ if

$[(w_1, w_2') \in \mathbb{B}, \text{ and } (w_1, w_2'') \in \mathbb{B}] \Rightarrow [(w_2' - w_2'') \to 0 \text{ for } t \to \infty]$
Btw

- A model is not a map, but a relation.
Btw

- A model is not a map, but a relation.
- A flow

\[
\frac{d}{dt} x = f(x) \quad \text{with or without} \quad y = h(x)
\]

is a very limited model class.

\sim \text{closed dynamical systems.}
Btw

- A model is not a map, but a relation.
- A flow is a very limited model class.
  \[ \sim \text{closed dynamical systems.} \]
- An open dynamical system is not an input/output map.

Heaviside  Wiener  Nyquist  Bode
Btw

- A model is not a map, but a relation.
- A flow is a very limited model class. \( \sim \) closed dynamical systems.
- An **open** dynamical system is not an input/output map.
- **input/state/output systems**, although still limited, are the first class of suitably general models

R.E. Kalman
Btw

- A model is not a map, but a relation.
- A flow is a very limited model class.
  $\sim$ closed dynamical systems.
- An open dynamical system is not an input/output map.
- input/state/output systems, although still limited, are the first class of suitably general models.
- Behaviors, including latent variables, are the first suitable general model class for physical applications and modeling by tearing, zooming, and linking.
A mathematical model = a subset

An important property of dynamical systems is stability.

Controllability: concatenability of trajectories
Observability: deducing one trajectory from another
Stabilizability: driving a trajectory to zero

The behavior is all there is. All properties in terms of the behavior. Equivalence, representations also.
• A mathematical model = a subset

• A dynamical system = a behavior
  = a family of trajectories
Summary

- A mathematical model = a subset
- A dynamical system = a **behavior** = a family of trajectories
- Latent variables are ubiquitous in models
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Stochastic models

We only consider deterministic models. Stochastic models:

there is a map $P$ (the ‘probability’)

$P : \mathcal{A} \rightarrow [0, 1]$

with $\mathcal{A}$ a ‘$\sigma$-algebra’ of subsets of $\mathcal{U}$.

$P(\mathcal{B}) =$ ‘degree of certainty’ (relative frequency, propensity, plausibility, belief) that outcomes are in $\mathcal{B}$; $\equiv$ the degree of validity of $\mathcal{B}$ as a model.
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\cong$ the degree of validity of $\mathcal{B}$ as a model.

**Determinism:** $P$ is a ‘$\{0, 1\}$-law’

$$\mathcal{A} = \{ \emptyset, \mathcal{B}, \mathcal{B}^{\text{complement}}, \mathcal{U} \}, \quad P(\mathcal{B}) = 1.$$
Fuzzy models: there is a map $\mu$ ('membership f’n')

$$\mu : \mathcal{U} \rightarrow [0, 1]$$

$\mu (x) =$ ‘the extent to which $x$ belongs to the model’s behavior’. 
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Determinism: $\mu$ is ‘crisp’:

$$\text{image } (\mu) = \{ 0, 1 \},$$

$$\mathcal{B} = \mu^{-1}(\{ 1 \}) := \{ x \in \mathcal{U} | \mu (x) = 1 \}$$
Every ‘good’ scientific theory is prohibition: it forbids certain things to happen...
The more a theory forbids, the better it is.

Karl Popper (1902-1994)

Replace ‘scientific theory’ by ‘mathematical model’!