The Behavioral Approach
to
Systems Theory

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Lecture 1: General Introduction

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Questions

- **What is a mathematical model, really?**

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  - **From basic laws: 'first principles' modeling**
  - **Combined with interconnection:**
    tearing, zooming, & linking
  - **From measured data: SYSID (system identification)**

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  - **Combined with interconnection: tearing, zooming, & linking**
  - **From measured data: SYSID (system identification)**
- **What is the role of (differential) equations?**
- **Importance of latent variables**

Static models

The seminal idea

Consider a ‘phenomenon’; produces ‘outcomes’, ‘events’.

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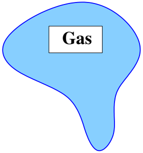
Before modeling: events in $\mathfrak{U}$ are possible
After modeling: only events in $\mathfrak{B}$ are possible

Sharper model $\rightsquigarrow$ smaller $\mathfrak{B}$.

Examples

Gas law



Phenomenon: A balloon filled with a gas

¡¡ Model the relation between volume, quantity, pressure, & temperature !!

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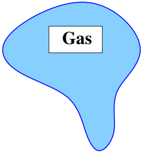

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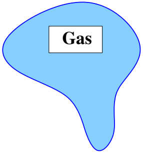

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Charles  **Boyle**  **and Avogadro** 

$\rightsquigarrow$ **model** $\dfrac{PV}{NT} =$ **a universal constant** $=: R$

$$\Rightarrow\Rightarrow \quad \mathfrak{B} = \left\{ (T, P, V, N) \in \mathbb{R}_+^4 \;\middle|\; \frac{PV}{NT} = R \right\} \quad \Leftarrow\Leftarrow$$

Examples

An economy **Phenomenon: trading of a product**

¡¡ Model the relation between

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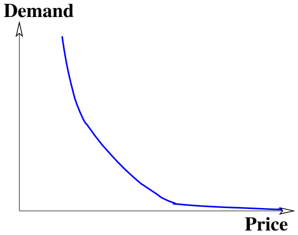
Typical model: $\mathfrak{B} =$ **graph of a curve**



Examples

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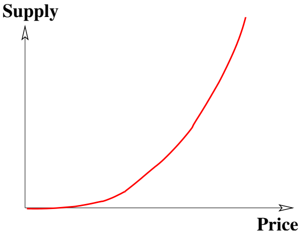

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An economy — **Phenomenon: trading of a product**

¡¡ Model the relation between

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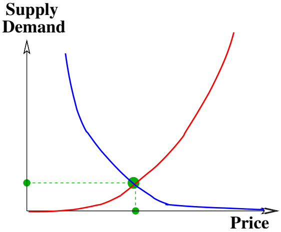
Event: $\left(\text{price } P,\ \text{demand } D,\ \text{supply } S\right) \rightsquigarrow \mathfrak{U} = \mathbb{R}^3_+$

$\mathfrak{B}$ = intersection of two graphs: $\rightsquigarrow$ usually point(s)



Examples

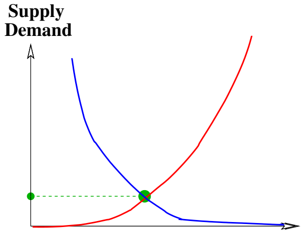
An economy **Phenomenon: trading of a product**

¡¡ Model the relation between
sales & production !! Price only to explain mechanism

Event: $\Big($ demand D, supply $S\Big)$ $\rightsquigarrow \mathfrak{U} = \mathbb{R}^2_+$

$\mathfrak{B} =$ intersection of two graphs : $\rightsquigarrow$ usually point(s)

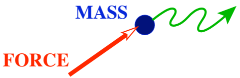
The price P becomes a ‘hidden’ variable. Modeling using ‘hidden’, ‘auxiliary’, ‘latent’ intermediate variables is very common.



How shall we deal with such variables?

Examples

Newton's 2-nd law



Phenomenon: A moving mass

¡¡ Model the relation between

force, mass, & acceleration !!

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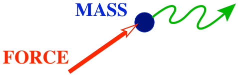

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$$\rightsquigarrow \mathfrak{U} = \mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R}^3$$

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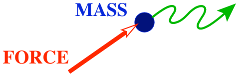

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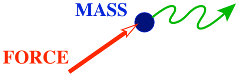
Model due to Newton:

$$F = ma$$

$$\Rightarrow\Rightarrow \quad \mathfrak{B} = \{ (F, m, a) \in \mathbb{R}^3 \times \mathbb{R}_+ \times \mathbb{R}^3 \mid F = ma \} \quad \Leftarrow\Leftarrow$$

Examples

Newton's 2-nd law



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But, the aim of Newton's law is really:

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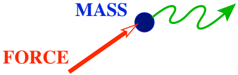

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But, the aim of Newton's law is really:

¡¡ Model the relation between force, mass, & position !!

Event: (force F, mass m, position q)

$$F = ma, \quad a = \frac{d^2}{dt^2}q$$

not 'instantaneous' relation between $F, m, q \rightsquigarrow$ dynamics

How shall we deal with this?

Dynamic models

Dynamical systems

Phenomenon produces ‘events’ that are functions of time

Mathematatization: It is convenient to distinguish

- **domain (‘independent’ variables) $\mathbb{T} \subseteq \mathbb{R}$ ‘time-axis’**
- **co-domain (‘dependent’ variables) $\mathbb{W}$ ‘signal space’**

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A dynamical system :=

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B}) \qquad \mathfrak{B} \subseteq (\mathbb{W})^{\mathbb{T}}$$ ***the behavior***

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$\mathbb{T} = \mathbb{R}, \mathbb{R}_+,$ **or interval in** $\mathbb{R}$**: continuous-time systems**

$\mathbb{T} = \mathbb{Z}, \mathbb{N}$**, etc.: discrete-time systems**

Later: set of independent variables = $\mathbb{R}^{n}$**, n** $>$ **1, PDE's.**

Dynamical systems

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$\mathbb{W} = \mathbb{R}^{\mathtt{w}}$**, etc. lumped systems**

$\mathbb{W} =$ **finite: finitary systems**

$\mathbb{T} = \mathbb{Z}$ **or** $\mathbb{N}$, $\mathbb{W}$ **finite: DES (discrete event systems)**

$\mathbb{W} =$ **function space: DPS (distributed parameter systems)**

Dynamical systems

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$\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ $\mathfrak{B} \subseteq (\mathbb{W})^{\mathbb{T}}$ *the behavior*

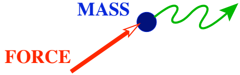
$\mathbb{W}$ vector space, $\mathfrak{B} \subset (\mathbb{W})^{\mathbb{T}}$ linear subspace: linear systems

controllability, observability, stabilizability, dissipativity, stability, symmetry, reversibility, (equivalent) representations, etc.: to be defined in terms of the behavior $\mathfrak{B}$

THE BEHAVIOR IS ALL THERE IS !

Examples

Newton's 2-nd law



¡¡ Model the relation between
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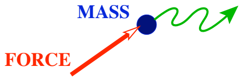

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Event: (force F (a f'n of time), position q (a f'n of time))

$$\rightsquigarrow \mathbb{T} = \mathbb{R}, \mathbb{W} = \mathbb{R}^3 \times \mathbb{R}^3$$

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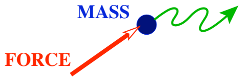

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Model:

$$F = ma, \quad a = \frac{d^2}{dt^2} q$$

$$\rightsquigarrow \qquad \Sigma = (\mathbb{R}, \mathbb{R}^3 \times \mathbb{R}^3, \mathfrak{B})$$

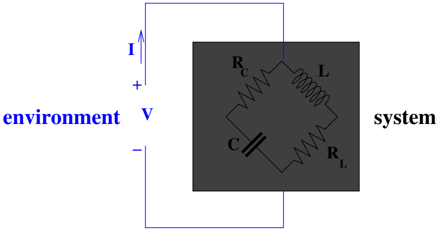
with

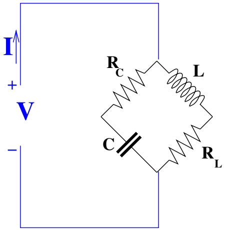
$$\Rightarrow\Rightarrow \quad \mathfrak{B} = \left\{ (F, q) : \mathbb{R} \rightarrow \mathbb{R}^3 \times \mathbb{R}^3 \mid F = m \frac{d^2}{dt^2} q \right\} \quad \Leftarrow\Leftarrow$$

Examples

RLC circuit

Phenomenon: the port voltage and current, f’ns of time





Model voltage/current histories as a f’n of time !

Examples

RLC circuit

$$\rightsquigarrow \qquad \Sigma = (\mathbb{R}, \mathbb{R}^2, \mathfrak{B})$$

behavior $\mathfrak{B}$ specified by:

Case 1: $CR_C \neq \dfrac{L}{R_L}$

$$\left(\frac{R_C}{R_L} + \left(1 + \frac{R_C}{R_L}\right) CR_C \frac{d}{dt} + CR_C \frac{L}{R_L} \frac{d^2}{dt^2}\right) V = \left(1 + CR_C \frac{d}{dt}\right)\left(1 + \frac{L}{R_L} \frac{d}{dt}\right) R_C I$$

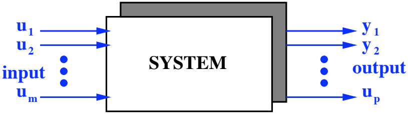
Case 2: $CR_C = \dfrac{L}{R_L}$

$$\left(\frac{R_C}{R_L} + CR_C \frac{d}{dt}\right) V = (1 + CR_C) \frac{d}{dt} R_C I$$

$\rightsquigarrow$ **behavior all solutions $(V, I) : \mathbb{R} \to \mathbb{R}^2$ of this ODE**

Examples

input/output models



$$y(t) = f(y(t-1), \cdots, y(t-\mathrm{n}), u(t), u(t-1), u(t-\mathrm{n})), \quad w = \begin{bmatrix} u \\ y \end{bmatrix}$$

Differential equation analogue

$$P(\frac{d}{dt})y = P(\frac{d}{dt})u, w = \begin{bmatrix} u \\ y \end{bmatrix}, \quad P, Q: \text{ polynomial matrices}$$

or matrices of rational functions as in $y = G(s)u$

How shall we define the behavior with the rational f'ns ?

Examples

input/output models

State models

R.E. Kalman

$$\frac{d}{dt}x = Ax + Bu,\ y = Cx + Du; \quad \frac{d}{dt}x = f\circ(x,u), y = h\circ(x,u)$$

¿¿ What is the behavior of this system ??

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State models

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¿¿ What is the behavior of this system ??

In applications, we care foremost about i/o pairs u, y

$$\rightsquigarrow \qquad \Sigma = (\mathbb{R}, \mathbb{U} \times \mathbb{Y}, \mathfrak{B})$$

$$\mathfrak{B} = \{(u,y) : \mathbb{R} \to \mathbb{U} \times \mathbb{Y} \mid$$

$$\exists x : \mathbb{R} \to \mathbb{X} \text{ such that } x = f \circ (x,u), y = h \circ (x,u)$$

So, here again, we meet auxiliary variables, the state x.

Latent variables

Latent variables

Auxiliary variables. We call them 'latent'. They are ubiquitous:

- **states in dynamical systems**
- **prices in economics**
- **the wave function in QM**
- **the basic probability space Ω**
- **potentials in mechanics, in EM**
- **interconnection variables**
- **driving variables in linear system theory**
- **etc., etc.**

Their importance in applications merits formalization.

Latent variables

Latent variable model := $(\mathfrak{U}, \mathfrak{L}, \mathfrak{B}_{\text{full}})$ with $\mathfrak{B}_{\text{full}} \subseteq (\mathfrak{U} \times \mathfrak{L})$

$\mathfrak{U}$: space of **manifest** variables
$\mathfrak{L}$: space of **latent** variables

$\mathfrak{B}_{\text{full}}$: 'full behavior'
$\mathfrak{B} = \{u \in \mathfrak{U} | \exists \ell \in \mathfrak{L} : (u, \ell) \in \mathfrak{B}_{\text{full}}\}$: 'manifest behavior'.

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This is readily generalized to dynamical systems.

A **latent variable dynamical system** :=

$(\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_{\text{full}})$ with $\mathfrak{B}_{\text{full}} \subseteq (\mathbb{W} \times \mathbb{L})^{\mathbb{T}}$

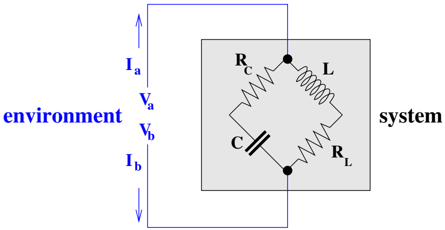
etc.

Example

The price in our economic example

Example

RLC circuit



Model voltage/current histories as a f’n of time !

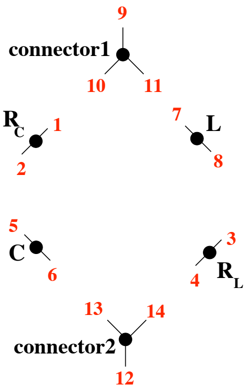
How do we actually go about this modeling ?

Emergence of latent variables.

Example

RLC circuit

TEARING



Example

RLC circuit

ZOOMING

The list of the modules & the associated terminals:

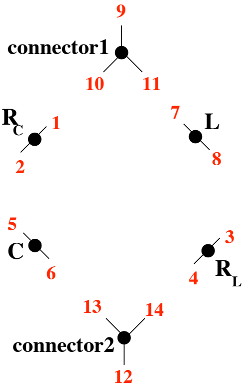
Module	Type	Terminals	Parameter
R_C	resistor	(1, 2)	in ohms
R_L	resistor	(3, 4)	in ohms
C	capacitor	(5, 6)	in farad
L	inductor	(7, 8)	in henry
connector1	3-terminal connector	(9, 10, 11)	
connector2	3-terminal connector	(12, 13, 14)	

Example

RLC circuit

TEARING

The interconnection architecture:



Pairing
{10, 1}
{11, 7}
{2, 5}
{8, 3}
{6, 13}
{4, 14}

Example

RLC circuit

Manifest variable assignment:

the variables

$$V_9, I_9, V_{12}, I_{12}$$

on the external terminals {9, 12}, i.e,

$$V_a = V_9, I_a = I_9, V_b = V_{12}, I_b = I_{12}.$$

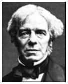
The internal terminals are

$$\{1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14\}$$

The variables (currents and voltages) on these terminals are our latent variables.

Example

RLC circuit



Faraday

Ohm

Henry

Coulomb

Kirchhoff

Equations for the full behavior:

Modules	Constitutive equations	
R_C	$I_1 + I_2 = 0$	$V_1 - V_2 = R_C I_1$
R_L	$I_7 + I_8 = 0$	$V_7 - V_8 = R_L I_7$
C	$I_5 + I_6 = 0$	$C \frac{d}{dt}(V_5 - V_6) = I_5$
L	$I_7 + I_8 = 0$	$V_7 - V_8 = L \frac{d}{dt} I_7$
connector1	$I_9 + I_{10} + I_{11} = 0$	$V_9 = V_{10} = V_{11}$
connector2	$I_{12} + I_{13} + I_{14} = 0$	$V_{12} = V_{13} = V_{14}$

Interconnection pair	Interconnection equations	
$\{10, 1\}$	$V_{10} = V_1$	$I_{10} + I_1 = 0$
$\{11, 7\}$	$V_{11} = V_7$	$I_{11} + I_7 = 0$
$\{2, 5\}$	$V_2 = V_5$	$I_2 + I_5 = 0$
$\{8, 3\}$	$V_8 = V_3$	$I_8 + I_3 = 0$
$\{6, 13\}$	$V_6 = V_{13}$	$I_6 + I_{13} = 0$
$\{4, 14\}$	$V_4 = V_{14}$	$I_4 + I_{14} = 0$

Example

RLC circuit

All these eq'ns combined define a latent variable system in the manifest 'external' variables

$$w = (V_a, I_a, V_b, I_b)$$

with 'internal' latent variables

$$\ell = (V_1, I_1, V_2, I_2, V_3, I_3, V_4, I_4, V_5, I_5, V_6, I_6, V_7, I_7, \\ V_8, I_8, V_{10}, I_{10}, V_{11}, I_{11}, V_{13}, I_{13}, V_{14}, I_{14}).$$

The manifest behavior $\mathfrak{B}$ is given by

$$\mathfrak{B} = \{(V_a, I_a, V_b, I_b) : \mathbb{R} \to \mathbb{R}^4 \mid \exists\ \ell : \mathbb{R} \to \mathbb{R}^{24} \ldots\}$$

Example

RLC circuit

Elimination:

Case 1: $CR_C \neq \frac{L}{R_L}$.

$$\left(\frac{R_C}{R_L} + \left(1 + \frac{R_C}{R_L}\right) CR_C \frac{d}{dt} + CR_C \frac{L}{R_L}\frac{d^2}{dt^2}\right)(V_a - V_b) = \left(1 + CR_C \frac{d}{dt}\right)\left(1 + \frac{L}{R_L}\frac{d}{dt}\right) R_C I_a.$$

$$I_a + I_b = 0$$

Case 2: $CR_C = \frac{L}{R_L}$.

$$\left(\frac{R_C}{R_L} + CR_C \frac{d}{dt}\right)(V_a - V_b) = \left(1 + CR_C \frac{d}{dt}\right) R_C I_a$$

$$I_a + I_b = 0$$

Perhaps 'port' variables: $V = V_a - V_b, I = I_a = -I_b$

Example

RLC circuit

Note: the eliminated equations are differential equations!
Does this follow from some general principle ?

Algorithms for elimination?

The modeling of this RLC circuit is an example of tearing, zooming & linking. It is the most prevalent way of modeling. See my website for formalization. Crucial role of latent variables.
Note: no input/output thinking;
systems in nodes, connections in edges.

Controllability & Observability

System properties

In this framework, system theoretic notions like

Controllability, observability, stabilizability,...

become simpler, more general, more convincing.

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Controllability, observability, stabilizability,...

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For simplicity, we consider only
time-invariant, continuous-time systems with $\mathbb{T} = \mathbb{R}$

time-invariant := $[\![\, w \in \mathfrak{B} \,]\!] \Rightarrow [\![\, w(t' + \cdot) \in \mathfrak{B} \;\; \forall\, t' \in \mathbb{R} \,]\!]$.

Controllability

The time-invariant system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$ is said to be

controllable

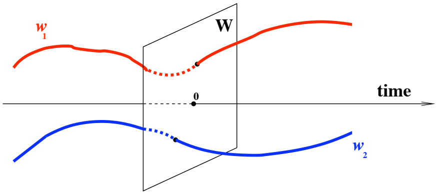
if for all $w_1, w_2 \in \mathfrak{B}\ \exists\ w \in \mathfrak{B}$ and $T \geq 0$ such that

$$w(t) = \begin{cases} w_1(t) & t < 0 \\ w_2(t-T) & t \geq T \end{cases}$$

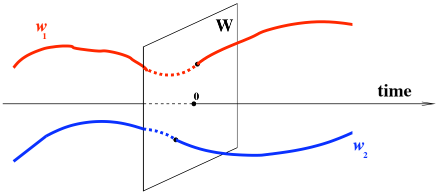
Controllability :$\Leftrightarrow$

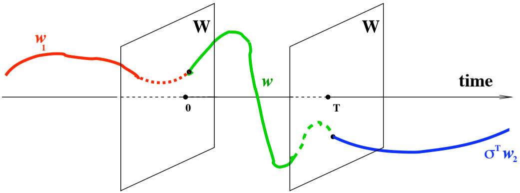
legal trajectories must be ‘patch-able’, ‘concatenable’.

Controllability



Controllability





Examples

$$\frac{d}{dt}x = Ax + Bu; \quad \frac{d}{dt}x = f \circ (x, u)$$

with $w = (x, u)$, controllable $\Leftrightarrow$ 'state point' controllable.

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likewise $\Leftrightarrow$ with $w = x$

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RLC circuit

Case 2: $CR_C = \frac{L}{R_L}$

$$\left(\frac{R_C}{R_L} + CR_C\frac{d}{dt}\right)(V_a - V_b) = \left(1 + CR_C\frac{d}{dt}\right)R_C I_a$$

$$I_a + I_b = 0$$

Assume also $R_C = R_L$. Controllable ?

$V_a - V_b = R_C I_a + constant \cdot e^{-\frac{t}{CR_C}}$. **Not controllable.**

Examples

$$\frac{d}{dt}x = Ax + Bu; \quad \frac{d}{dt}x = f \circ (x, u)$$

with $w = (x, u)$, controllable $\Leftrightarrow$ 'state point' controllable.

$$p(\frac{d}{dt})y = q(\frac{d}{dt})u$$

controllable $\Leftrightarrow$ p, q co-prime

Examples

$$\frac{d}{dt}x = Ax + Bu; \quad \frac{d}{dt}x = f \circ (x, u)$$

with $w = (x, u)$, controllable $\Leftrightarrow$ 'state point' controllable.

$$w = M(\frac{d}{dt})\ell$$

M a polynomial matrix, always has a controllable manifest behavior.

In fact, this characterizes the
controllable linear time-invariant differentiable systems ('image representation').

Note emergence of latent variables, ℓ.

Examples

$$w = M(\frac{d}{dt})\ell$$

M a polynomial matrix, always has a controllable manifest behavior. Likewise,

$$w = F(\frac{d}{dt})\ell$$

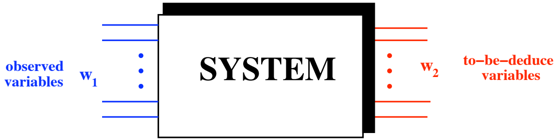
F matrix of rat. f'ns has controllable manifest behavior. But we need to give this 'differential equation' a meaning.

Whence

$$y = G(\frac{d}{dt})u, \quad w = \begin{bmatrix} u \\ y \end{bmatrix}$$

is always controllable.

Observability



¿ Is it possible to deduce w_2 from w_1 *and the model* $\mathfrak{B}$?

Observability

Consider the system $\Sigma = (\mathbb{T}, \mathbb{W}_1 \times \mathbb{W}_2, \mathfrak{B})$. Each element of $\mathfrak{B}$ hence consists of a pair of trajectories (w_1, w_2):

w_1 : observed;
w_2 : to-be-deduced.

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Definition: w_2 is said to be

observable from w_1

if $[\![(w_1, w_2') \in \mathfrak{B}$, and $(w_1, w_2'') \in \mathfrak{B}]\!] \Rightarrow [\![(w_2' = w_2'')]\!]$,
i.e., if on $\mathfrak{B}$, there exists a map $w_1 \mapsto w_2$.

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i.e., if on $\mathfrak{B}$, there exists a map $w_1 \mapsto w_2$.

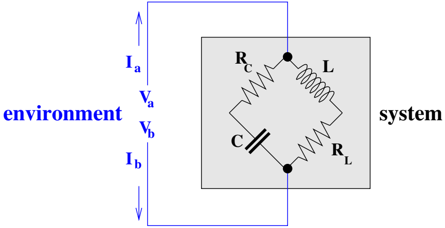
Very often manifest = observed, latent = to-be-deduced.
We then speak of an observable (latent variable) system.

Examples

$$\frac{d}{dt}x = Ax+Bu, y = Cx+Du; \quad \frac{d}{dt}x = f\circ(x,u), y = h\circ(x,u)$$

with $w_1 = (u,y)$**,** $w_2 = x$**, observable** $\Leftrightarrow$ **'state' observable.**

Examples

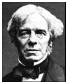


Controllability of this system (referring to external terminal variables) is a well-defined question.

Observability is not! No duality on the system's level.
Of course, there is a notion of $\mathfrak{B}^\perp$, and results connecting controllability of $\mathfrak{B}$ to state observability of $\mathfrak{B}^\perp$.

Examples

Equations for the full behavior:


Faraday


Ohm


Henry


Coulomb

Modules	Constitutive equations	
R_C	$I_1 + I_2 = 0$	$V_1 - V_2 = R_C I_1$
R_L	$I_7 + I_8 = 0$	$V_7 - V_8 = R_L I_7$
C	$I_5 + I_6 = 0$	$C \frac{d}{dt}(V_5 - V_6) = I_5$
L	$I_7 + I_8 = 0$	$V_7 - V_8 = L \frac{d}{dt} I_7$
connector1	$I_9 + I_{10} + I_{11} = 0$	$V_9 = V_{10} = V_{11}$
connector2	$I_{12} + I_{13} + I_{14} = 0$	$V_{12} = V_{13} = V_{14}$

Kirchhoff

Interconnection pair	Interconnection equations	
$\{10, 1\}$	$V_{10} = V_1$	$I_{10} + I_1 = 0$
$\{11, 7\}$	$V_{11} = V_7$	$I_{11} + I_7 = 0$
$\{2, 5\}$	$V_2 = V_5$	$I_2 + I_5 = 0$
$\{8, 3\}$	$V_8 = V_3$	$I_8 + I_3 = 0$
$\{6, 13\}$	$V_6 = V_{13}$	$I_6 + I_{13} = 0$
$\{4, 14\}$	$V_4 = V_{14}$	$I_4 + I_{14} = 0$

Examples

All these eq'ns combined define a latent variable system in the manifest variables

$$w = (V_a, I_a, V_b, I_b)$$

with latent variables

$$\ell = (V_1, I_1, V_2, I_2, V_3, I_3, V_4, I_4, V_5, I_5, V_6, I_6, V_7, I_7, V_8, I_8, V_{10}, I_{10}, V_{11}, I_{11}, V_{13}, I_{13}, V_{14}, I_{14}).$$

The manifest behavior $\mathfrak{B}$ is given by

$$\mathfrak{B} = \{(V_a, I_a, V_b, I_b) : \mathbb{R} \to \mathbb{R}^4 \mid \exists\ \ell : \mathbb{R} \to \mathbb{R}^{24} \ldots\}$$

Are the latent variables observable from the manifest ones?

$$\Leftrightarrow \quad CR_C \neq L/R_L$$

Examples

$$p(\frac{d}{dt})y = q(\frac{d}{dt})u$$

u is observable from y $\Leftrightarrow$ q = non-zero constant (no zeros).

A controllable linear time-invariant differential system always has an observable 'image' representation

$$w = M(\frac{d}{dt})\ell.$$

In fact, this again characterizes the controllable linear time-invariant differentiable systems.

Kalman definitions

<u>Special case</u>: classical Kalman definitions for

$$\frac{d}{dt}x = f \circ (x, u), \quad y = h \circ (x, u).$$

R.E. Kalman

Kalman definitions

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controllability: **variables = (input, state)**

If a system is not (state) controllable, why is it?

Insufficient influence of the control?

Or bad choice of the state?

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Insufficient influence of the control?

Or bad choice of the state?

observability: $\rightsquigarrow$ **observed = (input, output),**

to-be-deduced = state.

Why is it so interesting to try to deduce the state, of all things? The state is a derived notion, not a 'physical' one.

Stabilizability

The system $\Sigma = (\mathbb{T}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B})$ is said to be **stabilizable** if, for all $w \in \mathfrak{B}$, there exists $w' \in \mathfrak{B}$ such that

$$w(t) = w'(t) \text{ for } t < 0 \quad \text{and} \quad w'(t) \underset{t \to \infty}{\longrightarrow} 0.$$

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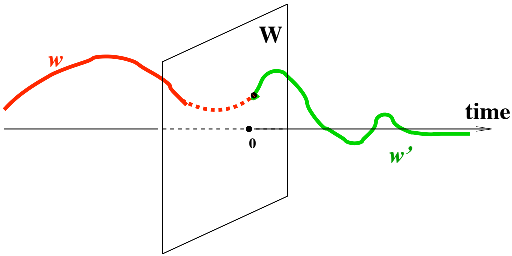
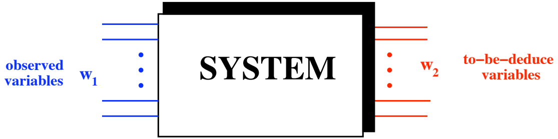
Stabilizability $:\Leftrightarrow$

legal trajectories can be steered to a desired point.

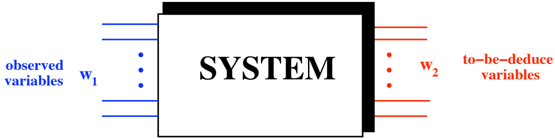


Detectability



¿ Is it possible to deduce w_2 asymptotically from w_1 ?

Detectability



¿ Is it possible to deduce w_2 asymptotically from w_1 ?

Definition: w_2 is said to be

detectable from w_1 if

$$[\![(w_1, w_2') \in \mathfrak{B}, \text{ and } (w_1, w_2'') \in \mathfrak{B}]\!]$$

$$\Rightarrow [\![(w_2' - w_2'') \to 0 \text{ for } t \to \infty]\!]$$

Summary

Btw

- **A model is not a map, but a relation.**

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- **A flow**

$$\frac{d}{dt}x = f(x) \text{ with or without } y = h(x)$$

is a very limited model class.

$\leadsto$ **closed dynamical systems.**

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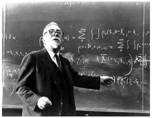
- **A flow is a very limited model class.**

  $\rightsquigarrow$ **closed dynamical systems.**
- **An open dynamical system is not an input/output map.**

Heaviside



Wiener

Nyquist

Bode

Btw

- **A model is not a map, but a relation.**
- **A flow is a very limited model class.**

  $\rightsquigarrow$ **closed dynamical systems.**
- **An open dynamical system is not an input/output map.**
- **input/state/output systems, although still limited, are the first class of suitably general models**

R.E. Kalman

Btw

- **A model is not a map, but a relation.**
- **A flow is a very limited model class.**
  $\rightsquigarrow$ closed dynamical systems.
- **An open dynamical system is not an input/output map.**
- **input/state/output systems, although still limited, are the first class of suitably general models**
- **Behaviors, including latent variables, are the first suitable general model class for physical applications and modeling by tearing, zooming, and linking**

Summary

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- **Important properties of dynamical systems**
  - **Controllability: concatenability of trajectories**
  - **Observability: deducing one trajectory from another**
  - **Stabilizability: driving a trajectory to zero**

Summary

- **A mathematical model = a subset**
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  = a family of trajectories
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- **Important properties of dynamical systems**
  - **Controllability: concatenability of trajectories**
  - **Observability: deducing one trajectory from another**
  - **Stabilizability: driving a trajectory to zero**
- **The behavior is all there is. All properties in terms of the behavior. Equivalence, representations also.**

Stochastic models

We only consider deterministic models. Stochastic models:

Laplace


Kolmogorov

there is a map P (the 'probability')

$$P : \mathfrak{A} \longrightarrow [0, 1]$$

with $\mathfrak{A}$ a 'σ-algebra' of subsets of $\mathfrak{U}$.

$P(\mathfrak{B}) =$ 'degree of certainty' (relative frequency, propensity, plausibility, belief) that outcomes are in $\mathfrak{B}$; $\cong$ the degree of validity of $\mathfrak{B}$ as a model.

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$\cong$ the degree of validity of $\mathfrak{B}$ as a model.

Determinism: P is a '$\{0,1\}$-law'

$$\mathfrak{A} = \{\varnothing, \mathfrak{B}, \mathfrak{B}^{\text{complement}}, \mathfrak{U}\}, P(\mathfrak{B}) = 1.$$

Fuzzy models

L. Zadeh

Fuzzy models: **there is a map μ (‘membership f’n’)**

$$\mu : \mathfrak{U} \longrightarrow [0,1]$$

$\mu(x) =$ **‘the extent to which x belongs to the model’s behavior’.**

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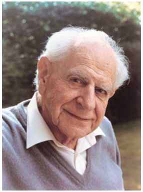
$\mu(x) =$ **'the extent to which x belongs to the model's behavior'.**

Determinism: μ is **'crisp'**:

$$\text{image}(\mu) = \{0, 1\},$$

$$\mathfrak{B} = \mu^{-1}(\{1\}) := \{x \in \mathfrak{U} \mid \mu(x) = 1\}$$

Every 'good' scientific theory is prohibition: it forbids certain things to happen...
The more a theory forbids, the better it is.



Karl Popper (1902-1994)

Replace 'scientific theory' by 'mathematical model' !