The Behavioral Approach
to
Systems Theory

Paolo Rapisarda, Un. of Southampton, U.K.
&
Jan C. Willems, K.U. Leuven, Belgium

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Lecture 2: Representations and annihilators of LTIDS

Lecturer: Jan C. Willems

Issues

- **What is a linear time-invariant differential system (LTIDS) ?**

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  - **Differential annihilators**
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- **Controllability, transfer functions, and image representations**

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  - **Differential annihilators**
  - **Rational annihilators**
- **Controllability, transfer functions, and image representations**
- **Representations using proper stable rational functions**

LTIDS

The class of systems

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$$[\![(w \in \mathfrak{B}) \wedge (t' \in \mathbb{R})]\!] \Rightarrow [\![\sigma^{t'} w \in \mathfrak{B})]\!]$$

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3. ***differential,*** **meaning**
   $\mathfrak{B}$ **consists of the sol'ns of a system of diff. eq'ns.**

The class of systems

w variables: $w_1, w_2, \ldots w_w$,
up to n-times differentiated,
g equations. $\leadsto$

$$\begin{array}{lcl}
\Sigma_{j=1}^{w} R_{1,j}^{0} w_j + \Sigma_{j=1}^{w} R_{1,j}^{1} \frac{d}{dt} w_j + \cdots + \Sigma_{j=1}^{w} R_{1,j}^{n} \frac{d^n}{dt^n} w_j & = & 0 \\
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\vdots \qquad\qquad \vdots & \vdots & \\
\Sigma_{j=1}^{w} R_{g,j}^{0} w_j + \Sigma_{j=1}^{w} R_{g,j}^{1} \frac{d}{dt} w_j + \cdots + \Sigma_{j=1}^{w} R_{g,j}^{n} \frac{d^n}{dt^n} w_j & = & 0
\end{array}$$

Coefficients R^k: 3 indices!

- $i = 1, \ldots, g$: **for the i-th differential equation,**
- $j = 1, \ldots, w$: **for the variable w_j involved,**
- $k = 1, \ldots, n$: **for the order $\frac{d^k}{dt^k}$ of differentiation.**

The class of systems

In vector/matrix notation:

$$w = \begin{bmatrix} w_1 \\ w_2, \\ \vdots \\ w_w \end{bmatrix}, \quad R_k = \begin{bmatrix} R^k_{1,1} & R^k_{1,2} & \cdots & R^k_{1,w} \\ R^k_{2,1} & R^k_{2,2} & \cdots & R^k_{2,w} \\ \vdots & \vdots & \cdots & \vdots \\ R^k_{g,1} & R^k_{g,2} & \cdots & R^k_{g,w} \end{bmatrix}.$$

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$$\boxed{R_0 w + R_1 \frac{d}{dt} w + \cdots + R_n \frac{d^n}{dt^n} w = 0,}$$

with $R_0, R_1, \cdots, R_n \in \mathbb{R}^{g \times w}$.

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with $R_0, R_1, \cdots, R_n \in \mathbb{R}^{g \times w}$. With polynomial matrix

$$R(\xi) = R_0 + R_1 \xi + \cdots + R_n \xi^n \in \mathbb{R}[\xi]^{g \times w}$$

we obtain the mercifully short notation

$$\boxed{R(\frac{d}{dt}) w = 0.}$$

Definition of the behavior

What shall we mean by the behavior of

$$R(\frac{d}{dt})w = 0 \text{ ?}$$

Solutions in $\mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$?
As many times differentiable as there appear derivatives appear in DE ?
Distributional solutions in $\mathcal{L}^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$?
In $\mathcal{L}_2^{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$?
Distributions?

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The easy way out

$$\mathfrak{B} := \{w \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \mid R(\frac{d}{dt})w(t) = 0 \; \forall\, t \in \mathbb{R}\}$$

Notation: $\mathfrak{B} = \ker(R(\frac{d}{dt}))$

Notation

$\mathbb{R}[\xi]$: **polynomials with real coeff., indeterminate** ξ

$\mathbb{R}[\xi]^{\mathtt{n}\times\mathtt{m}}$: **polynomial matrices**

$\mathbb{R}[\xi]^{\bullet\times\bullet}$: **appropriate number of rows, columns**

$\mathfrak{L}^{\mathtt{w}}, \mathfrak{L}^{\bullet}$: **linear differential systems**

$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}} := (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathfrak{B}) \in \mathfrak{L}^{\mathtt{w}}; \quad \mathfrak{B} = \ker(R(\frac{d}{dt}))$

$\mathbb{R}(\xi)$: **rational f'ns with real coeff., indeterminate** ξ

$\mathbb{R}(\xi)^{\mathtt{n}\times\mathtt{m}}$: **matrices of rat. f'ns**

$\mathbb{R}(\xi)^{\bullet\times\bullet}$: **appropriate number of rows, columns**

Rational symbols

We also want to give a meaning to

$$F\left(\frac{d}{dt}\right)w = 0$$

with $F \in \mathbb{R}(\xi)^{\bullet \times \mathtt{w}}$, i.e. a matrix of rational functions.
What do we mean by a solution?

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What do we mean by a solution?
We do this in terms of a left co-prime polynomial factorization.

$$F(\xi) = P(\xi)^{-1}Q(\xi)$$

with $P, Q \in \mathbb{R}[\xi]^{\bullet\times\bullet}$, $\det(P) \neq 0$, $\begin{bmatrix} P & Q \end{bmatrix}$ left prime.

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Define the behavior of this 'diff. eq'n' to be that of

$$Q(\frac{d}{dt})w = 0$$

Whence $\in \mathfrak{L}^\bullet$.

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One justification: Realize F as the t'f f'n of controllable system

$$\frac{d}{dt}x = Ax + Bu, \;\; y = Cx + D(\frac{d}{dt})u.$$

Consider 'output nulling' behavior

$$\frac{d}{dt}x = Ax + Bw, \;\; 0 = Cx + D(\frac{d}{dt})w.$$

This equals $Q(\frac{d}{dt})w = 0$

Elimination

Problem

Assume (w_1, w_2) governed by

$$R_1(\frac{d}{dt})w_1 = R_2(\frac{d}{dt})w_2$$

$R_1, R_2 \in \mathbb{R}[\xi]^{\bullet\times\bullet}$**. Behavior** $\mathfrak{B}$**. Obviously** $\mathfrak{B} \in \mathfrak{L}^\bullet$

Define the 'projection'

$$\mathfrak{B}_1 := \{w_1 \mid \exists\, w_2 \text{ such that } (w_1, w_2) \in \mathfrak{B}\}$$

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Does $\mathfrak{B}_1$ belong to $\mathfrak{L}^\bullet$?

Theorem: It does indeed, also with $R_1, R_2 \in \mathbb{R}(\xi)^{\bullet\times\bullet}$.

Algorithms?

Examples

The input/output behavior of

$$\frac{d}{dt}x = Ax + Bu, y = Cx + Du.$$

Every $\mathfrak{B} \in \mathfrak{L}^\bullet$ **admits such a representation** $\mathtt{w} \cong \begin{bmatrix} u \\ y \end{bmatrix}$.

Also representation

$$P(\frac{d}{dt})y = Q(\frac{d}{dt})u, \quad \mathtt{w} \cong \begin{bmatrix} u \\ y \end{bmatrix}.$$

Examples

The input/output behavior of

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The manifest behavior of

$$R(\frac{d}{dt})w = M(\frac{d}{dt})\ell, \quad R, M \in \mathbb{R}(\xi)^{\bullet \times \bullet}$$

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The port behavior of a circuit with (a finite number) linear resistors, capacitors, inductors, transformers, and gyrators.

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The port behavior of a circuit with (a finite number) linear resistors, capacitors, inductors, transformers, and gyrators.

Expect this to be a particular situation for LTIDS – but also holds for linear constant coefficient PDE's.

The annihilators

Polynomial annihilators

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, and $n \in \mathbb{R}[\xi]^{1\times\mathtt{w}}$.

Call n a **polynomial annihilator** of $\mathfrak{B}$:⇔

$$n(\frac{d}{dt})w = 0 \quad \forall\, w \in \mathfrak{B}, \text{ i.e. iff } n(\frac{d}{dt})\mathfrak{B} = 0.$$

Denote the set of annihilators by $\mathfrak{N}_{\mathfrak{B}}^{\mathbb{R}[\xi]}$.

The term **consequence** is also used.

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Easy: $\mathfrak{N}_{\mathfrak{B}}^{\mathbb{R}[\xi]}$ is an $\mathbb{R}[\xi]$-module. This means that

$$[\![n_1, n_2 \in \mathfrak{N}_{\mathfrak{B}}^{\mathbb{R}[\xi]} \text{ and } p \in \mathbb{R}[\xi]]\!]$$
$$\Rightarrow [\![n_1 + n_2 \in \mathfrak{N}_{\mathfrak{B}}^{\mathbb{R}[\xi]} \text{ and } pn_1 \in \mathfrak{N}_{\mathfrak{B}}^{\mathbb{R}[\xi]}]\!]$$

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Theorem:

1. Let $\mathfrak{B} = \ker(R(\frac{d}{dt}))$. Then $\mathfrak{N}_{\mathfrak{B}}^{\mathbb{R}[\xi]}$ is the $\mathbb{R}[\xi]$-module generated by the rows of R.
2. There is a 1:1 relation between $\mathfrak{L}^{\mathtt{w}}$ and the submodules of $\mathbb{R}[\xi]^{1\times \mathtt{w}}$, the correspondence being

$\mathfrak{B} \mapsto \mathfrak{N}_{\mathfrak{B}}^{\mathbb{R}[\xi]}$ $\qquad$ submodule $\mapsto \{w \mid n(\frac{d}{dt})w = 0 \ \forall n \in$ submodule$\}$

Properties of Polynomial Annihilators

Every submodule of $\mathbb{R}[\xi]^{1\times \mathtt{w}}$ is finitely generated.
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$R_1(\frac{d}{dt})w = 0$ **and** $R_2(\frac{d}{dt})w = 0$ **define the same system iff** $\exists\, F_1, F_2$ **such that** $R_2 = F_1 R_1$, $R_1 = F_2 R_2$

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$R(\frac{d}{dt})w = 0$ **has minimal number of rows among all kernel representations of same behavior iff** R **has full row rank.**

$R_1(\frac{d}{dt})w = 0$ **and** $R_2(\frac{d}{dt})w = 0$ **are minimal kernel repr. of the same system iff** $\exists$ **unimodular** F **such that** $R_2 = FR_1$.

$\rightsquigarrow$ **canonical forms, etc.**

Basically, therefore, polynomial kernel representations are unique up to unimodular pre-multiplication

Examples

$$p(\frac{d}{dt})w = 0 \quad p \in \mathbb{R}[\xi]$$

Polynomial annihilators: $q \in \mathbb{R}[\xi]$ with p as a factor: $\mathbb{R}[\xi]\,p$.

Canonical form: p monic.

There are also non-minimal representations, e.g.

$$p_1(\tfrac{d}{dt})w = 0$$

$$p_2(\tfrac{d}{dt})w = 0$$

with GCD(p_1, p_2)=p.

Exercise: What are the consequences of $\frac{d}{dt}w = Aw$?

Proof of elimination thm

'Fundamental principle'. When is the equation

$$F(x) = y \quad y \text{ given}, \quad x \text{ unknown}$$

solvable? In particular, when is

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Is this also sufficient, for a ‘small’ set of *N*’s? For example, for *F* a matrix. Then easy to see n.a.s.c. for solvability:

$$n \in \mathbb{R}^{\bullet}, \; nF = 0 \quad \Rightarrow \quad ny = 0$$

Proof of elimination thm

In particular, when is

$$F(\frac{d}{dt})x = y$$

solvable? N.a.s.c. for linear diff. eq'ns:

$$n(\frac{d}{dt})F(\frac{d}{dt}) = 0 \quad \Rightarrow \quad n(\frac{d}{dt})y = 0$$

These n's form a $\mathbb{R}[\xi]$-module: $n(\xi)$ such that $n(\xi)F(\xi) = 0$. Computable!

For what w's is $R(\frac{d}{dt})w = M(\frac{d}{dt})\ell$ solvable for ℓ?

Iff $nM = 0 \Rightarrow n(\frac{d}{dt})R(\frac{d}{dt})w = 0$.

$\rightsquigarrow$ condition $R'(\frac{d}{dt})w = 0$: elim'ion th'm + algorithm.

Proof of elimination thm

The fundamental principle and the elimination theorem also hold for linear constant coefficient PDE's!

Palamodov

Malgrange

Rational Annihilators

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, and $n \in \mathbb{R}(\xi)^{1\times\mathtt{w}}$.

Call n a **rational annihilator** of $\mathfrak{B}$ $:\Leftrightarrow$

$$n(\frac{d}{dt})w = 0 \ \forall w \in \mathfrak{B}, \text{ i.e. iff } n(\frac{d}{dt})\mathfrak{B} = 0.$$

Note what this means:

$n = p^{-1}\begin{bmatrix} q_1 & q_2 & \cdots & q_{\mathtt{w}} \end{bmatrix}$; $\quad p, q_1, q_2, \ldots, q_{\mathtt{w}}$ co-prime

$:\Leftrightarrow \quad q_1(\frac{d}{dt})w_1 + q_2(\frac{d}{dt})w_2 + \cdots + q_{\mathtt{w}}(\frac{d}{dt})w_{\mathtt{w}} = 0 \ \forall w \in \mathfrak{B}$.

Denote the set of rational annihilators by $\mathfrak{N}_{\mathfrak{B}}^{\mathbb{R}(\xi)}$.

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It is easy to see that $\mathfrak{N}_{\mathfrak{B}}^{\mathbb{R}(\xi)}$ is $\mathbb{R}[\xi]$-module. (Prove!)
But, now, a sub-module of $\mathbb{R}(\xi)^{1\times\mathtt{w}}$ viewed as a $\mathbb{R}[\xi]$-module.

Rational Annihilators

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$\mathfrak{B} \mapsto \mathfrak{N}_{\mathfrak{B}}^{\mathbb{R}(\xi)}$ submodule $\mapsto \{w \mid n(\frac{d}{dt})w = 0 \ \forall n \in$ submodule$\}$

Not a nice thm: refers to submodules of a vector space!

Examples

$$p(\frac{d}{dt})w = 0 \quad p \in \mathbb{R}[\xi]$$

Rational annihilators: $\frac{n_1}{n_2} \in \mathbb{R}(\xi)$ **with** n_1, n_2 **co-prime, and with** p **a factor of** n_1**.**

Examples

$$p(\frac{d}{dt})w_1 = q(\frac{d}{dt})w_2 \quad p, q \in \mathbb{R}[\xi]$$

Rational annihilators: $\frac{n_1}{n_2}\begin{bmatrix} p & -q \end{bmatrix}$**,**

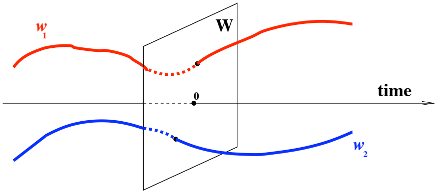
with $n_1, n_2 \in \mathbb{R}[\xi]$**, co-prime, and with** n_2, p, q **co-prime.**

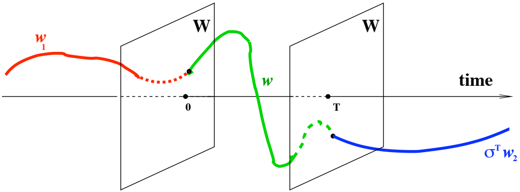
In the special case that p, q **are co-prime, this is actually the** $\mathbb{R}(\xi)$**-vector space generated by** $\begin{bmatrix} p & -q \end{bmatrix} \cong \begin{bmatrix} 1 & -\frac{q}{p} \end{bmatrix}$**!**

Why do we get a subspace instead of just a module?

Controllability & Stabilizability

Controllability

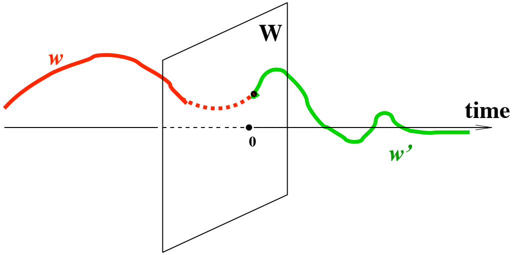




Stabilizability

Stabilizability :⇔

legal trajectories can be steered to a desired point.



Tests

Theorem:
$\mathfrak{B} = \ker(R(\frac{d}{dt})), R \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ **is controllable** $\Leftrightarrow$

$R(\lambda)$ **has the same rank for all** $\lambda \in \mathbb{C}$

Same result for rational symbols, but care should be taken in defining rank drop in situations where the symbol has zeros and poles in common points of the complex plane.

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$\mathfrak{B} = \ker(R(\frac{d}{dt})) \in \mathfrak{L}^{\mathtt{w}}$, $R \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ **is stabilizable** $\Leftrightarrow$

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Example 2: $y = G(\frac{d}{dt})u, w = \begin{bmatrix} u \\ y \end{bmatrix}$ is always controllable, and hence stabilizable.

Subspaces of annihilators

Characterization of controllability in terms of the structure of rational annihilators:

Theorem:

1. $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ **is controllable iff its rational annihilators** $\mathfrak{N}_{\mathfrak{B}}^{\mathbb{R}(\xi)}$ **form an** $\mathbb{R}(\xi)$**-subspace of** $\mathbb{R}(\xi)^{1\times\mathtt{w}}$**.**
2. **There is a one-to-one relation between the controllable systems in** $\mathfrak{L}^{\mathtt{w}}$ **and the** $\mathbb{R}(\xi)$**-subspaces of** $\mathbb{R}(\xi)^{1\times\mathtt{w}}$**.**

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The system

$$P(\frac{d}{dt})y = Q(\frac{d}{dt})u$$

is equal to

$$y = G(\frac{d}{dt})u \quad \text{with } G = P^{-1}Q$$

iff controllable (i.e., P, Q **left co-prime:** $\begin{bmatrix} P & Q \end{bmatrix}$ **left prime.**

Transfer functions deal with controllable systems (only) .

Kernels and images

Each element of $\mathfrak{L}^\bullet$ is by definition the kernel of a linear constant coefficient differential operator, i.e.

$$[\![\, \mathfrak{B} \in \mathfrak{L}^\bullet \,]\!] :\Leftrightarrow [\![\, \exists R \in \mathbb{R}[\xi]^{\bullet\times\bullet} \text{ such that } \mathfrak{B} = \ker(R(\tfrac{d}{dt})) \,]\!]$$

Consider the manifest behavior of

$$w = M(\frac{d}{dt})\ell, \text{ i.e. } \mathfrak{B} = \operatorname{im}(M(\frac{d}{dt}))$$

By the elimination theorem $\operatorname{im}(M(\frac{d}{dt})) \in \mathfrak{L}^\bullet$.

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Controllability !!

Image Representation

Theorem: **The following are equivalent for $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$:**

1. **it is controllable**
2. $\exists\, M \in \mathbb{R}[\xi]^{\bullet \times \bullet}$ **such that $\mathfrak{B}$ is the manifest behavior of**

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Controllable iff $\exists$ image representation. $\mathfrak{B} = \mathrm{im}(M(\frac{d}{dt}))$.

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We may assume WLOG these image repr. observable.

Controllable part

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The $\mathbb{R}(\xi)$-span of the rows of R in $\mathbb{R}(\frac{d}{dt})w = 0$ define the rational annihilators of the controllable part.

Prime representations

Primes in rings

A ring is closed under addition and multiplication.

Matrices, uni-modularity, etc.

Let $\mathcal{R}$ be a ring. A matrix $M \in \mathcal{R}^{\bullet \times \bullet}$ is left prime if $M = FM' \Rightarrow F$ is unimodular.

The matrices $M_1, M_2, \ldots, M_n, \in \mathcal{R}^{m \times \bullet}$ are said to be left coprime if $\begin{bmatrix} M_1 & M_2 & \cdots & M_n \end{bmatrix}$ is left prime.

There is an enormous zoology of rings with all sorts of properties...

Other rings

Consider

1. $\mathbb{R}[\xi]$: **polynomials**
2. $\mathbb{R}(\xi)$: **rational functions**
3. $\mathbb{R}(\xi)_{\text{proper}}$: **proper rational**
4. $\mathbb{R}(\xi)_{\text{proper/stable}}$: **proper (Hurwitz) stable rational**

These are all rings, with $\mathbb{R}(\xi)$ as field of fractions.
$\mathbb{R}(\xi)_{\text{proper/stable}}$ **is an Euclidean domain $\Rightarrow$ Bézout.**
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Every $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ admits by definition a 'kernel repr.' over $\mathbb{R}[\xi]$ i.e., $\exists\, R \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ such that $\mathfrak{B} = \ker(R(\frac{d}{dt}))$.

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How about the other rings? Should we care? Yes! Youla parametrization, dist. between systems, robustness, etc.

Ring representations

Relation between system properties and
prime representability over various rings.

Theorem: **Refers to 'kernel repr.' with rational symbols.**

1. $\mathfrak{B} \in \mathfrak{L}^\bullet$ **iff it admits a kernel representation with *R* in and left prime over** $\mathbb{R}(\xi)_{\text{proper}}$.
2. $\mathfrak{B} \in \mathfrak{L}^\bullet$ **is stabilizable iff it admits a kernel repr. with *R* in and left prime over** $\mathbb{R}(\xi)_{\text{proper/stable}}$.
3. $\mathfrak{B} \in \mathfrak{L}^\bullet$ **is controllable iff it admits a kernel representation with *R*** $\in \mathbb{R}[\xi]^{\bullet\times\bullet}$ **left prime over** $\mathbb{R}[\xi]$.

M. Vidyasagar

Unitary Representation

To close this lecture, a result on unitary representations.

Consider $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, controllable. Define $\mathfrak{B}_2 = \mathfrak{B} \cap \mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. $\mathfrak{B}_2$ is a closed linear subspace of $\mathcal{L}_2(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

Are there kernel or image representations that are adapted to this Hilbert space structure?

Unitary Representation

$G \in \mathbb{R}(\xi)^{\bullet\times\bullet}$, and consider the system

$$f_2 = G(\frac{d}{dt})f_1, \text{ with } f_1, f_2 \in \mathcal{L}_2(\mathbb{R}, \mathbb{R}^\bullet).$$

Is this a map $f_1 \mapsto f_2$?
If G is proper, no poles on the imaginary axis, then $f_2 = G(\frac{d}{dt})f_1$ defines a bounded linear operator from

$$f_1 \in \mathcal{L}_2(\mathbb{R}, \mathbb{R}^\bullet) \mapsto f_2 \in \mathcal{L}_2(\mathbb{R}, \mathbb{R}^\bullet).$$

Norm preserving ($:\Leftrightarrow ||f_1||^2 = ||f_2||^2$) iff

$$G^\top(-i\omega)G(i\omega) = I \quad \forall \omega \in \mathbb{R}.$$

Unitary Representation

$\mathfrak{B}$ (controllable) admits a rational kernel representation

$$R(\frac{d}{dt})w = 0$$

with R proper stable, left prime, and norm preserving.

$\mathfrak{B}$ (controllable) also admits a rational image representation

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Unitary Representation

$\mathfrak{B}$ **(controllable) admits a rational kernel representation**

$$R(\frac{d}{dt})w = 0$$

with R proper stable, left prime, and norm preserving.

Idea of proof: **start with minimal pol. repr. $R(\frac{d}{dt})w = 0$. Consider the polynomial matrix factorization equation**

$$R^\top(-\xi)R(\xi) = F^\top(-\xi)F(\xi).$$

Take Hurwitz sol'n H. Define the rational kernel repr.

$$G(\frac{d}{dt})w = 0 \quad \text{with} \quad G = RH^{-1}$$

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- Math. characterization of $\mathfrak{L}^{\mathtt{w}}$:
  - 1:1 relation between $\mathfrak{L}^{\mathtt{w}}$ and $\mathbb{R}[\xi]$-submodules
  - 1:1 relation between $\mathfrak{L}^{\mathtt{w}}_{\text{controllable}}$ and $\mathbb{R}(\xi)$-subspaces

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  - **1:1 relation between $\mathfrak{L}^{\mathtt{w}}_{\text{controllable}}$ and $\mathbb{R}(\xi)$-subspaces**
- **$\exists$ various more refined rational representations**

Discrete time systems

What changes for discrete time systems??

Ring

- **for $T = \mathbb{N}$ also $\mathbb{R}[\xi]$**
- **for $T = \mathbb{Z}$ instead $\mathbb{R}(\xi, \xi^{-1})$. This implies some differences.**

All major thms remain valid, mutatis mutandis.

Discrete time systems

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Take $\mathbb{T} = \mathbb{N}$. The following are equivalent.

- $\mathfrak{B}$ **linear, shift-inv., closed (pointwise conv.)**
- $\mathfrak{B}$ **linear, time-inv., complete ('prefix determined')**

$$:= [\![w \in \mathfrak{B}]\!] \Leftrightarrow [\![w_{[t_0,t_1]} \in \mathfrak{B}_{[t_0,t_1]} \ \forall t_0, t_1 \in \mathbb{N}]\!]$$

- $\exists\, R \in \mathbb{R}[\xi]^{\bullet\times\bullet}$ **(or** $\in \mathbb{R}(\xi)^{\bullet\times\bullet}$**) such that:**

$$\mathfrak{B} = \{ w : \mathbb{N} \to \mathbb{R}^{\bullet} \mid R(\sigma)w = 0 \}$$

- **and the many more traditional representations**