The Behavioral Approach to Systems Theory

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MTNS 2006
Kyoto, Japan, July 24–28, 2006

Lecture 3: State and state construction

Lecturer: Paolo Rapisarda

Outline

The axiom of state

Discrete-time systems

- First-order representations
- State maps
- The shift-and-cut map
- Algebraic characterization

Continuous-time systems

Computation of state-space representations

Questions

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- **How to construct a state from the equations?**
- **How to construct a state representation from the equations?**

The basic idea

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- **The state contains all the relevant information about the future behavior of the system**
- **The state is the memory of the system**
- **Independence of past and future given the state**

The axiom of state

$\boldsymbol{\Sigma} = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_{\text{full}})$ **is a *state system* if**

$$(w_1, x_1), (w_2, x_2) \in \mathfrak{B}_{\text{full}} \text{ and } x_1(T) = x_2(T)$$
$$\Downarrow$$
$$(w_1, x_1) \underset{T}{\wedge} (w_2, x_2) \in \mathfrak{B}_{\text{full}}$$

$\underset{T}{\wedge}$ **is *concatenation at* T:**

$$(f_1 \underset{T}{\wedge} f_2)(t) := \begin{cases} f_1(t) \text{ for } t < T \\ f_2(t) \text{ for } t \geq T \end{cases}$$

Graphically...

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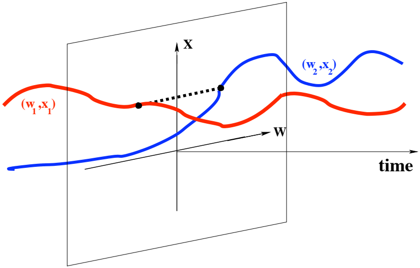

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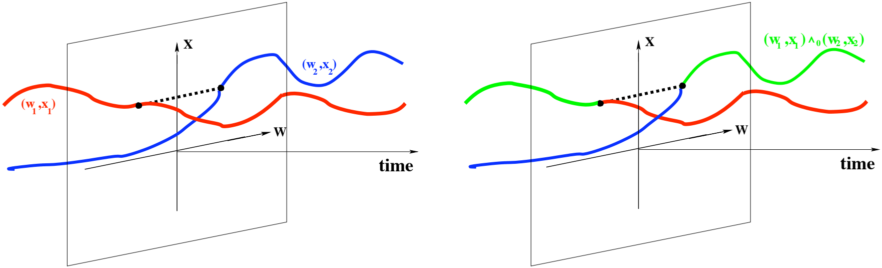

Example 1: discrete-time system

$\boldsymbol{\Sigma} = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{l}}, \mathfrak{B}_{\text{full}})$, with

$$\mathfrak{B}_{\text{full}} := \{(w, \ell) \mid F \circ (\sigma\ell, \ell, w) = 0\}$$

where

$$\sigma : (\mathbb{R}^{\mathtt{l}})^{\mathbb{Z}} \longrightarrow (\mathbb{R}^{\mathtt{l}})^{\mathbb{Z}}$$
$$(\sigma(\ell))(k) := \ell(k + 1)$$

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Special case: input-state-output equations

$$\begin{aligned} \sigma x &= f(x, u) \\ y &= h(x, u) \\ w &= (u, y) \end{aligned}$$

Example 2: continuous-time system

$\Sigma = (\mathbb{R}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{l}}, \mathfrak{B}_{\text{full}})$, with

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Special case: input-state-output equations

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First-order discrete-time representations

Theorem: **A ‘complete’ latent variable system**

$$\Sigma = (\mathbb{Z}, \mathbb{R}^{\mathtt{w}}, \mathbb{R}^{\mathtt{x}}, \mathfrak{B}_{\mathrm{full}})$$

is a state system iff $\mathfrak{B}_{\mathrm{full}}$ **can be described by**

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0-th order in *w*, 1st order in *x*

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1st order in *x* is equivalent to state property!

State construction: basic idea

Problem: **Given kernel or hybrid description, find a state representation**

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First **compute polynomial operator in the shift acting on system variables, inducing a state variable:**

$$X(\sigma)w = x$$

$$X(\sigma)\begin{bmatrix} w \\ \ell \end{bmatrix} = x$$

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Then **use original eqs. and** ***X*** **to obtain 1st order representation.**

State maps for kernel representations

$X \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi]$ induces a ***state map*** $X(\sigma)$ for $\mathtt{ker}(R(\sigma))$ if the behavior $\mathfrak{B}_{\text{full}}$ with latent variable x, described by

$$\begin{aligned} R(\sigma)w &= 0 \\ X(\sigma)w &= x \end{aligned}$$

satisfies the axiom of state.

Example

$$\mathfrak{B} = \{w \mid r(\sigma)w = 0\}$$

where $r \in \mathbb{R}[\xi]$, $\deg(r) = n$.

(Minimal) state map induced by

$$\begin{bmatrix} 1 \\ \xi \\ \vdots \\ \xi^{n-1} \end{bmatrix} \rightsquigarrow \begin{bmatrix} w \\ \sigma w \\ \vdots \\ \sigma^{n-1} w \end{bmatrix}$$

The axiom of state revisited

A *linear* system $\Sigma = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_{\text{full}})$ with latent variable x is a state system if

$$(w, x) \in \mathfrak{B}_{\text{full}} \text{ and } x(T) = 0$$

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$$(0, 0) \underset{T}{\wedge} (w, x) \in \mathfrak{B}_{\text{full}}$$

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- **Time-invariance $\Longrightarrow$ can choose $T = 0$;**

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- **Time-invariance $\Longrightarrow$ can choose $T = 0$;**
- **Concatenability with zero trajectory is key.**

When is $w \in \mathfrak{B}$ concatenable with zero?

$$R_0 w + R_1 \sigma w + \ldots + R_L \sigma^L w = 0$$

...	0	0	$w(0)$	$w(1)$	$w(2)$	$w(3)$	...
...	$k = -2$	$k = -1$	$k = 0$	$k = 1$	$k = 2$	$k = 3$	...

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$\ldots$	0	R_0	R_1	R_2	R_3	R_4	$\ldots$
$\ldots$	0	0	$w(0)$	$w(1)$	$w(2)$	$w(3)$	$\ldots$
$\ldots$	$k=-2$	$k=-1$	$k=0$	$k=1$	$k=2$	$k=3$	$\ldots$

$$R_0 w(0) + R_1 w(1) + \ldots + R_L w(L) = 0$$

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$\ldots$	R_0	R_1	R_2	R_3	R_4	R_5	$\ldots$
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The shift-and-cut map

$$\sigma_+ : \mathbb{R}[\xi] \rightarrow \mathbb{R}[\xi]$$

$$\sigma_+\left(\sum_{i=0}^{n} p_i \xi^i\right) := \sum_{i=0}^{n-1} p_{i+1} \xi^i$$

"Divide by ξ and take polynomial part"

Extended componentwise to vectors and matrices

Example

$$R(\xi) = R_0 + R_1\xi + \ldots + R_{L-1}\xi^{L-1} + R_L\xi^L$$

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$$\vdots = \vdots$$

$$\sigma_+^L(R(\xi)) = R_L$$

Shift-and-cut and concatenability with zero

***w* is concatenable with zero** $\Leftrightarrow$

$$\begin{aligned} (\sigma_+(R)(\sigma)w)(0) &= 0 \\ (\sigma_+^2(R)(\sigma)w)(0) &= 0 \\ \vdots &= \vdots \\ (\sigma_+^L(R)(\sigma)w)(0) &= 0 \end{aligned}$$

$\text{col}((\sigma_+^i(R))_{i=1,\ldots,L}(\sigma)$ is a state map!

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Example: scalar systems

$$r_0 w + r_1 \sigma w + \ldots + \sigma^n w = 0$$

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Observe w concatenable with zero iff $w = 0$. Indeed,

$$\begin{aligned}
\sigma_+^n(r)(\sigma)w &= w \\
\sigma_+^{n-1}(r)(\sigma)w &= r_{n-1}w + \sigma w \\
\vdots &= \vdots \\
\sigma_+(r)(\sigma)w &= r_1 w + \ldots + \sigma^{n-1} w
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\end{aligned}$$

Zero at $t = 0$ iff $(\sigma^k w)(0) = 0$ for $k = 0, \ldots, n-1$.

From kernel representation to state map

Denote $\text{col}((\sigma_+^i(R))_{i=1,\ldots,L} =: \Sigma_R$.

Theorem: Let $\mathfrak{B} = \mathtt{ker}(R(\sigma))$. Then

$$\begin{aligned} R(\sigma)w &= 0 \\ \Sigma_R(\sigma)w &= x \end{aligned}$$

is a state representation of $\mathfrak{B}$ with state variable x.

Algebraic characterization

Theorem: Let $\mathfrak{B} = \mathtt{ker}(R(\sigma))$, and define Σ_R as above. Then

$$\Xi_R := \{f \in \mathbb{R}^{1\times \mathtt{w}}[\xi] \;\; | \; \exists \; g \in \mathbb{R}^{1\times \bullet}[\xi], \alpha \in \mathbb{R}^{1\times \bullet} \text{ s.t. } f = \alpha \Sigma_R + gR\}$$

is a vector space over $\mathbb{R}$.

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$X \in \mathbb{R}^{\bullet\times \mathtt{w}}[\xi]$ is state map for $\mathfrak{B}$ iff row span$(X) = \Xi_R$

X is minimal if and only if its rows are a basis for Ξ_R.

Example

$$(\sigma^2 + 2\sigma + 3)y = (\sigma + 3)u$$

$$\left[\xi^2 + 2\xi + 3 \quad | \quad -\xi - 3\right]$$

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$$\sigma_+ \rightsquigarrow \left[\xi + 2 \quad | \quad -1\right] \rightsquigarrow \left[\sigma + 2 \quad | \quad -1\right]$$

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$$\sigma_+ \rightsquigarrow \left[\xi + 2 \quad | \quad -1\right] \rightsquigarrow \left[\sigma + 2 \quad | \quad -1\right]$$

If $(y, u) \in \mathfrak{B}$, then for all $g \in \mathbb{R}[\xi]$

$$\begin{aligned} \left[\sigma + 2 \quad | \quad -1\right] \begin{bmatrix} y \\ u \end{bmatrix} &= \left[\sigma + 2 \quad | \quad -1\right] \begin{bmatrix} y \\ u \end{bmatrix} \\ &+ \underbrace{g(\sigma)\left[\sigma^2 + 2\sigma + 3 \quad | \quad -\sigma - 3\right]}_{=0 \text{ on } \mathfrak{B}} \begin{bmatrix} y \\ u \end{bmatrix} \end{aligned}$$

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‘equivalence modulo *R*’

Example

$$(\sigma^2 + 2\sigma + 3)y = (\sigma + 3)u \qquad \begin{bmatrix} \xi^2 + 2\xi + 3 & | & -\xi - 3 \end{bmatrix}$$

$$\begin{aligned} \sigma_+ &\rightsquigarrow \begin{bmatrix} \xi + 2 & | & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \sigma + 2 & | & -1 \end{bmatrix} \\ \sigma_+^2 &\rightsquigarrow \begin{bmatrix} 1 & | & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & | & 0 \end{bmatrix} \end{aligned}$$

Example

$(\sigma^2 + 2\sigma + 3)y = (\sigma + 3)u$ $\quad$ $\begin{bmatrix} \xi^2 + 2\xi + 3 & \mid & -\xi - 3 \end{bmatrix}$

$$\sigma_+ \rightsquigarrow \begin{bmatrix} \xi + 2 & \mid & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} \sigma + 2 & \mid & -1 \end{bmatrix}$$
$$\sigma_+^2 \rightsquigarrow \begin{bmatrix} 1 & \mid & 0 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & \mid & 0 \end{bmatrix}$$

If $(y, u) \in \mathfrak{B}$, then for all $g \in \mathbb{R}[\xi]$

$$\begin{bmatrix} 1 & \mid & 0 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} 1 & \mid & 0 \end{bmatrix} \begin{bmatrix} y \\ u \end{bmatrix} + \underbrace{g(\sigma) \begin{bmatrix} \sigma^2 + 2\sigma + 3 & \mid & -\sigma - 3 \end{bmatrix}}_{=0 \text{ on } \mathfrak{B}} \begin{bmatrix} y \\ u \end{bmatrix}$$

Example

$$(\sigma^2 + 2\sigma + 3)y = (\sigma + 3)u \qquad \left[\xi^2 + 2\xi + 3 \quad | \quad -\xi - 3\right]$$

$$\sigma_+ \rightsquigarrow \left[\xi + 2 \quad | \quad -1\right] \rightsquigarrow \left[\sigma + 2 \quad | \quad -1\right]$$

$$\sigma_+^2 \rightsquigarrow \left[1 \quad | \quad 0\right] \rightsquigarrow \left[1 \quad | \quad 0\right]$$

$$\begin{aligned} \Xi_R = \{\alpha \left[\xi + 2 \quad | \quad -1\right] + g(\xi) \left[\xi^2 + 2\xi + 3 \quad | \quad -\xi - 3\right], \\ \beta \left[1 \quad | \quad 0\right] + f(\xi) \left[\xi^2 + 2\xi + 3 \quad | \quad -\xi - 3\right] \\ \alpha, \beta \in \mathbb{R}, f, g \in \mathbb{R}[\xi]\} \end{aligned}$$

Example

$(\sigma^2 + 2\sigma + 3)y = (\sigma + 3)u$ $\left[\xi^2 + 2\xi + 3 \;\; | \;\; -\xi - 3\right]$

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$$\Xi_R = \{\alpha \left[\xi + 2 \;\; | \;\; -1\right] + g(\xi) \left[\xi^2 + 2\xi + 3 \;\; | \;\; -\xi - 3\right],$$
$$\beta \left[1 \;\; | \;\; 0\right] + f(\xi) \left[\xi^2 + 2\xi + 3 \;\; | \;\; -\xi - 3\right]$$
$$\alpha, \beta \in \mathbb{R}, f, g \in \mathbb{R}[\xi]\}$$

Any set of generators of $\Xi_R \rightsquigarrow$ a state map

Outline

On the space of solutions

$\mathfrak{C}^\infty$-solutions to $R(\frac{d}{dt})w = 0$ too small $\rightsquigarrow \mathcal{L}_1^{\text{loc}}$

Equality in the sense of distributions:

$R(\frac{d}{dt})w = 0 \quad \Leftrightarrow \quad \int_{-\infty}^{+\infty} w(t)^\top (R(-\frac{d}{dt})^\top f)(t)dt = 0$

for all testing functions f.

The axiom of state revisited

$\Sigma = (\mathbb{T}, \mathbb{W}, \mathbb{X}, \mathfrak{B}_{\text{full}})$ **is a state system if**

$$(w_1, x_1), (w_2, x_2) \in \mathfrak{B}_{\text{full}} \text{ and } x_1(T) = x_2(T)$$

and x_1, x_2 continuous at T

$$\Downarrow$$

$$(w_1, x_1) \underset{T}{\wedge} (w_2, x_2) \in \mathfrak{B}_{\text{full}}$$

'State map' $\rightsquigarrow X(\frac{d}{dt})$

From kernel representation to state map

Denote $\text{col}((\sigma_+^i(R)))_{i=1,\ldots,L} =: \Sigma_R$.

Theorem: **Let** $\mathfrak{B} = \texttt{ker}(R(\frac{d}{dt}))$. **Then**

$$\begin{aligned} R(\frac{d}{dt})w &= 0 \\ \Sigma_R(\frac{d}{dt})w &= x \end{aligned}$$

is a state representation of $\mathfrak{B}$ **with state variable** x.

¿How to prove it?

When is $w \in \mathfrak{B}$ concatenable with zero?

$$0 \underset{0}{\wedge} w \in \mathfrak{B} \iff \int_{-\infty}^{+\infty} (0 \underset{0}{\wedge} w)(t)^\top (R(-\frac{d}{dt})^\top f)(t) dt = 0$$

$$\iff \int_{0}^{+\infty} w(t)^\top (R(-\frac{d}{dt})^\top f)(t) dt = 0$$

for all testing functions f

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$$\iff \int_{0}^{+\infty} w(t)^\top (R(-\frac{d}{dt})^\top f)(t)dt = 0$$

for all testing functions f

Integrating repeatedly by parts on f yields:

$$\sum_{k=1}^{\deg(R)} \sum_{j=k}^{\deg(R)} (-1)^{k-1} (\tfrac{d^{j-k}}{dt^{j-k}} w)(0)^\top R_j^\top (\tfrac{d^{k-1}}{dt^{k-1}} f)(0) + \int_0^{+\infty} (R(\tfrac{d}{dt}) w)(t)^\top f(t)dt = 0$$

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$$+ \int_0^{+\infty} (R(\frac{d}{dt})w)(t)^\top f(t)dt = 0$$

$w \in \mathfrak{B}$ concatenable with zero if and only if...

$$\sum_{k=1}^{\deg(R)} \sum_{j=k}^{\deg(R)} (-1)^{k-1} \left(\frac{d^{j-k}}{dt^{j-k}} w\right)(0)^\top R_j^\top \left(\frac{d^{k-1}}{dt^{k-1}} f\right)(0) = 0$$

$$\Updownarrow$$

$$\begin{bmatrix} f(0) \\ \left(\frac{d}{dt} f\right)(0) \\ \vdots \\ (-1)^{\deg(R)-1} \left(\frac{d^{\deg(R)-1}}{dt^{\deg(R)-1}} f\right)(0) \end{bmatrix}^\top \left(\Sigma_R\left(\frac{d}{dt}\right) w\right)(0) = 0$$

$$\Updownarrow$$

$$\left(\Sigma_R\left(\frac{d}{dt}\right) w\right)(0) = 0$$

The shift-and-cut state map!

Outline

From kernel representation to state representation

$$R \in \mathbb{R}^{g\times w}[\xi] \rightsquigarrow \textbf{state map } X \in \mathbb{R}^{n\times w}[\xi]$$

Find:

$E, F \in \mathbb{R}^{(n+g)\times n}$, $G \in \mathbb{R}^{(n+g)\times w}$

$T \in \mathbb{R}^{(n+g)\times g}[\xi]$ **with** $\mathrm{rank}(T(\lambda)) = g \;\forall \lambda \in \mathbb{C}$

satisfying

$$E\xi X(\xi) + FX(\xi) + G = T(\xi)R(\xi)$$

Linear equations, Gröbner bases computations!

From I/O representation to I/O/S representation

I/O representation $R = \begin{bmatrix} P & -Q \end{bmatrix}$ $\rightsquigarrow$ **state map** $\begin{bmatrix} X_y & X_u \end{bmatrix}$

Find:

$A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times p}$, $D \in \mathbb{R}^{p \times m}$

$T \in \mathbb{R}^{(n+p) \times p}[\xi]$ with $\mathrm{rank}(T(\lambda)) = g \ \forall \lambda \in \mathbb{C}$

satisfying

$$\begin{bmatrix} \xi X_y(\xi) & \xi X_u(\xi) \\ I_p & 0 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} X_y(\xi) & X_u(\xi) \\ 0 & I_m \end{bmatrix} + T(\xi) R(\xi)$$

On the choice of state map

State map
+
system equations

$\rightsquigarrow$

state-space equations

On the choice of state map

State map + **system equations** $\rightsquigarrow$ **state-space equations**

On the choice of state map

State map
+
system equations $\rightsquigarrow$ **state-space equations**

$$\left(\frac{d^2}{dt^2} + 2\frac{d}{dt} + 3\right)y = \left(\frac{d}{dt} + 3\right)u$$

$$\begin{bmatrix} \xi^2 + 2\xi + 3 & -\xi - 3 \end{bmatrix}$$

On the choice of state map

State map
+
system equations $\rightsquigarrow$ **state-space equations**

$$\left(\frac{d^2}{dt^2} + 2\frac{d}{dt} + 3\right)y = \left(\frac{d}{dt} + 3\right)u \qquad \begin{bmatrix}\xi^2 + 2\xi + 3 & -\xi - 3\end{bmatrix}$$

Take $X(\xi) = \begin{bmatrix} 1 & 0 \\ \xi + 2 & -1 \end{bmatrix}$ **('reverse shift-and-cut').**

Then

$$A = \begin{bmatrix} -2 & 1 \\ -3 & 0 \end{bmatrix} \quad B = \begin{bmatrix} -1 \\ -3 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

'observer canonical form'

On the choice of state map

State map
+
system equations $\rightsquigarrow$ **state-space equations**

$$\left(\frac{d^2}{dt^2} + 2\frac{d}{dt} + 3\right)y = \left(\frac{d}{dt} + 3\right)u \qquad \begin{bmatrix}\xi^2 + 2\xi + 3 & -\xi - 3\end{bmatrix}$$

Take $X(\xi) = \begin{bmatrix} 1 & 0 \\ \xi & -1 \end{bmatrix}$. **Then**

$$A = \begin{bmatrix} 0 & 1 \\ -3 & -2 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \quad D = \begin{bmatrix} 0 \end{bmatrix}$$

‘observable canonical form’

Summary

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- **Axiom of state**

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Summary

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- **State maps $\rightsquigarrow$ state-space equations**
- **Algorithms!**