The Behavioral Approach
to
Systems Theory

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Lecture 4: Bilinear and quadratic differential forms

Lecturer: Paolo Rapisarda

Part I: Basics

Outline

Dynamics and functionals in systems and control

Instances: **Lyapunov theory, performance criteria, etc.**

Linear case $\Longrightarrow$ *quadratic* and *bilinear* functionals.

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However, tearing and zooming $\not\Longrightarrow$ state space eq.s

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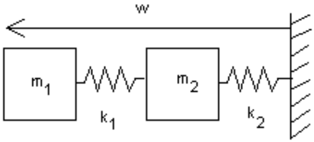
Usually: state-space equations, constant functionals.

However, tearing and zooming $\not\Longrightarrow$ **state space eq.s**

¡High-order differential equations!

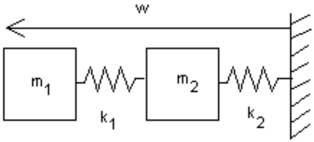
...involving also ***latent variables*...**

Example : a mechanical system



$$
\begin{aligned}
m_1 \frac{d^2 w_1}{dt^2} + k_1 w_1 - k_2 w_2 &= 0 \\
-k_1 w_1 + m_2 \frac{d^2 w_2}{dt^2} + (k_1 + k_2) w_2 &= 0
\end{aligned}
$$

Example : a mechanical system



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$$m_1 m_2 \frac{d^4}{dt^4} w_1 + (k_1 m_1 + k_2 m_1 + k_1 m_2) \frac{d^2}{dt^2} w_1 + k_1 k_2 w_1 = 0$$

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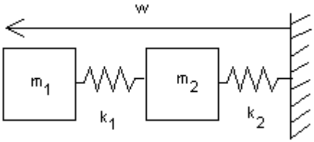

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¿Stability, stored energy, conservation laws?

Aim

An effective algebraic representation
of bilinear and quadratic functionals
of the system variables and their derivatives:

Operations/properties of functionals

$\Updownarrow$

algebraic operations/properties of representation

...a calculus of these functionals!

Outline

Motivation and aim

Definition

Two-variable polynomial matrices

The calculus of B/QDFs

Bilinear differential forms (BDFs)

$$\Phi := \left\{ \Phi_{k,\ell} \in \mathbb{R}^{w_1 \times w_2} \right\}_{k,\ell=0,\dots,L}$$

$$L_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{w_1}) \times \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{w_2}) \longrightarrow \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$$

$$L_\Phi(w_1, w_2) := \begin{bmatrix} w_1^\top & \frac{dw_1}{dt}^\top & \dots \end{bmatrix} \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots \\ \Phi_{1,0} & \Phi_{1,1} & \cdots \\ \vdots & \vdots & \cdots \\ \Phi_{k,0} & \Phi_{k,1} & \cdots \\ \vdots & \vdots & \cdots \end{bmatrix} \begin{bmatrix} w_2 \\ \frac{dw_2}{dt} \\ \vdots \end{bmatrix}$$

$$= \sum_{k,\ell} \left(\frac{d^k}{dt^k} w_1 \right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dt^\ell} w_2 \right)$$

Quadratic differential forms (QDFs)

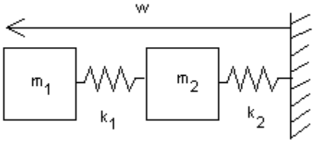
$$\Phi := \left\{\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}\right\}_{k,\ell=0,\ldots,L} \text{ symmetric, i.e. } \Phi_{k,\ell} = \Phi_{\ell,k}^{\top}$$

$$Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{w}}) \to \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$$

$$Q_\Phi(w) := \begin{bmatrix} w^\top & \frac{dw}{dt}^\top & \ldots \end{bmatrix} \begin{bmatrix} \Phi_{0,0} & \Phi_{0,1} & \cdots \\ \Phi_{1,0} & \Phi_{1,1} & \cdots \\ \vdots & \vdots & \cdots \\ \Phi_{k,0} & \Phi_{k,1} & \cdots \\ \vdots & \vdots & \cdots \end{bmatrix} \begin{bmatrix} w \\ \frac{dw}{dt} \\ \vdots \end{bmatrix}$$

$$= \sum_{k,\ell=0}^{L} \left(\frac{d^k}{dt^k} w\right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dt^\ell} w\right)$$

Example: total energy in mechanical system



$$\frac{1}{2}\left[m_1\left(\frac{d}{dt}w_1\right)^2 + m_2\left(\frac{d}{dt}w_2\right)^2\right] + \frac{1}{2}\left[k_1 w_1^2 + k_2 w_2^2\right]$$

$$\begin{bmatrix} w_1 & w_2 & \frac{d}{dt}w_1 & \frac{d}{dt}w_2 \end{bmatrix} \begin{bmatrix} \frac{1}{2}k_1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}k_2 & 0 & 0 \\ 0 & 0 & \frac{1}{2}m_1 & 0 \\ 0 & 0 & 0 & \frac{1}{2}m_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \frac{d}{dt}w_1 \\ \frac{d}{dt}w_2 \end{bmatrix}$$

Outline

Two-variable polynomial matrices for BDFs

$$\left\{\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}_1 \times \mathtt{w}_2}\right\}_{k,\ell=0,\ldots,L}$$

$$L_\Phi(w_1, w_2) = \sum_{k,\ell=0}^{L} \left(\frac{d^k}{dt^k} w_1\right)^\top \Phi_{k,\ell} \frac{d^\ell}{dt^\ell} w_2$$

$$\Phi(\zeta, \eta) = \textstyle\sum_{k,\ell=0}^{L} \Phi_{k,\ell}\, \zeta^k\, \eta^\ell$$

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2-variable polynomial matrix associated with L_Φ

Two-variable polynomial matrices for QDFs

$$\{\Phi_{k,\ell} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}\}_{k,\ell=0,\ldots,L} \text{ symmetric } (\Phi_{k,\ell} = \Phi_{\ell,k}^\top)$$

$$Q_\Phi(w) = \sum_{k,\ell=0}^{L} (\frac{d^k}{dt^k} w)^\top \Phi_{k,\ell} \frac{d^\ell}{dt^\ell} w$$

$$\Phi(\zeta, \eta) = \sum_{k,\ell=0}^{L} \Phi_{k,\ell} \zeta^k \eta^\ell$$

symmetric: $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^\top$

Example: total energy in mechanical system

$$Q_E(w_1, w_2) = \begin{bmatrix} w_1 & w_2 & \frac{d}{dt}w_1 & \frac{d}{dt}w_2 \end{bmatrix} \begin{bmatrix} \frac{1}{2}k_1 & 0 & 0 & 0 \\ 0 & \frac{1}{2}k_2 & 0 & 0 \\ 0 & 0 & \frac{1}{2}m_1 & 0 \\ 0 & 0 & 0 & \frac{1}{2}m_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ \frac{d}{dt}w_1 \\ \frac{d}{dt}w_2 \end{bmatrix}$$

$$E(\zeta, \eta) = \begin{bmatrix} \frac{1}{2}k_1 & 0 \\ 0 & \frac{1}{2}k_2 \end{bmatrix} + \begin{bmatrix} \frac{1}{2}\zeta\eta & 0 \\ 0 & \frac{1}{2}\zeta\eta \end{bmatrix}$$

Historical intermezzo

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stability tests (‘60s)

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path integrals (‘60s)

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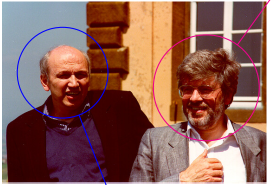
Historical intermezzo

Lyapunov functionals ('80s)



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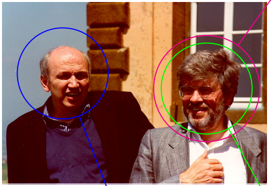  

path integrals ('60s)

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QDFs (1998)

Outline

The calculus of B/QDFs

Using powers of ζ and η as placeholders,

B/QDF $\leftrightsquigarrow$ two-variable polynomial matrix

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Using powers of ζ and η as placeholders,

B/QDF $\leftrightsquigarrow$ two-variable polynomial matrix

Operations and properties of B/QDF	$\leftrightsquigarrow$	**algebraic operations/properties on two-variable matrix**

Differentiation

$\Phi \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. $\overset{\bullet}{\Phi}$ **derivative** of Q_Φ:

$$Q_{\overset{\bullet}{\Phi}} : \mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \rightarrow \mathfrak{C}^\infty(\mathbb{R},\mathbb{R})$$

$$Q_{\overset{\bullet}{\Phi}}(w) := \frac{d}{dt}(Q_\Phi(w))$$

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$$\overset{\bullet}{\Phi}(\zeta,\eta) = (\zeta+\eta)\Phi(\zeta,\eta)$$

Two-variable version of Leibniz's rule

Integration

$\mathfrak{D}(\mathbb{R}, \mathbb{R}^\bullet)$ $\mathfrak{C}^\infty$**-compact-support trajectories**

$$L_\Phi : \mathfrak{D}(\mathbb{R}, \mathbb{R}^{w_1}) \times \mathfrak{D}(\mathbb{R}, \mathbb{R}^{w_2}) \longrightarrow \mathfrak{D}(\mathbb{R}, \mathbb{R})$$

$$\int L_\Phi : \mathfrak{D}(\mathbb{R}, \mathbb{R}^{w_1}) \times \mathfrak{D}(\mathbb{R}, \mathbb{R}^{w_2}) \longrightarrow \mathbb{R}$$

$$\int L_\Phi(w_1, w_2) := \int_{-\infty}^{+\infty} L_\Phi(w_1, w_2)dt$$

Analogous for QDFs

Part II: Applications

Outline

Lyapunov theory

Dissipativity theory

Balancing and model reduction

Nonnegativity and positivity along a behavior

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Prop.: Let $\mathfrak{B} = \texttt{ker} R(\frac{d}{dt})$. Then $Q_\Phi \overset{\mathfrak{B}}{\geq} 0$ iff there exist $D \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$, $X \in \mathbb{R}^{\bullet\times\mathtt{w}}[\zeta,\eta]$ such that

$$\Phi(\zeta,\eta) = \underbrace{D(\zeta)^\top D(\eta)}_{\geq 0 \text{ for all } w} + \underbrace{R(\zeta)^\top X(\zeta,\eta) + X(\eta,\zeta)^\top R(\eta)}_{=0 \text{ if evaluated on } \mathfrak{B}}$$

Lyapunov theory

$\mathfrak{B}$ autonomous is *asymptotically stable*
if $\lim_{t\to\infty} w(t) = 0 \ \forall \ w \in \mathfrak{B}$

$$\mathfrak{B} = \mathtt{ker} R(\tfrac{d}{dt}) \text{ stable} \iff \det(R) \text{ Hurwitz}$$

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Theorem: $\mathfrak{B}$ **asymptotically stable iff exists** Q_Φ **such that** $Q_\Phi \overset{\mathfrak{B}}{\geq} 0$ **and** $Q_{\overset{\bullet}{\Phi}} \overset{\mathfrak{B}}{<} 0$

Example

$$\mathfrak{B} = \mathtt{ker}\left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right)$$

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Example

$\mathfrak{B} = \texttt{ker}\left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right)$ $r(\xi) = \xi^2 + 3\xi + 2$

Choose $\Psi(\zeta,\eta)$ **s.t.** $Q_\Psi \overset{\mathfrak{B}}{<} 0$**, e.g.** $\Psi(\zeta,\eta) = -\zeta\eta$**;**

Find $\Phi(\zeta,\eta)$ **s.t.** $\frac{d}{dt}Q_\Phi(w) = Q_\Psi(w)$ **for all** $w \in \mathfrak{B}$**:**

$$(\zeta+\eta)\Phi(\zeta,\eta) = \Psi(\zeta,\eta) + \underbrace{r(\zeta)x(\eta) + x(\zeta)r(\eta)}_{=0 \text{ on } \mathfrak{B}}$$

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Equivalent to solving polynomial Lyapunov equation

$$0 = \underbrace{\Psi(-\xi,\xi)}_{\xi^2} + \underbrace{r(-\xi)}_{\xi^2-3\xi+2}x(\xi) + x(-\xi)\underbrace{r(\xi)}_{\xi^2+3\xi+2}$$

$$\rightsquigarrow x(\xi) = \frac{1}{6}\xi$$

Example

$\mathfrak{B} = \texttt{ker}\left(\frac{d^2}{dt^2} + 3\frac{d}{dt} + 2\right)$ $\quad$ $r(\xi) = \xi^2 + 3\xi + 2$

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$$\begin{aligned}\Phi(\zeta,\eta) &= \frac{-\zeta\eta + (\zeta^2+3\zeta+2)\frac{1}{6}\eta + \frac{1}{6}\zeta(\eta^2+3\eta+2)}{\zeta+\eta} \\ &= \frac{1}{6}\zeta\eta + \frac{1}{3} > 0\end{aligned}$$

State-space case

$$\left(\frac{d}{dt}I_{\mathtt{x}} - A\right)x = 0 \quad \rightsquigarrow \quad R(\xi) = \xi I_{\mathtt{x}} - A$$

- Choose $Q < 0$;
- Solve polynomial Lyapunov equation

$$(\xi I_{\mathtt{x}} - A)^\top P + P(\xi I_{\mathtt{x}} - A) = -A^\top P - PA = Q$$

  equivalent with *matrix* Lyapunov equation!
- Lyapunov functional is

$$x^\top(-P)x$$

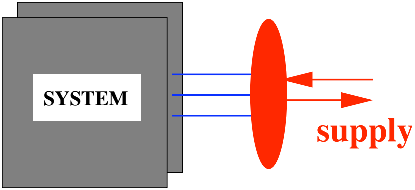
Outline

Lyapunov theory

Dissipativity theory

Balancing and model reduction

Dissipativity theory



Power is supplied

$\rightsquigarrow$ **energy is stored**

RLC circuits **Power** $V^\top I$

Storage in capacitors and inductors

Mechanical system **Power** $F^\top v + (\frac{d}{dt}\vartheta)^\top T$

Potential+kinetic

Setting the stage

Controllable system

$$w = M(\tfrac{d}{dt})\ell \rightsquigarrow M(\xi)$$

Power ('supply rate')

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$Q_{\Phi'}$ acts on free variable ℓ, i.e. $\mathfrak{C}^\infty$

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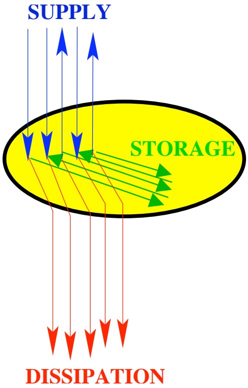
Dissipation inequality

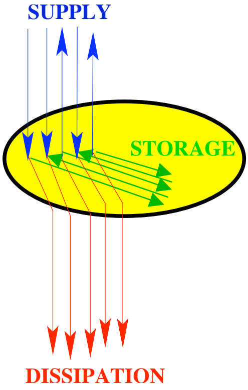


Dissipation inequality

Q_Ψ is storage function for the supply Q_Φ if

$$\frac{d}{dt} Q_\Psi \leq Q_\Phi$$

Rate of storage increase $\leq$ supply



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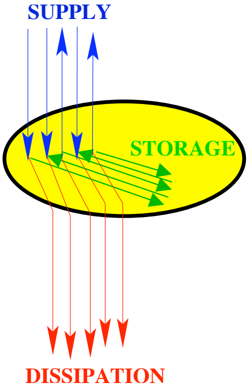
Q_Δ is **dissipation function** for Q_Φ if

$$Q_\Delta \geq 0 \text{ and } \int Q_\Delta dt = \int Q_\Phi dt$$



Characterizations of dissipativity

Theorem: **The following conditions are equivalent:**

- $\int_{-\infty}^{+\infty} Q_\Phi(\ell)dt \geq 0$ **for all** $\mathfrak{C}^\infty$ **compact-support** ℓ**;**
- Q_Φ **admits a storage function;**
- Q_Φ **admits a dissipation function**

Also, storage and dissipation functions are one-one:

$$\frac{d}{dt}Q_\Psi = Q_\Phi - Q_\Delta$$

$$(\zeta+\eta)\Psi(\zeta,\eta) = \Phi(\zeta,\eta) - \Delta(\zeta,\eta)$$

Example: mechanical systems

$$M\frac{d^2}{dt^2}q + D\frac{d}{dt}q + Kq = F \qquad \begin{bmatrix} F \\ q \end{bmatrix} = \begin{bmatrix} M\frac{d^2}{dt^2} + D\frac{d}{dt} + K \\ I_3 \end{bmatrix} \ell$$

Example: mechanical systems

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Supply rate: power

$$F^\top \left(\frac{d}{dt}q\right) = \left(M\frac{d^2}{dt^2}\ell + D\frac{d}{dt}\ell + K\ell\right)^\top \left(\frac{d}{dt}\ell\right)$$

corresponding to

$$\Phi(\zeta,\eta) = \frac{1}{2}(M\zeta^2 + D\zeta + K)^\top \eta + \frac{1}{2}\zeta(M\eta^2 + D\eta + K)$$

Example: mechanical systems

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$$\Phi(\zeta,\eta) = \tfrac{1}{2}(M\zeta^2 + D\zeta + K)^\top \eta + \tfrac{1}{2}\zeta(M\eta^2 + D\eta + K)$$

If dissipation inequality

$$\Phi(\zeta,\eta) = (\zeta+\eta)\Psi(\zeta,\eta) + \Delta(\zeta,\eta)$$

holds, then

$$\Phi(-\xi,\xi) = -\frac{1}{2}\xi^2(D^\top + D) = \Delta(-\xi,\xi)$$

$$\Longrightarrow \Delta(\zeta,\eta) = \frac{1}{2}(D^\top + D)\zeta\eta$$

Spectral factorization of $\Phi(-\xi,\xi)$ is key

Example: mechanical systems

$$M\frac{d^2}{dt^2}q + D\frac{d}{dt}q + Kq = F \qquad \begin{bmatrix} F \\ q \end{bmatrix} = \begin{bmatrix} M\frac{d^2}{dt^2} + D\frac{d}{dt} + K \\ I_3 \end{bmatrix} \ell$$

$$\Phi(\zeta,\eta) = \tfrac{1}{2}(M\zeta^2 + D\zeta + K)^\top \eta + \tfrac{1}{2}\zeta(M\eta^2 + D\eta + K)$$

$$\Delta(\zeta,\eta) = \tfrac{1}{2}(D^\top + D)\zeta\eta$$

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Storage function

$$\Psi(\zeta,\eta) = \frac{\Phi(\zeta,\eta) - \Delta(\zeta,\eta)}{\zeta+\eta} = \frac{1}{2}M\zeta\eta + \frac{1}{2}K$$

Total energy

Outline

Lyapunov theory

Dissipativity theory

Balancing and model reduction

Balancing

A minimal and stable realization (A, B, C, D) is balanced if exist $\sigma_i \in \mathbb{R}$ such that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n > 0$ and moreover

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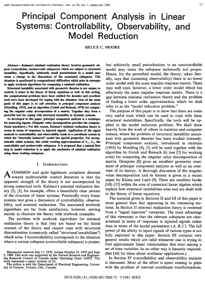
IEEE TRANSACTIONS ON AUTOMATIC CONTROL, VOL. AC-26, NO. 1, FEBRUARY 1981 17

Principal Component Analysis in Linear Systems: Controllability, Observability, and Model Reduction

BRUCE C. MOORE

Abstract—Kalman's minimal realization theory involves geometric objects (controllable, unobservable subspaces) which are subject to structural instability. Specifically, arbitrarily small perturbations in a model may cause a change in the dimensions of the associated subspaces. This situation is manifested in computational difficulties which arise in attempts to apply textbook algorithms for computing a minimal realization.

Structural instability associated with geometric theories is not unique to control; it arises in the theory of linear equations as well. In this setting, the computational problems have been studied for decades and excellent tools have been developed for coping with the situation. One of the main goals of this paper is to call attention to *principal component analysis* (Hotelling, 1933), and an algorithm (Golub and Reinsch, 1970) for computing the *singular value decomposition* of a matrix. Together they form a powerful tool for coping with structural instability in dynamic systems.

As developed in this paper, principal component analysis is a technique for analyzing signals. (Singular value decomposition provides the computational machinery.) For this reason, Kalman's minimal realization theory is recast in terms of responses to injected signals. Application of the signal analysis to controllability and observability leads to a coordinate system in which the "internally balanced" model has special properties. For asymptotically stable systems, this yields working approximations of X_c, $X_{\bar{o}}$, the controllable and unobservable subspaces. It is proposed that a natural first step in model reduction is to apply the mechanics of minimal realization using these working subspaces.

I. Introduction

A COMMON and quite legitimate complaint directed toward multivariable control literature is that the apparent strength of the theory is not accompanied by strong numerical tools. Kalman's minimal realization theory [2], [3], for example, offers a beautifully clear picture of the structure of linear systems. Practically every linear systems text gives a discussion of controllability, observability, and minimal realization. The associated textbook algorithms are far from satisfactory, however, serving mainly to illustrate the theory with textbook examples.

The problem with textbook algorithms for minimal realization theory is that they are based on the literal content of the theory and cannot cope with structural discontinuities (commonly called "structural instabilities") which arise. Uncontrollability corresponds to the situation where a certain subspace (controllable subspace) is proper, but arbitrarily small perturbations in an uncontrollable model may make the subspace technically not proper. Hence, for the perturbed model, the theory, taken literally, says that (assuming observability) there is no lower order model with the same impulse response matrix. There may well exist, however, a lower order model which has effectively the same impulse response matrix. There is a gap between minimal realization theory and the problem of finding a lower order approximation, which we shall refer to as the "model reduction problem."

The purpose of this paper is to show that there are some very useful tools which can be used to cope with these structural instabilities. Specifically, the tools will be applied to the model reduction problem. We shall draw heavily from the work of others in statistics and computer science, where the problem of structural instability associated with geometric theories has been studied intensely. Principal component analysis, introduced in statistics (1933) by Hotelling [4], [5] will be used together with the algorithm by Golub and Reinsch [6] (see [7] for working code) for computing the singular value decomposition of matrix. Dempster [8] gives an excellent geometric treatment of principal component analysis as well as an overview of its history. A thorough discussion of the singular value decomposition and its history is given in a recent paper by Klema and Laub [9]. There are excellent books [10]–[15] within the area of numerical linear algebra which explain how structural instabilities arise and are dealt with in the theory of linear equations.

The material given in Sections II and III of this paper is more general than that appearing in the remaining sections. In Section II minimal realization theory is reviewed from a "signal injection" viewpoint. The main advantage of this viewpoint is that the relevant subspaces are characterized in terms of responses to injected signals rather than in terms of the model parameters (A, B, C). The full power of the ability to inject signals of various types is not fully exploited in this paper. Section III contains very general results which are valid whenever one is trying to find approximate linear relationships that exist among a set of time variables. In no other way is linearity required. (See [16] for ideas about nonlinear applications.)

In Section IV controllability and observability analysis is discussed. Most of the effort is spent coming to grips with the problem of internal coordinate transformations.

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Balancing

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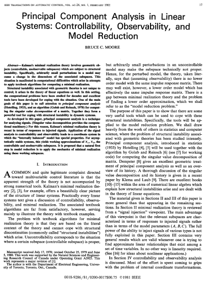
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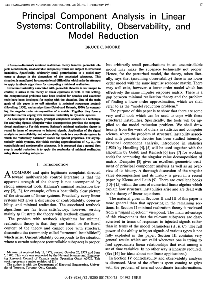
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Balancing $\equiv$ choice of basis of state space diagonalizing the Gramians

$\equiv$ choice of state map!

The controllability Gramian K

$$p(\tfrac{d}{dt})y = q(\tfrac{d}{dt})u \qquad \begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} q(\tfrac{d}{dt}) \\ p(\tfrac{d}{dt}) \end{bmatrix} \ell$$

where $GCD(p, q) = 1$, p stable, $\deg(q) \leq \deg(p) =: n$

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In state-space framework, K is defined as

$$\inf_u \int_{-\infty}^{0} u(t)^2 dt =: x_0^\top K x_0$$

where u is such that $x(-\infty) \rightsquigarrow x(0) = x_0$

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In our framework: let $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$. Then Q_K is QDF such that

$$\inf_{\ell'} \int_{-\infty}^{0} \left(p(\frac{d}{dt})\ell' \right)^2 dt =: Q_K(\ell)(0)$$

where $\ell' \in \mathfrak{C}^\infty(\mathbb{R}_+, \mathbb{R})$ is such that $\ell'_{|[0,+\infty)} = \ell_{|[0,+\infty)}$

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¿How to compute $K(\zeta,\eta)$?

Computation of $K(\zeta, \eta)$

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Since $p(-\xi)p(\xi) = p(\xi)p(-\xi)$, exists $K \in \mathbb{R}[\zeta, \eta]$ s.t.

$$p(\zeta)p(\eta) - p(-\zeta)p(-\eta) = (\zeta + \eta)K(\zeta, \eta)$$

Computation of $K(\zeta, \eta)$

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Consequently,

$$\int_{-\infty}^{0} \left(p(\frac{d}{dt})\ell' \right)^2 dt = \int_{-\infty}^{0} \left(p(-\frac{d}{dt})\ell' \right)^2 dt + Q_K(\ell')(0)$$

minimized for the ℓ' in $\texttt{ker}\; p(-\frac{d}{dt})$ with the given initial conditions.

Computation of $K(\zeta, \eta)$

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Q_K is quadratic function of the state:
for every state map $X(\frac{d}{dt})$ there exists K_X such that

$$Q_K(\ell) = \left(X(\frac{d}{dt})\ell \right)^\top K_X \left(X(\frac{d}{dt})\ell \right)$$

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In state-space framework, W is defined as

$$\int_0^{+\infty} y(t)^2 dt =: x_0^\top W x_0$$

where y is the free response emanating from $x(0) = x_0$

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In our framework: let $\ell \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R})$. Then Q_W is

$$Q_W(\ell)(0) := \int_0^{+\infty} \left(q(\frac{d}{dt})\ell'\right)^2 dt$$

where $\ell' \in \mathfrak{C}^\infty(\mathbb{R}_+, \mathbb{R})$ is such that

- $\ell'_{|(-\infty,0]} = \ell_{|(-\infty,0]}$
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- **If $\ell_{\mathtt{k}}$ is such that $X(\ell_{\mathtt{k}})(0)$ is the $\mathtt{k}$-th canonical basis vector, then**

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- $Q_W(\ell_1)(0) \geq Q_W(\ell_2)(0) \geq \ldots \geq Q_W(\ell_{\mathtt{n}})(0) > 0$

  or equivalently

  $$0 < Q_K(\ell_1)(0) \leq Q_K(\ell_2)(0) \leq \ldots \leq Q_K(\ell_{\mathtt{n}})(0)$$

  'first who contributes most'

Balancing with QDFs

Linear algebra $\Longrightarrow$ **there is basis** $\{x_i^b \in \mathbb{R}_{n-1}[\xi]\}_{i=1,\dots,n}$ **and** $\sigma_i \in \mathbb{R}$ **such that** $\sigma_1 \geq \sigma_2 \geq \dots \sigma_n$ **such that**

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$\sigma_i \rightsquigarrow$ **(classical) Hankel singular values**

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Model reduction by balancing follows

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- **Lyapunov theory, dissipativity, model reduction by balancing.**