The Behavioral Approach
to
Systems Theory

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MTNS 2006
Kyoto, Japan, July 24–28, 2006

Lecture 5: Multidimensional systems

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Part I: Basics

Outline

Motivation and aim

Basic definitions

Examples

Elimination of latent variables

Controllability

Motivation

In many situations, system variables depend not only on time but also on space:

- **Heat diffusion processes**
- **Electromagnetism**
- **Vibration of structures**
- **...**

¿How to incorporate these systems in the behavioral framework ?

Aim

Develop a behavioral framework for systems described by Partial Differential Equations (PDEs).

Issues:

- **Definitions consistent with 1-D case and basic tenets of behavioral approach**
- **Calculus of representations**
- **System properties, B/QDFs, etc.**

Outline

Linear differential distributed systems

$$\boldsymbol{\Sigma = (\mathbb{T}, \mathbb{W}, \mathfrak{B})}$$

$\mathbb{T}$: **independent variables**, $\mathbb{T} = \mathbb{R}^{\mathtt{n}}$ **with** $\mathtt{n} > 1$

$\mathbb{W}$: **external variables**, $\mathbb{W} = \mathbb{R}^{\mathtt{w}}$

$\mathfrak{B} \subseteq (\mathbb{W})^{\mathbb{T}}$: **behavior, solution set of system of linear, constant-coefficient PDEs**

w $\in \mathfrak{B} \implies$ ***w*** **is compatible with the dynamics**

The behavior

$\mathfrak{B}$ **is a n-D linear differential behavior if it is**

linear: $w_1, w_2 \in \mathfrak{B} \Rightarrow \alpha_1 w_1 + \alpha_2 w_2 \in \mathfrak{B}\ \forall\ \alpha_1, \alpha_2 \in \mathbb{R}$;

shift-invariant: $w \in \mathfrak{B} \Rightarrow \sigma^x w \in \mathfrak{B}$, **where** $x = (x_1, \ldots, x_\mathtt{n})$ **and**

$$(\sigma^x w)(x'_1, \ldots, x'_\mathtt{n}) := w(x_1 + x'_1, \ldots, x_\mathtt{n} + x'_\mathtt{n})$$

differential: $\mathfrak{B}$ **is solution set of a system of PDEs.**

Notation: $\mathfrak{B} \in \mathfrak{L}^\mathtt{w}_\mathtt{n}$

Outline

Vibrating membrane (**2** − ***D*** wave) equation

$\mathbb{T}$ **(independent variables):** $\mathbb{R} \times \mathbb{R}^2$

$\mathbb{W}$ **(dependent variables):** $\mathbb{R}$

$$\mathfrak{B} := \{w \text{ satisfying } \rho_0 \frac{\partial^2 w}{\partial t^2} - \tau^2 \nabla^2 w = 0\}$$

ρ_0= mass density; τ=magnitude of tensile force

Maxwell's equations

Set $\mathbb{T}$ **of independent variables:** $\mathbb{R} \times \mathbb{R}^3$

Set $\mathbb{W}$ **of dependent variables:** $\mathbb{R}^{10}$

$\mathfrak{B} := \{(\vec{E}, \vec{B}, \vec{j}, \rho)$ **satisfying Maxwell's equations**$\}$

$$
\begin{aligned}
\nabla \cdot \vec{E} - \frac{1}{\varepsilon_0}\rho &= 0 \\
\nabla \times \vec{E} + \frac{\partial}{\partial t}\vec{B} &= 0 \\
\nabla \cdot \vec{B} &= 0 \\
c^2 \nabla \times \vec{B} - \frac{1}{\varepsilon_0}\vec{j} - \frac{\partial}{\partial t}\vec{E} &= 0
\end{aligned}
$$

$\mathtt{n}$-variable polynomial matrices

$R \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \ldots, \xi_{\mathtt{n}}]$ **induces**

$$R(\tfrac{\partial}{\partial x_1}, \cdots, \tfrac{\partial}{\partial x_{\mathtt{n}}})w = 0$$

a kernel representation of

$$\mathfrak{B} := \{w \in \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}}) \mid R(\tfrac{\partial}{\partial x_1}, \cdots, \tfrac{\partial}{\partial x_{\mathtt{n}}})w = 0\}$$

$\mathfrak{C}^\infty$ **solutions mainly (but not only!) for convenience...**

Example: $\mathbf{2}-\boldsymbol{D}$ wave equation

$$\mathfrak{B} = \{w \in \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^1) \text{ satisfying } \rho_0 \frac{\partial^2 w}{\partial t^2} - \tau^2 \nabla^2 w = 0\}$$

Here $\mathtt{n} = 3$ (time, space), $\mathtt{w} = 1$. Consequently,

$$R[\xi_t, \xi_x, \xi_y]$$

$$R(\xi_t, \xi_x, \xi_y) = \rho_0 \xi_t^2 - \tau^2 \xi_x^2 - \tau^2 \xi_y^2$$

$$\underbrace{(\rho_0 \frac{\partial^2}{\partial t^2} - \tau^2 \frac{\partial^2}{\partial x^2} - \tau^2 \frac{\partial^2}{\partial y^2})}_{R(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y})} w = 0$$

Example: Maxwell's equations

$\boldsymbol{w} = (\vec{E}, \vec{B}, \vec{j}, \rho) \in \mathfrak{C}^\infty(\mathbb{R}^4, \mathbb{R}^{10})$, **8 equations**

$$R(\xi_t, \xi_x, \xi_y, \xi_z) = \begin{bmatrix} \xi_x & \xi_y & \xi_z & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\varepsilon_0} \\ 0 & -\xi_z & \xi_y & \xi_t & 0 & 0 & 0 & 0 & 0 & 0 \\ \xi_z & 0 & -\xi_x & 0 & \xi_t & 0 & 0 & 0 & 0 & 0 \\ -\xi_y & \xi_x & 0 & 0 & 0 & \xi_t & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_x & \xi_y & \xi_z & 0 & 0 & 0 & 0 \\ \xi_t & 0 & 0 & 0 & \xi_z & -\xi_y & \frac{1}{\varepsilon_0} & 0 & 0 & 0 \\ 0 & \xi_t & 0 & -\xi_z & 0 & \xi_x & 0 & \frac{1}{\varepsilon_0} & 0 & 0 \\ 0 & 0 & \xi_t & \xi_y & -\xi_x & 0 & 0 & 0 & \frac{1}{\varepsilon_0} & 0 \end{bmatrix}$$

$$R\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) w = 0$$

Linear differential latent variable distributed systems

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_f)$$

$\mathbb{L}$: **latent variables**, $\mathbb{L} = \mathbb{R}^{\mathtt{l}}$

$\mathfrak{B}_f \subseteq \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}\times\mathtt{l}}$: **full behavior**

$$\{(w,\ell) \mid R(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_{\mathtt{n}}})w = M(\frac{\partial}{\partial x_1},\cdots,\frac{\partial}{\partial x_{\mathtt{n}}})\ell\}$$

Linear differential latent variable distributed systems

$$\Sigma = (\mathbb{T}, \mathbb{W}, \mathbb{L}, \mathfrak{B}_f)$$

$\mathbb{L}$: **latent variables**, $\mathbb{L} = \mathbb{R}^{1}$

$\mathfrak{B}_f \subseteq \mathfrak{L}_{\mathtt{n}}^{\mathtt{w}\times 1}$: **full behavior**

$$\{(w, \ell) \mid R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\ell\}$$

Σ induces $\Sigma_e = (\mathbb{T}, \mathbb{W}, \mathfrak{B})$, the **manifest system**

$\mathfrak{B}$: **manifest behavior**

$$\mathfrak{B} := \{w \mid \exists \ell \text{ s.t. } (w, \ell) \in \mathfrak{B}_f\}$$

Maxwell's equations and latent variables

Dependent variables: $(\vec{E}, \vec{j}, \rho)$ **i.e.** $\mathbb{W} = \mathbb{R}^7$

Latent variable: $\vec{B}$ **i.e.** $\mathbb{L} = \mathbb{R}^3$

$$\begin{aligned}\mathfrak{B}_f &:= \{(\vec{E}, \vec{j}, \rho, \vec{B}) \text{ satisfying Maxwell's equations}\}\\ \mathfrak{B} &:= \{(\vec{E}, \vec{j}, \rho) \mid \exists \vec{B} \text{ s.t. } (\vec{E}, \vec{j}, \rho, \vec{B}) \in \mathfrak{B}_f\}\end{aligned}$$

**¿Is $\mathfrak{B}$ also described
by linear, constant-coefficient PDE's?**

Algebraic characterization of behaviors

Different $\mathtt{n}$-variable polynomial matrices may represent the same behavior

$$\mathfrak{N}_{\mathfrak{B}} := \{r \in \mathbb{R}^{1\times \mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}] \mid r(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}})\mathfrak{B} = 0\}$$

Module of annihilators of $\mathfrak{B}$

$< R >$**:= module generated by the rows of R.**

Of course $[\mathfrak{B} = \mathtt{ker}(R)] \Longrightarrow [< R > \subseteq \mathfrak{N}_{\mathfrak{B}}]$;
for $\mathfrak{C}^\infty$ trajectories, also converse holds:

$$< R > = \mathfrak{N}_{\mathfrak{B}}$$

Calculus of representations

$\mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ **is one-one with** $\{$**modules of** $\mathbb{R}^{1\times \mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]\}$**:**

- $\mathtt{ker}(R_1) = \mathtt{ker}(R_2)$ **iff** $< R_1 > = < R_2 >$
- $\mathtt{ker}(R_1) \subseteq \mathtt{ker}(R_2)$ **iff** $< R_1 > \supseteq < R_2 >$
- $\mathtt{ker}(R_1) \cap \mathtt{ker}(R_2) \rightsquigarrow < R_1 > \cup < R_2 >$

$\mathbb{R}[\xi_1, \cdots, \xi_{\mathtt{n}}]$ **is not a Euclidean domain!**

For example, no Smith form...

Outline

Elimination of latent variables

$$\mathfrak{B}_f = \left\{ (w, \ell) \mid R(\tfrac{\partial}{\partial x_1}, \cdots, \tfrac{\partial}{\partial x_n}) w = M(\tfrac{\partial}{\partial x_1}, \cdots, \tfrac{\partial}{\partial x_n}) \ell \right\}$$

$$\Big| \quad \pi_w \quad \downarrow$$

$$\mathfrak{B} = \left\{ w \mid \exists \ell \text{ s.t. } R(\tfrac{\partial}{\partial x_1}, \cdots, \tfrac{\partial}{\partial x_n}) w = M(\tfrac{\partial}{\partial x_1}, \cdots, \tfrac{\partial}{\partial x_n}) \ell \right\}$$

¿$\exists R' \in \mathbb{R}^{\bullet \times \mathtt{w}}[\xi_1, \cdots, \xi_\mathtt{n}]$ s.t. $\mathfrak{B} = \mathtt{ker}(R'(\tfrac{\partial}{\partial x_1}, \cdots, \tfrac{\partial}{\partial x_n}))$?

Yes! $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$: follows from the Fundamental Principle

The Fundamental Principle for static equations

¿ Given $M \in \mathbb{R}^{\bullet \times \bullet}$ and $y \in \mathbb{R}^{\bullet}$, is there x s.t. $Mx = y$?

The Fundamental Principle for static equations

¿ Given $M \in \mathbb{R}^{\bullet \times \bullet}$ and $y \in \mathbb{R}^{\bullet}$, is there x s.t. $Mx = y$?

Assume x exists, and consider

$$\mathfrak{N}_M := \{v \in \mathbb{R}^{\bullet} \mid v^\top M = 0\}$$

Then

$$v \in \mathfrak{N}_M \Rightarrow v^\top y = 0$$

The Fundamental Principle for static equations

¿ Given $M \in \mathbb{R}^{\bullet\times\bullet}$ and $y \in \mathbb{R}^{\bullet}$, is there x s.t. $Mx = y$?

Conversely, assume

$$v \in \mathfrak{N}_M \Rightarrow v^\top y = 0$$

for all $v \in \mathfrak{N}_M$. Then

$$(\mathrm{Im}(M))^\perp \subseteq y^\perp$$

which implies

$$y \in \mathrm{Im}(M)$$

i.e. the existence of x such that $Mx = y$.

The Fundamental Principle for static equations

¿ Given $M \in \mathbb{R}^{\bullet \times \bullet}$ and $y \in \mathbb{R}^{\bullet}$, is there x s.t. $Mx = y$?

There exists x s.t. $Mx = y$	$\Longleftrightarrow$	**$v \in \mathfrak{N}_M \Rightarrow v^T y = 0$ for all $v \in \mathfrak{N}_M$**

Now for polynomial differential operators...

The fundamental principle (Ehrenpreis-Palamodov)

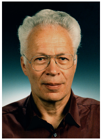
Let $f \in \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{f}})$. There exists ℓ in $\mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{l}})$ s.t.

$$M\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right)\ell = f$$

if and only if

$$n\left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_{\mathtt{n}}}\right)f = 0$$

for all $n \in \{v \in \mathbb{R}^{1\times\bullet}[\xi_1, \cdots, \xi_{\mathtt{n}}] \mid v \cdot M = 0\}$.




$\{v \in \mathbb{R}^{1\times\bullet}[\xi_1, \cdots, \xi_{\mathtt{n}}] \mid v \cdot M = 0\}$:
left syzygy of *M*, a module

The fundamental principle and elimination

Exists ℓ **s.t.** $M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell = R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w$ **IFF**

$$n(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = 0$$

for all n **in the left syzygy of** M**.**

How: compute, e.g. with Gröbner bases, a generator matrix F **for the left syzygy of** M**. Then** $w \in \mathfrak{B}$ **if and only if**

$$(FR)(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = 0$$

Example: Maxwell's equations

Eliminating $\vec{B}$ and ρ: compute left syzygy of

$$\begin{bmatrix} 0 & 0 & 0 & \frac{1}{\varepsilon_0} \\ \xi_t & 0 & 0 & 0 \\ 0 & \xi_t & 0 & 0 \\ 0 & 0 & \xi_t & 0 \\ \xi_x & \xi_y & \xi_z & 0 \\ 0 & \xi_z & -\xi_y & 0 \\ -\xi_z & 0 & \xi_x & 0 \\ \xi_y & -\xi_x & 0 & 0 \end{bmatrix}$$

Leads to

$$\begin{aligned} \varepsilon_0 \frac{\partial}{\partial t} \nabla \vec{E} + \nabla \vec{j} &= 0 \\ \varepsilon_0 \frac{\partial^2}{\partial t^2} \nabla \vec{E} + \varepsilon_0 c^2 \nabla \times \nabla \times \vec{E} + \frac{\partial}{\partial t} \vec{j} &= 0 \end{aligned}$$

Outline

Image representations

$$w = M(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})\ell$$

From Fundamental Principle $\exists\ R^{\bullet\times\bullet}[\xi_1, \cdots, \xi_n]$ s.t.

$$R(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})w = 0$$

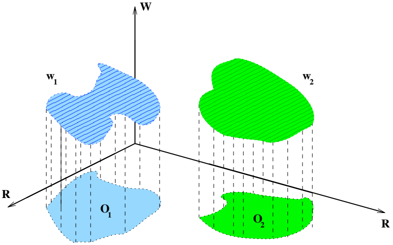
¿What kernels of polynomial partial differential operators are also images?

Controllable systems

$\mathfrak{B} \in \mathfrak{L}_{n}^{w}$ is **controllable** if for every $w_1, w_2 \in \mathfrak{B}$ and any open $O_1, O_2 \subseteq \mathbb{R}^n$ such that $\overline{O_1} \cap \overline{O_2} = \emptyset$, there exists $w \in \mathfrak{B}$ such that

$$w_{|O_1} = w_1 \text{ and } w_{|O_2} = w_2$$

"Patching" of trajectories is key:



Controllable systems

$\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ is **controllable** if for every $w_1, w_2 \in \mathfrak{B}$ and any open $O_1, O_2 \subseteq \mathbb{R}^{\mathtt{n}}$ such that $\overline{O_1} \cap \overline{O_2} = \emptyset$, there exists $w \in \mathfrak{B}$ such that

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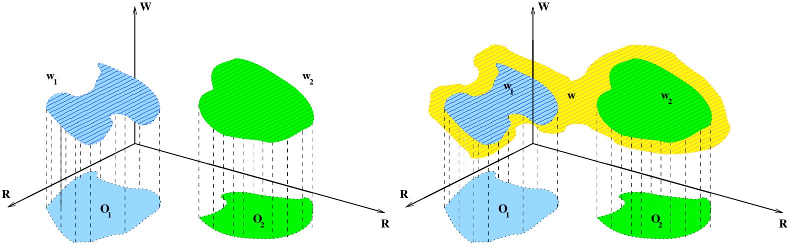

Characterizations of controllable $\mathtt{n} - D$ systems

Theorem: **Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$. The following statements are equivalent:**

1. **$\mathfrak{B}$ is controllable;**
2. **$\mathfrak{B}$ admits an image representation;**
3. **$\mathbb{R}^{1\times\mathtt{w}}[\xi_1, \cdots, \xi_{\mathtt{n}}]/\mathfrak{N}_{\mathfrak{B}}$ is torsion-free.**

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¡Controllability $\equiv$ image representation!

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¡Controllability $\equiv$ image representation!

Torsion-free property computable via Gröbner bases

Part II: Combining dynamics and functionals

Outline

B- and QDFs for $\mathtt{n}-\boldsymbol{D}$ systems

The calculus of $\mathtt{n}-\boldsymbol{D}$ B/QDFs

Losslessness

Dissipativity

Example: vibrating string

$$\frac{\partial^2}{\partial t^2} w - c^2 \frac{\partial^2}{\partial x^2} w = 0$$

$$\frac{d}{dt}\left(\underbrace{\frac{1}{2}\left(\frac{\partial}{\partial t} w\right)^2}_{\text{kinetic energy}} + \underbrace{\frac{c^2}{2}\left(\frac{\partial}{\partial x} w\right)^2}_{\text{potential energy}} \right) = 0$$

¿How to formalize this in the behavioral setting?

Bilinear differential forms

$$L_\Phi : \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{w_1}) \times \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^{w_2}) \longrightarrow \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$$

$$L_\Phi(v, w) := \sum_{k,\ell} \left(\frac{d^k}{dx^k} w_1\right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dx^\ell} w_2\right)$$

$$k := (k_1, \ldots, k_n) \in \mathbb{N}^n \qquad \ell := (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n$$

$$\frac{d^k}{dx^k} := \frac{\partial^{k_1+\cdots+k_n}}{\partial x_1^{k_1} \cdots \partial x_n^{k_n}} \qquad \frac{d^\ell}{dx^\ell} := \frac{\partial^{\ell_1+\cdots+\ell_n}}{\partial x_1^{\ell_1} \cdots \partial x_n^{\ell_n}}$$

$$\Phi_{k,\ell} \in \mathbb{R}^{w_1 \times w_2}$$

$\mathbf{2n}$-variable polynomial representation

$$L_\Phi(v, w) := \sum_{k,\ell} \left(\frac{d^k}{dx^k} v \right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dx^\ell} w \right)$$

$k := (k_1, \ldots, k_n) \in \mathbb{N}^n$ $\ell := (\ell_1, \ldots, \ell_n) \in \mathbb{N}^n$

$\zeta := (\zeta_1, \ldots, \zeta_n)$ $\eta := (\eta_1, \ldots, \eta_n)$

$$\sum_{k,\ell} \Phi_{k,\ell} \zeta^k \eta^\ell$$

Quadratic differential forms

$$Q_\Phi : \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R}^w) \to \mathfrak{C}^\infty(\mathbb{R}^n, \mathbb{R})$$

$$Q_\Phi(w) := \sum_{k,\ell} \left(\frac{d^k}{dx^k} w\right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dx^\ell} w\right)$$

W.l.o.g. **symmetry**: $\Phi_{k,\ell} = \Phi_{k,\ell}^\top$

$$\sum_{k,\ell} \Phi_{k,\ell} \zeta^k \eta^\ell$$

2n-variable polynomial associated with Q_Φ

Outline

B- and QDFs for $\mathtt{n} - \boldsymbol{D}$ systems

The calculus of $\mathtt{n} - \boldsymbol{D}$ B/QDFs

Losslessness

Dissipativity

Divergence

Vector of QDFs (VQDF) $Q_\Phi = \text{col}(Q_{\Phi_i})_{i=1,\ldots,n}$

$$(\text{div}\,(Q_\Phi))\,(w) := \frac{\partial}{\partial x_1} Q_{\Phi_1}(w) + \ldots + \frac{\partial}{\partial x_n} Q_{\Phi_n}(w)$$

¿What $2n$-variable polynomial corresponds to div Q_Φ?

$$(\zeta_1 + \eta_1)\Phi_1(\zeta,\eta) + \ldots + (\zeta_n + \eta_n)\Phi_n(\zeta,\eta)$$

Divergence

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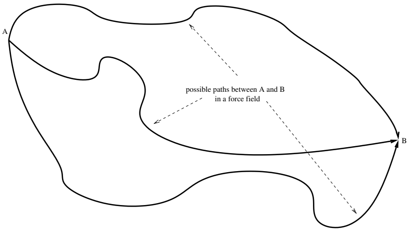
$$(\zeta_1 + \eta_1)\Phi_1(\zeta,\eta) + \ldots + (\zeta_n + \eta_n)\Phi_n(\zeta,\eta)$$

Also denoted with div(Φ)

Path independence

Let $\Omega \subseteq \mathbb{R}^{\mathtt{n}}$ be closed and bounded.
$\int_{\Omega} Q_{\Phi}(w)dx$ is independent of path if it depends only on the values of w and its derivatives on $\partial\Omega$.



Theorem: The following statements are equivalent.

1. $\int_{\Omega} Q_{\Phi}$ path independent $\forall$ closed bounded $\Omega \subseteq \mathbb{R}^{\mathtt{n}}$
2. $\int Q_{\Phi} = 0$ (on compact support trajectories)
3. $\Phi(-\xi, \xi) = 0$
4. $\exists$ VQDF $\Psi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta, \eta]^{\mathtt{n}}$, s.t. div $(\Psi) = \Phi$

Outline

B- and QDFs for $\mathtt{n}-\boldsymbol{D}$ systems

The calculus of $\mathtt{n}-\boldsymbol{D}$ B/QDFs

Losslessness

Dissipativity

Lossless systems

Supply rate Q_Φ: "energy" delivered to the system, positive when absorbed.

A controllable $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ is lossless with respect to Q_Φ if

$$\int Q_\Phi(w)\,dx = 0$$

for all $w \in \mathfrak{B}$ of compact support.

$\int Q_\Phi$ is net supply over all $\mathbb{R}^{\mathtt{n}}$ ("time" and "space").

Algebraic characterization

Theorem. Let $\mathfrak{B} = \mathrm{im}(M(\frac{d}{dx}))$. Let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$, and define $\Phi'(\zeta,\eta) := M(\zeta)^\top \Phi(\zeta,\eta) M(\eta)$. The following statements are equivalent:

1. $\mathfrak{B}$ is lossless w.r.t. Q_Φ;
2. $\int_\Omega Q_\Phi(w)dx$ is independent of path
   for all bounded and closed $\Omega \subseteq \mathbb{R}^{\mathtt{n}}$ and all $w \in \mathfrak{B}$;
3. $\int Q_{\Phi'}$ is a path integral;
4. $\exists$ VQDF Ψ s.t. for all (w,ℓ) s.t. $w = M(\frac{d}{dx})\ell$, holds

$$\mathrm{div}\ (Q_\Psi)(w) = Q_{\Phi'}(\ell) = Q_\Phi(w)$$

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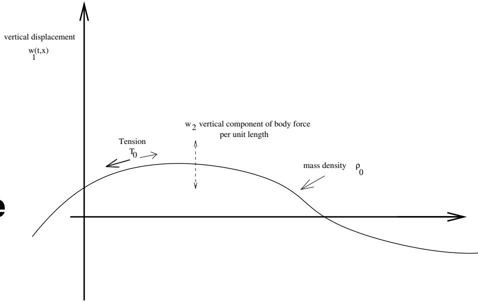
$$\mathrm{div}\,(Q_\Psi)(w) = Q_{\Phi'}(\ell) = Q_\Phi(w)$$

conservation equation

Example: vibrating string

$$\rho_0 \frac{\partial^2}{\partial t^2} w_1 - T_0 \frac{\partial^2}{\partial x^2} w_1 = w_2$$

ρ_0 **density,** T_0 **tension**
w_1 **position,** w_2 **(vertical) force**



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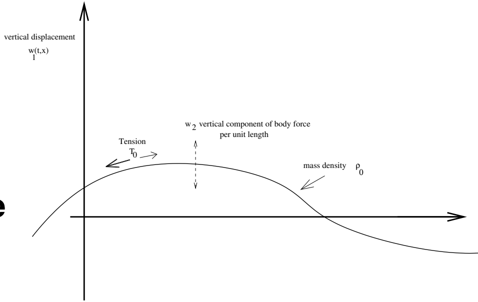

$$R(\xi_t, \xi_x) = \begin{bmatrix} \rho_0 \xi_t^2 - T_0 \xi_x^2 & -1 \end{bmatrix}$$

Image representation $w = M(\frac{d}{dx})\ell$ **induced by**

$$M(\xi_t, \xi_x) := \begin{bmatrix} 1 \\ \rho_0 \xi_t^2 - T_0 \xi_x^2 \end{bmatrix}$$

Example: vibrating string

Supply rate is $\frac{\partial}{\partial t} w_1 \cdot w_2$, represented by

$$\frac{1}{2}\begin{bmatrix}1 & \rho_0\zeta_t^2 - T_0\zeta_x^2\end{bmatrix}\begin{bmatrix}0 & \zeta_t \\ \eta_t & 0\end{bmatrix}\begin{bmatrix}1 \\ \rho_0\zeta_t^2 - T_0\zeta_x^2\end{bmatrix}$$

Example: vibrating string

$$\Phi(\zeta,\eta) = \frac{1}{2}\left(\rho_0\zeta_t^2\eta_t - T_0\zeta_x^2\eta_t + \rho_0\zeta_t\eta_t^2 - T_0\zeta_t\eta_x^2\right)$$
$$= (\zeta_t + \eta_t)\tfrac{1}{2}(\rho_0\zeta_t\eta_t + T_0\zeta_x\eta_x)$$
$$+(\zeta_x + \eta_x)\tfrac{1}{2}(-T_0\zeta_t\eta_x - T_0\eta_t\zeta_x)$$

$$Q_\Phi(w_1) = \frac{\partial}{\partial t}\left[\underbrace{\frac{1}{2}\rho_0\left(\frac{\partial}{\partial t}w_1\right)^2}_{\text{kinetic energy}} + \underbrace{\frac{1}{2}T_0\left(\frac{\partial}{\partial x}w_1\right)^2}_{\text{potential energy}}\right]$$
$$+\frac{\partial}{\partial x}\left[\underbrace{-\frac{1}{2}T_0\left(\frac{\partial}{\partial x}w_1\right)\left(\frac{\partial}{\partial t}w_1\right)}_{\text{flux}}\right]$$

Flux: infinitesimal tensile force times velocity (infinitesimal power) per unit time per unit length.

Outline

B- and QDFs for $\mathtt{n}-\boldsymbol{D}$ systems

The calculus of $\mathtt{n}-\boldsymbol{D}$ B/QDFs

Losslessness

Dissipativity

Dissipative systems

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathtt{n}}$ be controllable, and let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$.
$\mathfrak{B}$ is dissipative w.r.t. Q_Φ if

$$\int Q_\Phi(w)dx \geq 0$$

for all $w \in \mathfrak{B} \cap \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{w}})$ of compact support.

Dissipative systems

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power

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energy

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Energy goes into the system

Dissipative systems

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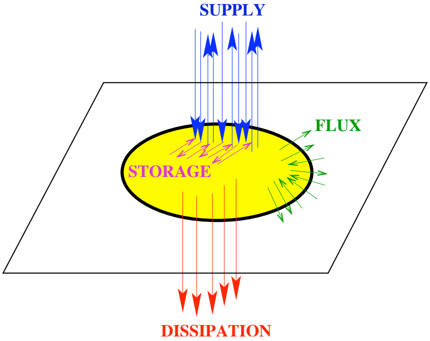
Energy is dissipated, but local flow can be negative.

¡Energy must be locally stored!



Storage and dissipation functions

$\mathfrak{B}$ **represented as** $w = M\left(\frac{d}{dx}\right)\ell$**, let** $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$**.**

VQDF $\Psi = (\Psi_1, \ldots, \Psi_{\mathtt{n}})$ **is storage function (flux) for** $\mathfrak{B}$ **w.r.t.** Q_Φ **if**

$$\text{div } Q_\Psi(\ell) \leq Q_\Phi(w)$$

$\forall\ \ell \in \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}}, \mathbb{R}^{\mathtt{l}})$ **of compact support and** $(w, \ell) \in \mathfrak{B}_f$**.**

Storage and dissipation functions

$\mathfrak{B}$ represented as $w = M\left(\frac{d}{dx}\right)\ell$, let $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$.

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$\Delta \in \mathbb{R}^{\mathtt{l}\times\mathtt{l}}[\zeta,\eta]$ is **dissipation function** for $\mathfrak{B}$ w.r.t. Q_Φ if

$$Q_\Delta \geq 0 \text{ and } \int Q_\Delta(\ell) = \int Q_\Phi(w)$$

$\forall\, \ell \in \mathfrak{C}^\infty(\mathbb{R}^{\mathtt{n}},\mathbb{R}^{\mathtt{l}})$ of compact support and $(w,\ell) \in \mathfrak{B}_f$.

Characterizations of dissipativity

Theorem: **Let $\mathfrak{B}$ be controllable, and $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta,\eta]$. Then $\mathfrak{B}$ admits an image representation $w = M(\frac{d}{dx})\ell$ s.t. the following conditions are equivalent:**

- **$\mathfrak{B}$ is dissipative w.r.t. Q_Φ (acting on w);**
- **$\exists$ a storage function Q_Ψ (acting on ℓ);**
- **$\exists$ a dissipation function Q_Δ (acting on ℓ).**

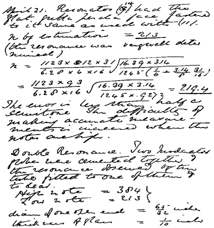
Also, the following dissipation equality holds:

$$\begin{aligned} \text{div } Q_\Psi(\ell) + Q_\Delta(\ell) &= Q_\Phi(w) \\ \text{div } \Psi(\zeta,\eta) + \Delta(\zeta,\eta) &= M(\zeta)^\top \Phi(\zeta,\eta) M(\eta) \end{aligned}$$

Example: damped vibrating string

$$\rho_0 \frac{\partial^2}{\partial t^2} w_1 - T_0 \frac{\partial^2}{\partial x^2} w_1 + \beta \frac{\partial}{\partial t} w_1 = w_2$$

$\beta > 0$ friction coefficient,
w_1 position, w_2 (vertical) force



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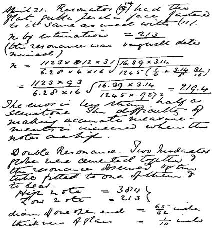

$$R(\xi_t, \xi_x) = \begin{bmatrix} \rho_0 \xi_t^2 - T_0 \xi_x^2 + \beta \xi_t & -1 \end{bmatrix}$$

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Example: damped vibrating string

$$\begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \rho_0 \frac{\partial^2}{\partial t^2} - T_0 \frac{\partial^2}{\partial x^2} + \beta \frac{\partial}{\partial t} \end{bmatrix} \ell$$

Supply rate is $\frac{\partial}{\partial t} w_1 \cdot w_2$, represented by

$$\frac{1}{2}\left(\rho_0 \zeta_t^2 \eta_t - T_0 \zeta_x^2 \eta_t + 2\beta \zeta_t \eta_t + \rho_0 \zeta_t \eta_t^2 - T_0 \zeta_t \eta_x^2\right) =: \Phi(\zeta_t, \zeta_x, \eta_t, \eta_x)$$

$\Phi(-\xi_t, -\xi_x, \xi_t, \xi_x) = -2\beta \xi_t^2 \Longrightarrow$ **dissipation rate** is

$$\sqrt{2\beta}\zeta_t \sqrt{2\beta}\eta_t$$

Example: damped vibrating string

Simple algebra leads to the storage function

$$(\zeta_t+\eta_t)\tfrac{1}{2}(\rho_0\zeta_t\eta_t+T_0\zeta_x\eta_x)+(\zeta_x+\eta_x)\tfrac{1}{2}(-T_0\zeta_t\eta_x-T_0\eta_t\zeta_x)$$

corresponding to

$$\frac{\partial}{\partial t}\left[\underbrace{\frac{1}{2}\rho_0\left(\frac{\partial}{\partial t}w_1\right)^2}_{\text{kinetic energy}}+\underbrace{\frac{1}{2}T_0\left(\frac{\partial}{\partial x}w_1\right)^2}_{\text{potential energy}}\right]$$
$$+\frac{\partial}{\partial x}\left[\underbrace{-\frac{1}{2}T_0\left(\frac{\partial}{\partial x}w_1\right)\left(\frac{\partial}{\partial t}w_1\right)}_{\text{flux}}\right]$$

Factorization of multivariable polynomial matrices

$$Q_{\Delta}(\ell) \geq 0$$

for all $\ell \in \mathfrak{C}^{\infty}(\mathbb{R}^{n}, \mathbb{R}^{1})$

of compact support

$$\Longleftrightarrow \quad \Delta(-\xi, \xi) = D(-\xi)^{\top} D(\xi)$$

Factorization of multivariable polynomial matrices

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For $\mathtt{n} = 1$, this is a spectral factorization problem, with known solvability conditions.

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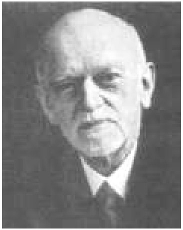
For $n = 1$, this is a spectral factorization problem, with known solvability conditions.

Hilbert's 17th problem:

given $p \in \mathbb{R}[\xi_1, \ldots, \xi_n]$, write it as the sum-of-squares

$$p = p_1^2 + \ldots + p_k^2$$



Factorization of multivariable polynomial matrices

$$Q_{\Delta}(\ell) \geq 0 \quad \text{for all } \ell \in \mathfrak{C}^{\infty}(\mathbb{R}^{n}, \mathbb{R}^{1}) \text{ of compact support} \quad \Longleftrightarrow \quad \Delta(-\xi, \xi) = D(-\xi)^{\top} D(\xi)$$

If $n > 1$, it is not possible in general to factorize with a polynomial D.

However, it is possible with D a rational function.

On the storage function

Storage function is not unique; in the damped vibrating string example, another choice is

$$(\zeta_t + \eta_t)\frac{1}{2}(\rho_0\zeta_t\eta_t - T_0\zeta_x^2 - T_0\eta_x^2 - T_0\zeta_x\eta_x)$$
$$+(\zeta_x + \eta_x)\frac{1}{2}(T_0\zeta_t\zeta_x + T_0\eta_t\eta_x)$$

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$$+(\zeta_x + \eta_x)\frac{1}{2}(T_0\zeta_t\zeta_x + T_0\eta_t\eta_x)$$

Non-uniqueness of storage function arises from

- **The non-uniqueness of $D(\xi)$ in the factorization of $\Delta(-\xi,\xi) = D(-\xi)^\top D(\xi)$;**
- **If $\mathtt{n} > 1$, there is no one-one correspondence between storage- and dissipation function**

On the storage function

Storage function is not unique; in the damped vibrating string example, another choice is

$$(\zeta_t + \eta_t)\frac{1}{2}(\rho_0\zeta_t\eta_t - T_0\zeta_x^2 - T_0\eta_x^2 - T_0\zeta_x\eta_x)$$
$$+(\zeta_x + \eta_x)\frac{1}{2}(T_0\zeta_t\zeta_x + T_0\eta_t\eta_x)$$

Storage function depends on hidden latent variables, that may be nonobservable.

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- **Dissipativity.**