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On the linear quadratic data-driven control

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Abstract— **The classical approach for solving control problems is model based: first a model representation is derived from given data of the plant and then a control law is synthesized using the model and the control specifications. We present an alternative approach that circumvents the explicit identification of a model representation. The considered control problem is finite horizon linear quadratic tracking. The results are derived assuming exact data and the optimal trajectory is constructed off-line.**

I. Introduction

We consider a finite horizon linear quadratic tracking problem that takes as input data a trajectory of the to-be-controlled plant instead of an input/state/output representation. Such a formulation is considered to be closer to a real-life control problem, because in practice one rarely has an input/state/output representation but often has measurements (i.e., an observed trajectory) of the plant. In addition, as shown in this paper, our formulation gives more freedom in the choice of the approach for solving the problem.

The classical approach for solving the control problem is model based. First, a plant model is explicitly identified from the data and, second, a controller that achieves the desired specifications is synthesized using the model. Thus, the control problem is split into two independent stages:

1) identification and
2) model-based synthesis.

The control objectives are not taken into account in the identification part and once the model is computed from the data, the data is not used in the synthesis of the controller. Both system identification and controller synthesis are mature research areas, however, their interplay in solving the overall problem from data to control has only recently been addressed in a new field, called identification for control.

Identification for control aims to determine the “best” model to be used with a given model-based synthesis method. Presently, there are only partial results in solving this problem. In its full generality, the question what is the best model for control seems to be as hard as the original problem that aims to derive optimal control directly from the available data.

An alternative to the model based paradigm is the derivation of the optimal control input or the optimal controller directly from the data. This paradigm has also been explored in the literature. Different authors call it with different names: data-based, data-driven, unfalsified [1], [2], model-free [3], [4], and model-less control. In this paper, we refer to the direct construction of the control from data as data-driven control.

Perhaps the first data-driven control method is the Ziegler-Nichols procedure for tuning PID controllers. It is based, however, on the plant step response, which is a very special response. In addition, the method is graphical and does not generalize to other control problems. A procedure for deriving multivariable linear quadratic Gaussian controller, using the plant impulse response, is proposed in [5]. Data-driven synthesis methods using an arbitrary response are proposed in [6] (linear quadratic regulation) and [7], [8] (linear quadratic tracking).

In this paper we consider a data-driven finite horizon linear quadratic tracking problem, where the given trajectory $w_{\rm d}$ is assumed to be exact and the plant $\mathscr{B}$ is assumed to be a linear time-invariant system. (The more realistic but harder to deal with situation when the data is perturbed will be treated elsewhere.) In Section II, we present three solutions for this problem. The first one is the classical model-based control that first computes an input/state/output representation of the plant and then synthesizes the controller by solving the corresponding Riccati equation. The second approach computes an impulse response representation of the plant and then finds the optimal trajectory by solving a weighted least squares problem. These approaches derive explicitly a representation of the plant. The third approach computes the optimal trajectory directly from the given data without computing a representation of the plant. The idea is to project the reference trajectory on the subbehavior $\mathscr{B}_0$ of the plant, consisting of all zero initial conditions trajectories. Under certain specified conditions, $\mathscr{B}_0$ can be computed from $w_{\rm d}$, which makes the procedure implementable.

The idea behind the third approach is similar to the one presented in [7], [8]. We give, however, sufficient conditions under which $\mathscr{B}_0$ can be computed from $w_{\rm d}$. Such conditions are missing in [7], [8]. Also we employ ideas from [9] and [10], and derive a different algorithm for data-driven control than the one suggested in [8]. In Section IV, we show simulation examples that illustrate the equivalence of the three approaches. In the conclusions we comment about the suitability of the different algorithms for on-line implementation and list some open questions.

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Preliminaries and notation

We use the behavioral language. A discrete-time dynamical system $\mathscr{B}$ is a subset of the signal space $(\mathbb{R}^{\mathtt{w}})^{\mathbb{N}}$. The integer $\mathtt{w}$ is the number of variables and the set of natural numbers $\mathbb{N}$ is the time axis. We consider linear, time-invariant, and finite dimensional systems, so that $\mathscr{B}$ is a closed shift-invariant subspace of $(\mathbb{R}^{\mathtt{w}})^{\mathbb{N}}$. In addition, we assume that a trajectory w of $\mathscr{B}$ has an input/output partition $\mathrm{col}(u,y)$. (In general one needs to permute the variables in order to have the inputs as the first variables.) $\mathscr{B}|_{[t_1,t_2]}$ denotes the restriction of the behavior on the interval $[t_1,t_2]$, i.e.,

$$\mathscr{B}|_{[t_1,t_2]} := \{\, w \in (\mathbb{R}^{\mathtt{w}})^{t_2-t_1} \mid \text{ there are } w_{\mathrm{p}} \text{ and } w_{\mathrm{f}}, \text{ such that } \mathrm{col}(w_{\mathrm{p}}, w, w_{\mathrm{f}}) \in \mathscr{B} \,\}.$$

σ denotes the backwards shift operator $\sigma w(t) = w(t+1)$.

The Hankel matrix with t block rows, composed of the sequence $w \in (\mathbb{R}^{\mathtt{w}})^T$ is denoted by

$$\mathscr{H}_{t_1,t_2}(w) := \begin{bmatrix} w(1) & w(2) & \cdots & w(t_2) \\ w(2) & w(3) & \cdots & w(t_2+1) \\ w(3) & w(4) & \cdots & w(t_2+2) \\ \vdots & \vdots & & \vdots \\ w(t_1) & w(t_1+1) & \cdots & w(t_1+t_2-1) \end{bmatrix}. \quad (1)$$

If the index t_2 is skipped, it is assumed to have the maximal possible value $T - t_1 + 1$. The lower-triangular Toeplitz matrix with t block rows, composed of the sequence $h = \big(h(0), h(1), \dots, h(t-1)\big)$ is denoted by

$$\mathscr{T}_t(h) := \begin{bmatrix} h(0) & & & & \\ h(1) & h(0) & & & \\ h(2) & h(1) & h(0) & & \\ \vdots & \ddots & \ddots & \ddots & \\ h(t-1) & \cdots & \cdots & h(1) & h(0) \end{bmatrix}. \quad (2)$$

The integer $\mathbf{n}(\mathscr{B})$ is the order of $\mathscr{B}$ and $\mathbf{l}(\mathscr{B})$ is the lag of $\mathscr{B}$ (i.e., the observability index of $\mathscr{B}$). Throughout the paper, $\mathtt{m}$ denotes the number of inputs and $\mathtt{p}$ the number of output of $\mathscr{B}$.

The time series $u = \big(u(1), \dots, u(T)\big)$ is persistently exciting of order L if the Hankel matrix $\mathscr{H}_L(u)$ is of full row rank. "row dim" denotes the number of *block* rows of a matrix or vector, and A^+ denotes the Moore-Penrose pseudoinverse of the matrix A.

II. Linear quadratic tracking

In a linear quadratic tracking problem the objective is to choose the control inputs in such a way that the plant $\mathscr{B}$ follows as close as possible in the sense of the quadratic error criterion

$$J(w_{\mathrm{r}}, w) := \sum_{t=1}^{T_{\mathrm{r}}} \big(w_{\mathrm{r}}(t) - w(t)\big)^{\top} \Phi \big(w_{\mathrm{r}}(t) - w(t)\big), \quad (3)$$

a given reference trajectory $w_{\mathrm{r}} \in (\mathbb{R}^{\mathtt{w}})^{T_{\mathrm{r}}}$. In the definition of the criterion J, $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ is a positive definite weight matrix and T_{r} is the tracking horizon.

In the special case when the reference trajectory is the zero trajectory, the tracking problem becomes the regulation problem. In this case, the optimal tracking, aiming solely at minimizing the criterion $J(w_{\mathrm{r}}, \cdot)$ over all trajectories of $\mathscr{B}$, has a trivial solution—the zero trajectory. The regulation problem is meaningful, when a nonzero initial condition is specified for the plant $\mathscr{B}$. Therefore, we will introduce initial conditions specification in the general tracking problem.

In a representation free setting, we specify initial condition by requiring the system to follow a given initial trajectory $w_{\mathrm{ini}} \in (\mathbb{R}^{\mathtt{w}})^{T_{\mathrm{ini}}}$. If $w_{\mathrm{ini}} \in \mathscr{B}$ is $\mathbf{l}(\mathscr{B})$ or more samples long, following w_{ini}, the system has a uniquely determined final state. This final state serves as an initial condition x_{ini} for the tracking problem. (See (4) and (5), where the minimality of the state representation and $T_{\mathrm{ini}} \geq \mathbf{l}(\mathscr{B})$ ensure that the system of equations for the initial state $x(1)$ has a unique solution, which determines a unique final state $x(T_{\mathrm{ini}}+1) =: x_{\mathrm{ini}}$.)

The classical formulation of the tracking problem starts with a given input/state/output representation of the system $\mathscr{B}$. In the context of data-driven tracking, we start instead from a given trajectory $w_{\mathrm{d}} \in (\mathbb{R}^{\mathtt{w}})^T$ of $\mathscr{B}$, which under the conditions of [11], uniquely specifies $\mathscr{B}$. In the solution of the data-driven tracking problem, we aim at finding the optimal trajectory without explicitly computing a representation (in particular an input/state/output representation) of $\mathscr{B}$.

Problem 1 (Linear quadratic tracking)**.** Given

1) a trajectory $w_{\mathrm{d}} = \big(w_{\mathrm{d}}(1), \dots, w_{\mathrm{d}}(T)\big)$ of a linear time-invariant system $\mathscr{B}$,
2) a reference trajectory $w_{\mathrm{r}} = \big(w_{\mathrm{r}}(1), \dots, w_{\mathrm{r}}(T_{\mathrm{r}})\big)$,
3) an initial trajectory $w_{\mathrm{ini}} = \big(w_{\mathrm{ini}}(1), \dots, w_{\mathrm{ini}}(T_{\mathrm{ini}})\big) \in \mathscr{B}$,
4) a positive definite matrix $\Phi \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$,

find a trajectory of $\mathscr{B}$ that is optimal with respect to the performance criterion $J(w_{\mathrm{r}}, \cdot)$ (see (3)) and has as a prefix the initial trajectory w_{ini}, i.e., solve the problem

$$\min_{w_{\mathrm{f}}} J(w_{\mathrm{r}}, w_{\mathrm{f}}) \quad \text{subject to} \quad \mathrm{col}(w_{\mathrm{ini}}, w_{\mathrm{f}}) \in \mathscr{B}.$$

A. Solution using an input/state/output representation

The classical but indirect solution of Problem 1 is to compute first a state space representation

$$\sigma x = Ax + Bu, \quad y = Cx + Du$$

of $\mathscr{B}$ from the data w_{d} and then using (A,B,C,D) to compute the optimal trajectory w_{f}^*. This is the well known model-based approach that we summarize for completeness.

By assumption the data w_{d} is an exact trajectory of the unknown system $\mathscr{B}$. Therefore, we are dealing with an exact (deterministic) identification problem $w_{\mathrm{d}} \mapsto \mathscr{B}$. Sufficient conditions for identifiability are (see [11]):

1) $\mathscr{B}$ is controllable,
2) u_{d} is persistently exciting of order $\mathbf{n}(\mathscr{B}) + \mathbf{l}(\mathscr{B}) + 1$, i.e., $\mathscr{H}_{\mathbf{n}(\mathscr{B})+\mathbf{l}(\mathscr{B})+1}(u_{\mathrm{d}})$ is full rank.

(A)

Under these conditions, there are algorithms for exact identification, see [12, Chaper 2] and [10, Chapter 8], that compute an input/state/output representation of the system $\mathscr{B}$.

Once the parameters (A,B,C,D) of a minimal input/state/output representation of $\mathscr{B}$ are available we can find the initial condition x_{ini} for the tracking problem that is induced by the initial trajectory w_{ini}. This is an observer design problem. Let $\operatorname{col}(y_{\text{ini}}, u_{\text{ini}})$ be an input/output partitioning of w_{ini} and let h be the impulse response of $\mathscr{B}$, i.e.,

$$h(0) = D, \qquad h(t) = CA^{t-1}B, \quad \text{for } t = 1,2,\ldots.$$

Then

$$y_{\text{ini}} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{T_{\text{ini}}-1} \end{bmatrix} x(1) + \mathscr{T}_{T_{\text{ini}}}(h) u_{\text{ini}} \tag{4}$$

defines a system of equations for the initial state $x(1)$. This system has a unique solution since by assumption w_{ini} is an exact trajectory of $\mathscr{B}$ (existence) and (A,B,C,D) is a minimal representation (uniqueness). The initial condition x_{ini} for the tracking problem is equal to the final state $x(T_{\text{ini}}+1)$ of $\mathscr{B}$, following w_{ini}, i.e.,

$$\begin{aligned} x_{\text{ini}} = x(T_{\text{ini}}+1) = CA^{T_{\text{ini}}} x(1) + \\ \begin{bmatrix} h(T_{\text{ini}}-1) & h(T_{\text{ini}}-2) & \cdots & h(0) \end{bmatrix} u_{\text{ini}}. \end{aligned} \tag{5}$$

Once the state space representation and the initial state are available, Problem 1 becomes

$$\begin{aligned} &\min_{x,u,y} \sum_{t=1}^{T_{\text{r}}} \left(w_{\text{r}}(t) - \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \right)^{\top} \Phi \left(w_{\text{r}}(t) - \begin{bmatrix} u(t) \\ y(t) \end{bmatrix} \right) \\ &\text{subject to} \quad \begin{aligned} &x(t+1) = Ax(t) + Bu(t), \quad x(1) = x_{\text{ini}} \\ &y(t) = Cx(t) + Du(t), \quad \text{for } t = 1,\ldots,T_{\text{r}}. \end{aligned} \end{aligned} \tag{6}$$

The solution leads to a difference Riccati equation that depends only on the A,B,C,D matrices and a backward in time recursion that depends on the reference signal w_{r} and the initial condition x_{ini}. The formulas for the continuous-time case are given in [13, Theorem 1]. We were not able, however, to find in the literature the solution for the discrete-time tracking problem (6), so we give next the solution for the finite-horizon linear quadratic regulation, i.e., for the special case of $w_{\text{r}} = 0$.

We can eliminate the variable y in (6) by substitution. This gives the following linear quadratic regulation problem with complete state information

$$\begin{aligned} &\min_{x,u} \sum_{t=1}^{T_{\text{r}}} \begin{bmatrix} u(t) \\ x(t) \end{bmatrix}^{\top} \begin{bmatrix} I & 0 \\ D & C \end{bmatrix}^{\top} \Phi \begin{bmatrix} I & 0 \\ D & C \end{bmatrix} \begin{bmatrix} u(t) \\ x(t) \end{bmatrix} \\ &\text{subject to} \quad x(t+1) = Ax(t) + Bu(t), \quad x(1) = x_{\text{ini}} \\ &\qquad\qquad \text{for } t = 1,\ldots,T_{\text{r}}. \end{aligned} \tag{7}$$

Define the partitioning

$$\Phi =: \begin{bmatrix} \Phi_u & \Phi_{uy} \\ \Phi_{yu} & \Phi_y \end{bmatrix}.$$

The solution of (7) is (see, e.g., [14, Theorem 11.1])

$$\begin{aligned} x^*(t+1) &= (A - BL_t) x^*(t), \quad x(1) = x_{\text{ini}}, \\ w_{\text{f}}^*(t) &= \begin{bmatrix} -L_t \\ C - DL_t \end{bmatrix} x^*(t), \end{aligned} \tag{8}$$

where

$$\begin{aligned} L_t := \big(B^{\top} S_{t+1} B + \Phi_u + \Phi_{uy} D + D^{\top} \Phi_{uy}^{\top} + D^{\top} \Phi_y D\big)^{-1} \\ \times \big(B^{\top} S_{t+1} A + \Phi_{uy} C + D^{\top} \Phi_y C\big) \end{aligned} \tag{9}$$

and

$$\begin{aligned} S_t = A^{\top} S_{t+1} A + C^{\top} \Phi_y C - \big(B^{\top} S_{t+1} A + \Phi_{uy} C + D^{\top} \Phi_y C\big)^{\top} \\ \times \big(B^{\top} S_{t+1} B + \Phi_u + \Phi_{uy} D + D^{\top} \Phi_{uy}^{\top} + D^{\top} \Phi_y D\big)^{-1} \\ \times \big(B^{\top} S_{t+1} A + \Phi_{uy} C + D^{\top} \Phi_y C\big), \qquad S_{T_{\text{r}}} = 0. \end{aligned} \tag{10}$$

In summary, an algorithm for solving Problem 1, in the special case $w_{\text{r}} = 0$, using an input/state/output representation of the plant is:

1) $w_{\text{d}} \xrightarrow{\text{Identification [10, Algorithm 8.5]}} (A,B,C,D)$
2) $(w_{\text{ini}}, A,B,C,D) \xrightarrow{\text{Observer (4) and (5)}} x_{\text{ini}}$
3) $(\Phi, w_{\text{r}}, x_{\text{ini}}, A,B,C,D) \xrightarrow{\text{Synthesis (8,9,10)}} w_{\text{f}}^*$

Note 2. Algorithm 8.5 of [10] needs, in addition to the data w_{d}, an upper bound $\mathtt{l}_{\max}$ for the system lag $\mathbf{l}(\mathscr{B})$. The same is true for Algorithms 8.7 and 8.9 of [10] and Algorithm 2 of [15], which are referred to later in this paper. Although we do not write it explicitly, we do assume that such an upper bound is given as part of the problem formulation and is passed to the identification algorithms.

B. Solution using an impulse response representation

Another approach is to compute the impulse response h of $\mathscr{B}$ from the data w_{d} and then using h to compute w^*. Condition (A) is sufficient for being able to derive h from w_{d}, see [10, Section 8.6]. Moreover, there are algorithms for doing this. Although this approach does not derive the classical input/state/output representation of the system $\mathscr{B}$ for computing w^*, it is not data driven either (the impulse response is a representation of $\mathscr{B}$).

Let the columns of $\mathscr{O}$ be $\mathbf{n}(\mathscr{B})$ linearly independent zero-input trajectories of $\mathscr{B}$. (Such a matrix can also be computed from data, see [10, Section 8.8].) Any zero-input trajectory of $\mathscr{B}$ can be written as $\mathscr{O}x_{\text{ini}}$, for some $x_{\text{ini}} \in \mathbb{R}^{\mathtt{n}}$. Define

$$\begin{aligned} \tilde{h}(0) &= \operatorname{col}\big(I_{\mathtt{m}}, h(0)\big), \quad \text{and} \\ \tilde{h}(t) &= \operatorname{col}\big(0_{\mathtt{m}}, h(t)\big), \quad \text{for } t = 1,2,\ldots, \end{aligned}$$

where $I_{\mathtt{m}}$ is the $\mathtt{m} \times \mathtt{m}$ identity matrix and $0_{\mathtt{m}}$ is the $\mathtt{m} \times \mathtt{m}$ zero matrix. Then for any trajectory

$$w := \operatorname{col}(w_{\text{ini}}, w_f) \in \mathscr{B}|_{[1, T_{\text{ini}}+T_{\text{r}}]}$$

there is a corresponding initial condition x_{ini} and an input sequence $\operatorname{col}(u_{\text{ini}}, u_{\text{f}})$, such that

$$\begin{bmatrix} w_{\text{ini}} \\ w_{\text{f}} \end{bmatrix} = \mathscr{O}x(1) + \mathscr{T}_{T_{\text{ini}}+T_{\text{r}}}(\tilde{h}) \begin{bmatrix} u_{\text{ini}} \\ u_{\text{f}} \end{bmatrix}. \tag{11}$$

(See (2) for the definition of $\mathscr{T}$.) Define the partitionings

$$\begin{bmatrix} \mathscr{O}_1 \\ \mathscr{O}_2 \end{bmatrix} \qquad \text{and} \qquad \mathscr{T}_{T_{\text{ini}}+T_{\text{r}}}(\tilde{h}) = \begin{bmatrix} T_{11} & 0 \\ T_{21} & \mathscr{T}_{T_{\text{r}}}(\tilde{h}) \end{bmatrix},$$

that are conformable with the partitionings of $\operatorname{col}(w_{\text{ini}}, w_{\text{f}})$ and $\operatorname{col}(u_{\text{ini}}, u_{\text{f}})$. The equations in (11) with left-hand-side w_{ini}

$$w_{\text{ini}} = \mathscr{O}_1 x(1) + T_{11} u_{\text{ini}} \tag{12}$$

are decoupled from the other equations and can be solved independently in terms of the initial state $x(1)$ and initial input sequence u_{ini}. The remaining equations

$$w_{\text{f}} = \mathscr{O}_2 x(1) + T_{21} u_{\text{ini}} + \mathscr{T}_{T_{\text{r}}}(\tilde{h}) u_{\text{f}}$$

specify w_{f} in terms of the input u_{f}. Define

$$w_{\text{f},1} := \mathscr{O}_2 x(1) + T_{21} u_{\text{ini}}. \tag{13}$$

Note that $w_{\text{f},1}$ is the free response of $\mathscr{B}$, caused by the initial trajectory w_{ini} (or equivalently by the initial condition x_{ini}), i.e., it is of the form $w_{\text{f},1} = \operatorname{col}(0, y_{\text{f},1})$.

The optimal tracking problem becomes

$$\min_{u_{\text{f}}} J\big(w_{\text{r}}, w_{\text{f},1} + \mathscr{T}_{T_{\text{r}}}(\tilde{h}) u_{\text{f}}\big),$$

which is a standard weighted least squares problem. Its solution is

$$u_{\text{f}}^* = \big(\mathscr{T}_{T_{\text{r}}}^\top(\tilde{h}) \tilde{\Phi} \mathscr{T}_{T_{\text{r}}}(\tilde{h})\big)^{-1} \mathscr{T}_{T_{\text{r}}}^\top(\tilde{h}) \tilde{\Phi}(w_{\text{r}} - w_{\text{f},1}), \tag{14}$$

where $\tilde{\Phi} = \operatorname{diag}(\Phi, \dots, \Phi) \in \mathbb{R}^{T_{\text{r}}\mathtt{w} \times T_{\text{r}}\mathtt{w}}$, so that

$$w_{\text{f}}^* = \mathscr{T}_{T_{\text{r}}}(\tilde{h}) u_{\text{f}}^* + w_{\text{f},1}. \tag{15}$$

In summary, an algorithm for solving Problem 1, using an impulse response representation of the plant is:

1) $(w_{\text{d}}, T_{\text{r}}) \xrightarrow{\text{[10, Algorithms 8.7 and 8.9]}} (h, \mathscr{O})$
2) $(w_{\text{ini}}, h, \mathscr{O}) \xrightarrow{(12,13)} w_{\text{f},1}$
3) $(\Phi, w_{\text{r}}, w_{\text{f},1}, h) \xrightarrow{(14,15)} w_{\text{f}}^*$

C. Data-driven solution

A third possibility, which gives a truly data-driven solution, is to project the trajectory $w_{\text{r}} - w_{\text{f},1}$ on the zero initial conditions subbehavior of $\mathscr{B}$:

$$\mathscr{B}_0|_{[1,T_{\text{r}}]} := \big\{ w \in (\mathbb{R}^{\mathtt{w}})^{T_{\text{r}}} \mid \operatorname{col}(0_{\mathbf{l}(\mathscr{B})\mathtt{w}\times 1}, w) \in \mathscr{B}|_{[1,\mathbf{l}(\mathscr{B})+T_{\text{r}}]} \big\}.$$

The definition says that $\mathscr{B}_0$ consists of all trajectories of $\mathscr{B}$ that when extended with $\mathbf{l}(\mathscr{B})$ zero samples are still trajectories of $\mathscr{B}$. The $\mathbf{l}(\mathscr{B})$ trailing zero samples specify zero initial conditions, so $\mathscr{B}_0$ is indeed the subspace of $\mathscr{B}$ consisting of all zero initial conditions trajectories.

Theorem 3. *Let $W_0 \in \mathbb{R}^{T_{\text{r}}\mathtt{w}\times \bullet}$ be a matrix, such that*

$$\operatorname{image}\big(W_0\big) = \mathscr{B}_0|_{[1,T_{\text{r}}]}.$$

Then the solution of Problem 1 is given by

$$w_{\text{f}}^* = W_0\big(W_0^\top \tilde{\Phi} W_0\big)^+ W_0^\top \tilde{\Phi}(w_{\text{r}} - w_{\text{f},1}) + w_{\text{f},1}, \tag{16}$$

where $w_{\text{f},1}$ is the free response of $\mathscr{B}$, caused by the initial trajectory w_{ini} and $(\cdot)^+$ is the Moore-Penrose pseudoinverse. If W_0 defines a basis for $\mathscr{B}_0|_{[1,T_{\text{r}}]}$, then the pseudoinverse in (16) can be replaced by inverse.

Proof: Any zero initial conditions trajectory $w = \operatorname{col}(u, y) \in (\mathbb{R}^{\mathtt{w}})^{T_{\text{r}}}$ is of the form $w = \mathscr{T}_{T_{\text{r}}}(\tilde{h}) u$. Therefore,

$$\mathscr{B}_0|_{[1,T_{\text{r}}]} = \operatorname{image}\big(\mathscr{T}_{T_{\text{r}}}(\tilde{h})\big) = \operatorname{image}\big(W_0\big).$$

Consider the space $\mathscr{W} = (\mathbb{R}^{\mathtt{w}})^{T_{\text{r}}}$ with inner product defined by $\langle w_1, w_2 \rangle = w_1^\top \tilde{\Phi} w_2$. The projector on $\mathscr{B}_0|_{[1,T_{\text{r}}]}$ in $\mathscr{W}$ is

$$\begin{aligned} &\mathscr{T}_{T_{\text{r}}}(\tilde{h})\big(\mathscr{T}_{T_{\text{r}}}^\top(\tilde{h})\tilde{\Phi}\mathscr{T}_{T_{\text{r}}}(\tilde{h})\big)^{-1}\mathscr{T}_{T_{\text{r}}}^\top(\tilde{h})\tilde{\Phi} \\ &\qquad = W_0\big(W_0^\top \tilde{\Phi} W_0\big)^+ W_0^\top \tilde{\Phi}. \end{aligned}$$

Then (16) follows from (14,15). □

Theorem 3 is based on the fact that the optimal solution w_{f}^* depends only on the subspace $\mathscr{B}_0|_{[1,T_{\text{r}}]}$, the metric, given by the weight matrix Φ, and the free response $w_{\text{f},1}$, initiated by w_{ini}, and not on the particular basis of $\mathscr{B}_0|_{[1,T_{\text{r}}]}$. In (14,15), we used as a basis for $\mathscr{B}_0|_{[1,T_{\text{r}}]}$ the columns of the Toeplitz matrix $\mathscr{T}_{T_{\text{r}}}(\tilde{h})$ constructed from the impulse response of $\mathscr{B}$. Suppose, however, that we are able to find from the given data w_{d} another basis W_0 for $\mathscr{B}_0$. Then the optimal trajectory would be given by (16). In addition, the free response $w_{\text{f},1}$ can be computed directly from the data w_{d}, using the data driven simulation algorithm of [15], so we would completely circumvent the need to compute the impulse response h. A procedure for construction of a matrix W_0 which columns span $\mathscr{B}_0$ is given in the next section.

In summary, a data-driven algorithm for solving Problem 1 is:

1) $(w_{\text{d}}, T_{\text{r}}) \xrightarrow{\text{Section III, Algorithm 1}} W_0$
2) $(w_{\text{ini}}, w_{\text{d}}, T_{\text{r}}) \xrightarrow{\text{[15, Algorithm 2]}} w_{\text{f},1}$
3) $(\Phi, w_{\text{r}}, w_{\text{f},1}, W_0) \xrightarrow{(16)} w_{\text{f}}^*$

III. Computation of a basis for $\mathscr{B}_0$

Under the assumptions of Theorem 1 from [11]:

1) $\mathscr{B}$ controllable,
2) $w_{\text{d}} \in \mathscr{B}|_{[1,T]}$, and
3) u_{d} persistently exciting of order $\mathtt{l}_{\max} + T_{\text{r}} + \mathbf{n}(\mathscr{B})$),

we have that

$$\operatorname{image}\big(\mathscr{H}_{\mathtt{l}_{\max}+T_{\text{r}}}(w_{\text{d}})\big) = \mathscr{B}|_{[1,\mathtt{l}_{\max}+T_{\text{r}}]}.$$

(See (1) for the definition of $\mathscr{H}$.) Therefore, any $\mathtt{l}_{\max} + T_{\text{r}}$-samples long trajectory of $\mathscr{B}$ can be constructed as a linear combination of the columns of $\mathscr{H}_{\mathtt{l}_{\max}+T_{\text{r}}}(w_{\text{d}})$,

$$w \in \mathscr{B}|_{[1,\mathtt{l}_{\max}+T_{\text{r}}]} \iff \text{there is } g \in \mathbb{R}^{T-\mathtt{l}_{\max}-T_{\text{r}}+1} \text{ such that } w = \mathscr{H}_{\mathtt{l}_{\max}+T_{\text{r}}}(w_{\text{d}}) g. \tag{17}$$

Assuming that $\mathtt{l}_{\max} \geq \mathbf{l}(\mathscr{B})$, see Note 2, the first $\mathtt{l}_{\max}$ samples of w (referred to as the "past") can be used to set up initial conditions for the remaining response (referred to as the "future"). From (17) and the definition of $\mathscr{B}_0|_{[1,T_{\text{r}}]}$ it follows that

$$w \in \mathscr{B}_0|_{[1,T_{\text{r}}]} \iff \text{there is } g \in \mathbb{R}^{T-\mathtt{l}_{\max}-T_{\text{r}}+1}, \text{ such that } \operatorname{col}(0_{\mathtt{l}_{\max}\mathtt{w}\times 1}, w) = \mathscr{H}_{\mathtt{l}_{\max}+T_{\text{r}}}(w_{\text{d}}) g.$$

Next we introduce some notation. Define the Hankel matrices from the input and the output

$$U := \mathscr{H}_{\mathtt{l}_{\max}+T\mathrm{r}}(u_{\mathrm{d}}), \qquad Y := \mathscr{H}_{\mathtt{l}_{\max}+T\mathrm{r}}(y_{\mathrm{d}})$$

and their past/future partitionings

$$U =: \begin{bmatrix} U_{\mathrm{p}} \\ U_{\mathrm{f}} \end{bmatrix}, \qquad Y =: \begin{bmatrix} Y_{\mathrm{p}} \\ Y_{\mathrm{f}} \end{bmatrix},$$

where $\operatorname{row dim}(U_{\mathrm{p}}) = \operatorname{row dim}(Y_{\mathrm{p}}) = \mathtt{l}_{\max}$ and $\operatorname{row dim}(U_{\mathrm{f}}) = \operatorname{row dim}(Y_{\mathrm{f}}) = T_{\mathrm{r}}$. Permuting the equations of the system

$$\mathscr{H}_{\mathtt{l}_{\max}+T_{\mathrm{r}}}(w_{\mathrm{d}})g = \operatorname{col}(0_{\mathtt{l}_{\max}\mathtt{w}\times 1}, w)$$

we have the equivalent system

$$\begin{bmatrix} U_{\mathrm{p}} \\ Y_{\mathrm{p}} \\ U_{\mathrm{f}} \\ Y_{f} \end{bmatrix} g = \begin{bmatrix} 0_{\mathtt{l}_{\max}\mathtt{m}\times 1} \\ 0_{\mathtt{l}_{\max}\mathtt{p}\times 1} \\ u \\ y \end{bmatrix}.$$

Given an arbitrary input vector u, we can compute the corresponding output vector y, such that $\operatorname{col}(u,y) \in \mathscr{B}_0$ as follows:

1) solve the system of equations for g

$$\begin{bmatrix} U_{\mathrm{p}} \\ Y_{\mathrm{p}} \\ U_{\mathrm{f}} \end{bmatrix} g = \begin{bmatrix} 0_{\mathtt{l}_{\max}\mathtt{m}\times 1} \\ 0_{\mathtt{l}_{\max}\mathtt{p}\times 1} \\ u \end{bmatrix}$$

2) define $y = Y_f g$.

This gives us a procedure for computing an element of $\mathscr{B}_0$.

In order to compute a set of generators for $\mathscr{B}_0|_{[1,T_{\mathrm{r}}]}$, we need to compute at least $\dim(\mathscr{B}_0|_{[1,T_{\mathrm{r}}]}) = T_{\mathrm{r}}\mathtt{m}$ linearly independent elements of $\mathscr{B}_0|_{[1,T_{\mathrm{r}}]}$. This can be done as outlined in Algorithm 1.

Algorithm 1 Block computation of a basis for $\mathscr{B}_0|_{[1,T_{\mathrm{r}}]}$

Input: u_{d}, y_{d}, $\mathtt{l}_{\max}$, and T_{r}.

1: Solve the system of equations for G

$$\begin{bmatrix} U_{\mathrm{p}} \\ Y_{\mathrm{p}} \\ U_{\mathrm{f}} \end{bmatrix} G = \begin{bmatrix} 0_{\mathtt{l}_{\max}\mathtt{m}\times T_{\mathrm{r}}\mathtt{m}} \\ 0_{\mathtt{l}_{\max}\mathtt{p}\times T_{\mathrm{r}}\mathtt{m}} \\ \mathscr{H}_{T_{\mathrm{r}},T_{\mathrm{r}}\mathtt{m}}(u_{\mathrm{d}}) \end{bmatrix}.$$

2: Compute $Y_0 = Y_f G$.

Output: a basis $\operatorname{col}\left(\mathscr{H}_{T_{\mathrm{r}},T_{\mathrm{r}}\mathtt{m}}(u_{\mathrm{d}}), Y_0\right)$ for $\mathscr{B}_0|_{[1,T_{\mathrm{r}}]}$.

Due to the persistency of excitation of u_{d}, the Hankel matrix $\mathscr{H}_{T_{\mathrm{r}},T_{\mathrm{r}}\mathtt{m}}(u_{\mathrm{d}})$ is full rank and the computed responses, given by the columns of $\operatorname{col}\left(\mathscr{H}_{T_{\mathrm{r}},T_{\mathrm{r}}\mathtt{m}}(u_{\mathrm{d}}), Y_0\right)$, are linearly independent. Therefore, a desired matrix W_0, such that $\operatorname{image}(W_0) = \mathscr{B}_0|_{[1,T_{\mathrm{r}}]}$, can be obtained from $\operatorname{col}\left(\mathscr{H}_{T_{\mathrm{r}},T_{\mathrm{r}}\mathtt{m}}(u_{\mathrm{d}}), Y_0\right)$ by permutation of the rows.

Note 4. Algorithm 1 corresponds to the block algorithm of [9] for the computation of the impulse response. It requires persistency of excitation of order $\mathtt{l}_{\max} + T_{\mathrm{r}} + \mathbf{n}(\mathscr{B})$ for u_{d}. A recursive version of Algorithm 1, derived along the lines of the recursive algorithm of [9] for the computation of the impulse response, however, requires persistency of excitation for u_{d} of order $\mathtt{l}_{\max} + \mathbf{n}(\mathscr{B}) + 1$. Note that this is the same condition that is required for identifiability of $\mathscr{B}$.

IV. Simulation examples

The aim of the simulation examples, shown in this section, is to illustrate numerically the equivalence of the three methods for data-driven control, presented in the paper. The to-be-controlled plant $\mathscr{B}$ is a linear time-invariant system of order $\mathtt{n} = 2$, with $\mathtt{m} = 1$ input and $\mathtt{p} = 1$ output. It is induced by the transfer function

$$\bar{H}(z) = \frac{(z-0.7847)(z+1.17)}{z^2 - 1.615z + 0.6972}.$$

The data w_{d}, used by the algorithms, is a random trajectory of $\mathscr{B}$ with $T = 200$ samples. It is the same in all simulations. A reference trajectory w_{r} with T_{r} samples and an initial trajectory w_{ini} with $T_{\mathrm{ini}} = \mathbf{l}(\mathscr{B}) = 2$ samples are chosen as follows:

- Experiment 1: data-driven regulation $T_{\mathrm{r}} = 30$

$$w_{\mathrm{r}} = 0 \quad \text{and} \quad w_{\mathrm{ini}} = (1,1),$$

- Experiment 2: data-driven step tracking $T_{\mathrm{r}} = 60$

$$u_{\mathrm{r}} = 0, \quad y_{\mathrm{r}}(t) = \begin{cases} 0, & \text{for } t = 1,2\ldots,30, \\ 1, & \text{for } t = 31,52,\ldots,60, \end{cases}$$

and $w_{\mathrm{ini}} = (1,1)$.

In both experiments, Φ is the 2×2 identity matrix.

In the first experiment the three methods compute the same optimal trajectory, see Figure 1. The corresponding optimal value of the cost functional is $J(0, w_{\mathrm{f}}^*) = 1.1139$.

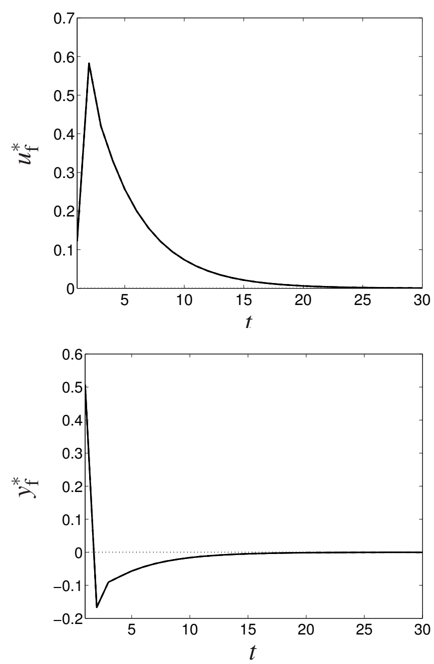

Fig. 1. First experiment: w_{r} dotted line, w_f^* solid line.

In the second experiment, we compare only the second and the third approach. (We do not have the solution of the general tracking problem by the input/state/output approach.) The two solutions coincide and are shown on Figure 2. The corresponding optimal value of the cost functional is $J(w_{\mathrm{r}}, w_{\mathrm{f}}^*) = 2.1034$.

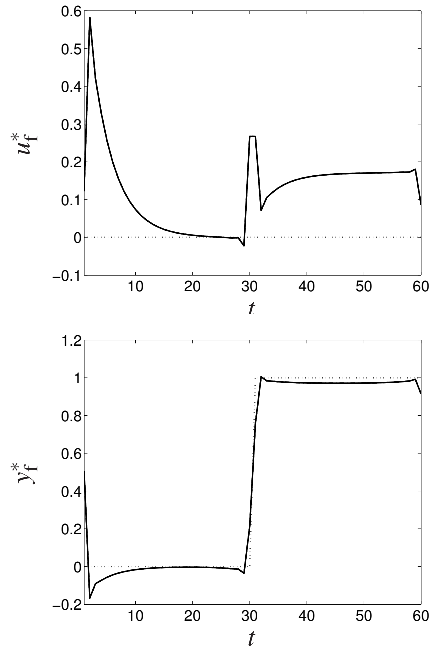

Fig. 2. Second experiment: w_r dotted line, w_f^* solid line.

V. Conclusions

We considered a finite horizon linear quadratic tracking problem, where the given data is assumed exact, and presented three solutions to the problem. All solutions need the same basic assumptions: 1) the plant $\mathscr{B}$ is controllable, and 2) an input component of the given trajectory is persistently exciting of order $\mathbf{n}(\mathscr{B})+\mathbf{l}(\mathscr{B})+1$. The solution given by the input/state/outout approach, however, is in the form of a feedback, while the other solutions compute off-line the optimal trajectory. In [5] a procedure for computing the optimal controller from the impulse response of the plant is described, however, the question "How to compute the optimal controller directly from data?" is yet unsolved.

Another important issue that we did not discuss is "How to compute the optimal trajectory or the optimal controller recursively?" In [9] a procedure for recursive computation of the impulse response is presented. Combined with recursive least squares for computing the optimal trajectory, given by (16), we obtain a recursive algorithm. Recursive implementation of the algorithm, however, does not necessarily imply suitability for on-line implementation. The algorithm should in addition be causal, i.e., operating in real time it should use only past data.

Apart from the on-line implementation, another important issue is to adapt the methods, to "work well" with perturbed data. In this paper a restrictive assumption is that the given trajectory of the plant is exact and the plant is a low-order linear time-invariant system. In practice, the data is noisy and the plant is likely to be nonlinear and time-varying. This makes it necessary to modify the algorithms in order to allow for approximation. The final goal of this work is to obtain approximate recursive algorithms for data-driven control.

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