Precise BER Formulas for Asynchronous QPSK-Modulated DS-CDMA Systems Using Random Quaternary Spreading Over Rayleigh Channels

Xiang Liu and Lajos Hanzo

Abstract—Precise bit-error-ratio (BER) analysis of an asynchronous QPSK-modulated direct-sequence code-division multiple-access system using random quaternary spreading sequences for transmission over Rayleigh channels is performed based on the characteristic-function approach. Its accuracy is verified by our numerical simulation results and also compared with those of the Gaussian approximation.

Index Terms—Asynchronous direct-sequence code-division multiple-access (DS-CDMA), bit-error-ratio (BER), precise, QPSK, quarternary spreading, Rayleigh.

I. INTRODUCTION

TRADITIONALLY, the in-phase and quadrature-phase components of QPSK-modulated direct-sequence code-division multiple-access (DS-CDMA) systems are separately spread [1]. Quaternary spreading sequences have been applied in the wideband CDMA systems characterized in [2]. As a benefit of their relative simplicity compared with other polyphase sequences [3], a number of quaternary spreading sequences, which are synonymously also referred to as complex-valued spreading sequences, have been proposed in [4]–[6]. Some of them were claimed to outperform gold and m-sequences in asynchronous DS-CDMA systems [5], [6]. A receiver architecture using quaternary spreading sequences and QPSK modulation was proposed in [7].

The bit-error-ratio (BER) performance of the family of DS-CDMA systems using quaternary spreading and QPSK modulation has been investigated using a variety of techniques, including precise analysis [6], [7], approximations [6], [8], [9], bounding techniques [9], or simulations [5], [8]. The achievable BER performance over both additive white Gaussian noise (AWGN) [5], [7]–[9] and Rayleigh [6] channels was also documented. The BER performance attainable using both deterministic quaternary spreading sequences [5]–[7], [9] and random quaternary spreading sequences [8] was also studied.

In contrast to the performance analysis of [6] and [7], we provide the precise BER analysis of the asynchronous DS-CDMA systems using the random quaternary spreading sequences and the QPSK modulation for transmission over the Rayleigh channels rather than employing the deterministic quaternary spreading sequences [6], [7] for transmission over the AWGN channels [7]. Furthermore, we do not assume the independence between the real and imaginary parts of the received signal, which was implicitly assumed in [7] when calculating symbol error ratio. In contrast to the performance analysis of [9], we use quadriphase data rather than binary data.

II. SYSTEM MODEL

We consider a general asynchronous QPSK-modulated DS-CDMA system using the random quaternary spreading and a rectangular chip waveform for transmission over a Rayleigh-fading channel. We assume that there are K simultaneously transmitting users, and the zeroth user's signal is the desired one. Since exactly the same system model has been described in [7], we commence our discussions from the decision statistics $\widetilde{Z}$ of the received signal at the output of the correlation receiver matched to the zeroth user's signal

$$\widetilde{Z} = h_0 L \widetilde{b}_{0,0} + \sum_{k=1}^{K-1} \widetilde{X}_k h_k e^{j\Delta_k} + \widetilde{\eta} \tag{1}$$

where L is the length of the spreading sequences. The signal amplitudes $\{h_k\}_{k=0}^{K-1}$ of the kth user are assumed to be independently Rayleigh-distributed with a second uncentered moment of $\{2\sigma_k^2\}$, whereas the quaternary data sequences $\{\widetilde{b}_{k,m}\}_{m=-\infty}^{\infty}$ are assumed to be mutually independent and are uniformly distributed with a probability of $P\{\widetilde{b}_{k,m} = \pm 1 \pm j\} = 1/4$. Finally, $\widetilde{\eta}$ is the noise component, which is a zero-mean complex-valued Gaussian variable. The phase-shift difference $\{\Delta_k\}_{k=1}^{K-1}$ between the kth and zeroth users is defined as $\Delta_k = -\omega_c(\tau_k - \tau_0) + (\theta_k - \theta_0)$, where ω_c is the common carrier frequency, T_c is the chip duration, and the phase shifts $\{\theta_k\}_{k=0}^{K-1}$ and the time delays $\{\tau_k\}_{k=0}^{K-1}$ are

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independently and uniformly distributed in $[0, 2\pi)$ and $[0, LT_{\mathrm{c}})$, respectively. The random variable $\widetilde{X}_k$ may be further expressed as [10], [11]

$$\widetilde{X}_k = \sum_{m=0}^{L-2} \widetilde{Y}_{k,m} \left[(1-\nu_k) + \frac{1}{2}\widetilde{a}_{0,m}\widetilde{a}_{0,m+1}^* \nu_k \right]$$

$$+ \widetilde{Y}_{k,L-1}\nu_k + \widetilde{Y}_{k,L}(1-\nu_k) \quad (2)$$

where the $(L+1)$ random variables $\{\widetilde{Y}_{k,m}\}_{m=0}^{L}$ are mutually independent and uniformly distributed, which are conditioned on the zeroth user's spreading sequence $\{\widetilde{a}_{0,m}\}_{m=0}^{L-1}$. Furthermore, the relative chip shifts $\{\nu_k\}_{k=1}^{K-1}$ between the kth and zeroth users, which are normalized by the chip duration, are mutually independent and uniformly distributed in $[0, 1)$ [10], [11].

III. BER ANALYSIS

The $(L-1)$ possible chip combinations of $\{(1/2)\widetilde{a}_{0,m} \widetilde{a}_{0,m+1}^*\}_{m=0}^{L-2}$ can be categorized into four sets according to the relative change of the chip value. Let A, B, C, and D denote the number of the relative phase changes of $0°$, $180°$, $270°$, and $90°$ between the adjacent two chips' values within the zeroth user's spreading sequence, respectively. Then, we have $A + B + C + D = L - 1$. Upon defining the cochannel interference (CCI) $\widetilde{I}_k = \widetilde{X}_k h_k e^{j\Delta_k}$ incurred by the kth user, the characteristic function of $\Re\{\widetilde{I}_k\}$ conditioned on A, B, and C can be expressed as

$$\Phi_{\Re\{\widetilde{I}_k\}|A,B,C}(\omega)$$

$$= 2^{-2(L+1)} \sum_{\widetilde{d}_1 \in \mathcal{A}} \sum_{\widetilde{d}_2 \in \mathcal{B}} \sum_{\widetilde{d}_3 \in \mathcal{C}} \sum_{\widetilde{d}_4 \in \mathcal{D}}$$

$$\times \left(\frac{A}{\frac{\Re\{\widetilde{d}_1\}+A}{2}} \right) \left(\frac{A}{\frac{\Im\{\widetilde{d}_1\}+A}{2}} \right) \left(\frac{B}{\frac{\Re\{\widetilde{d}_2\}+B}{2}} \right) \left(\frac{B}{\frac{\Im\{\widetilde{d}_2\}+B}{2}} \right)$$

$$\times \left(\frac{C}{\frac{\Re\{\widetilde{d}_3\}+C}{2}} \right) \left(\frac{C}{\frac{\Im\{\widetilde{d}_3\}+C}{2}} \right) \left(\frac{D}{\frac{\Re\{\widetilde{d}_4\}+D}{2}} \right) \left(\frac{D}{\frac{\Im\{\widetilde{d}_4\}+D}{2}} \right)$$

$$\times \sum_{\widetilde{Y}_{k,L-1}, \widetilde{Y}_{k,L}} \Phi_{\Re\{\widetilde{I}_k\}|\lambda_0,\lambda_1,\lambda_2}(\sigma_k\omega) \quad (3)$$

where the sets $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, and $\mathcal{D}$ are defined as

$$\mathcal{A} = \left\{ \widetilde{d}_1 \Big| \Re\left\{\widetilde{d}_1\right\}, \Im\left\{\widetilde{d}_1\right\} \in \{-A, -(A-2), \dots, A\} \right\}$$

$$\mathcal{B} = \left\{ \widetilde{d}_2 \Big| \Re\left\{\widetilde{d}_2\right\}, \Im\left\{\widetilde{d}_2\right\} \in \{-B, -(B-2), \dots, B\} \right\}$$

$$\mathcal{C} = \left\{ \widetilde{d}_3 \Big| \Re\left\{\widetilde{d}_3\right\}, \Im\left\{\widetilde{d}_3\right\} \in \{-C, -(C-2), \dots, C\} \right\}$$

$$\mathcal{D} = \left\{ \widetilde{d}_4 \Big| \Re\left\{\widetilde{d}_4\right\}, \Im\left\{\widetilde{d}_4\right\} \in \{-D, -(D-2), \dots, D\} \right\}.$$

$$(4)$$

The function $\Phi_{\Re\{\widetilde{I}_k\}|\lambda_0,\lambda_1,\lambda_2}(\omega)$ is defined as (5), shown at the bottom of the page, where the coefficients λ_0, λ_1, and λ_2 are given by

$$\lambda_0 = \left| \widetilde{d}_1 + \widetilde{d}_2 + \widetilde{d}_3 + \widetilde{d}_4 + \widetilde{Y}_{k,L} \right|^2$$

$$\lambda_1 = \Re\left\{ \left(\widetilde{d}_1 + \widetilde{d}_2 + \widetilde{d}_3 + \widetilde{d}_4 + \widetilde{Y}_{k,L} \right) \right.$$

$$\left. \times \left[-2\widetilde{d}_2 - (1-j)\widetilde{d}_3 - (1+j)\widetilde{d}_4 + \widetilde{Y}_{k,L-1} - \widetilde{Y}_{k,L} \right]^* \right\}$$

$$\lambda_2 = \left| -2\widetilde{d}_2 - (1-j)\widetilde{d}_3 - (1+j)\widetilde{d}_4 + \widetilde{Y}_{k,L-1} - \widetilde{Y}_{k,L} \right|^2. \quad (6)$$

Since the CCI $\{\widetilde{I}_k\}_{k=1}^{K-1}$ conditioned on A, B, and C is independent [10], [11], the BER of the zeroth user's real component conditioned on A, B, and C can be shown as [10]

$$P_{e|A,B,C}^{\mathrm{r}} = \frac{1}{2} - \frac{\sigma_0 L}{\sqrt{2\pi}} \int_0^\infty \exp\left[-\frac{1}{2}\omega^2 \left(\sigma_0^2 L^2 + \sigma_{\Re\{\widetilde{\eta}\}}^2 \right) \right]$$

$$\times \prod_{k=1}^{K-1} \Phi_{\Re\{\widetilde{I}_k\}|A,B,C}(\omega) d\omega. \quad (7)$$

Then, the BER of the zeroth user's real component is obtained by averaging $P_{e|A,B,C}^{\mathrm{r}}$ over all spreading sequences [10]. Following the same approach, we may conclude that the average BER of the zeroth user's imaginary component P_e^{i} has the same value as P_e^{r}. Finally, we arrive at the overall BER P_e averaged over both the real and imaginary components of the zeroth user yielding

$$P_e = P_e^{\mathrm{r}} = 4^{-(L-1)} \sum_{A=0}^{L-1} \sum_{B=0}^{L-1-A} \sum_{C=0}^{L-1-A-B} \binom{L-1}{A}$$

$$\times \binom{L-1-A}{B} \binom{L-1-A-B}{C} P_{e|A,B,C}^{\mathrm{r}}. \quad (8)$$

$$\Phi_{\Re\{\widetilde{I}_k\}|\lambda_0,\lambda_1,\lambda_2}(\omega) = \begin{cases} \exp\left(-\frac{1}{2}\lambda_0\omega^2\right), & \text{if } \lambda_2 = 0 \\ \frac{1}{\omega}\sqrt{\frac{\pi}{2\lambda_2}} \exp\left[\frac{1}{2}\left(\frac{\lambda_1^2}{\lambda_2} - \lambda_0\right)\omega^2\right] \\ \quad \times \left\{ \mathrm{erfc}\left(\frac{\lambda_1\omega}{\sqrt{2\lambda_2}}\right) - \mathrm{erfc}\left[\sqrt{\frac{\lambda_2}{2}}\left(1 + \frac{\lambda_1}{\lambda_2}\right)\omega\right] \right\}, & \text{if } \lambda_2 \neq 0 \end{cases} \quad (5)$$
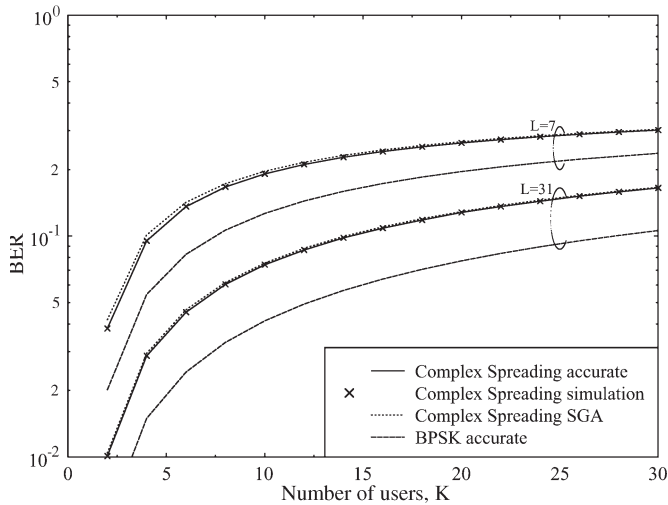
Fig. 1. BER versus the number of users K in the asynchronous QPSK-modulated DS-CDMA system using the random quaternary spreading sequences. The lengths of the random quaternary spreading sequences are $L = 7$ and 31, respectively, the average power of all users at the receiver is equal, and the background noise is ignored, i.e., when $\gamma_{SNR} = \infty$.
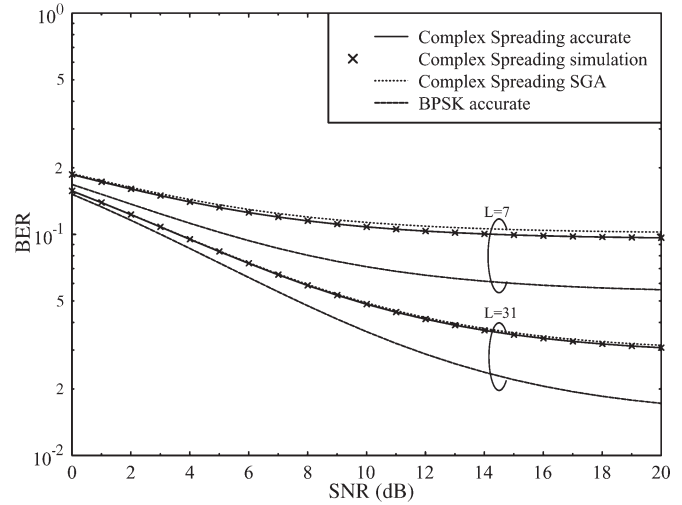

Fig. 2. BER versus per-bit SNR in the asynchronous QPSK-modulated DS-CDMA system using the random quaternary spreading sequences. The lengths of the random quaternary spreading sequences are $L = 7$ and 31, respectively, and the average power of all users at the receiver is equal. The number of users is $K = 4$.

IV. NUMERICAL RESULTS

We will compare the BER results acquired by our precise analysis derived in Section III with that of the standard Gaussian approximation (SGA) [10], [11], with that of the BPSK system of [10], and with our simulations described in this section. Similar to the discussions in [10] and [11], the complexity of our analysis, the SGA, and the simulation is on the order of $\Theta(L^3)$, $\Theta(1)$, and $\Theta(4^L)$, respectively, where Θ is the big-Theta notation.[1] Fig. 1 shows that the results obtained by our precise analysis exactly match those obtained by simulations for two different-length random spreading sequences when using $L = 7$ and 31. However, the SGA overestimates the BER, particularly in the scenario where there is a low number of interfering users and when short spreading sequences are used. The BER of the QPSK system is higher than that of the BPSK systems due to the crosstalk between the real and imaginary components. Fig. 2 also confirms that the BER results obtained by our precise analysis match those obtained by simulation for both of the different-length random spreading sequences, i.e., for $L = 7$ and 31. By contrast, the SGA slightly overestimates the BER, particularly when the SNR is high and short spreading sequences are used. The BER of the QPSK systems is higher than that of the BPSK systems due to the crosstalk between the real and imaginary components.

V. CONCLUSION

In this paper, we provided exact BER formulas for the asynchronous QPSK-modulated DS-CDMA system using the random quaternary spreading sequences for transmission over

the Rayleigh channels. Its accuracy was verified by our numerical simulation results and also compared with those of the Gaussian approximation.

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[1] A function is in $\Theta(f(n))$ if it does not significantly increase faster or slower than $f(n)$, i.e., there exist two positive real constants c_1 and c_2 and a positive integer n_0 so that $c_1 n \leq \Theta(f(n)) \leq c_2 n$ for $\forall n \geq n_0$.

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