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Control and filtering for discrete linear repetitive processes with $\mathscr{H}_\infty$ and ℓ_2–ℓ_∞ performance

Ligang Wu · James Lam · Wojciech Paszke · Krzysztof Gałkowski · Eric Rogers · Anton Kummert

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Abstract Repetitive processes are characterized by a series of sweeps, termed passes, through a set of dynamics defined over a finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This can lead to oscillations which increase in amplitude in the pass to pass direction and cannot be controlled by standard control laws. Here we give new results on the design of physically based control laws for the sub-class of so-called discrete linear repetitive processes which arise in applications areas such as iterative learning control. The main contribution is to show how control law design can be undertaken within the framework of a general robust filtering problem with guaranteed levels of performance. In particular, we develop algorithms for the design of an $\mathscr{H}_\infty$ and ℓ_2–ℓ_∞ dynamic output feedback controller and filter which guarantees that the resulting

During the period when the research reported here was undertaken, Krzysztof Gałkowski was a Gerhard Mercator Guest Professor in the University of Wuppertal, Germany.

L. Wu
Space Control and Inertial Technology Research Center, Harbin Institute of Technology, Harbin 150001, People's Republic of China
e-mail: ligangwu@hit.edu.cn

Present Address:
L. Wu
Department of Mechanical Engineering, University of Hong Kong, Pokfulam Road, Hong Kong, China

J. Lam (✉)
Department of Mechanical Engineering, University of Hong Kong, Pokfulam Road, Hong Kong, China
e-mail: james.lam@hku.hk

W. Paszke
Control Systems Technology Group, Eindhoven University of Technology, PO Box 513, 5600 MB Eindhoven, The Netherlands
e-mail: W.Paszke@tue.nl

K. Gałkowski
Institute of Control and Computation Engineering, University of Zielona Góra, ul. Podgórna 50, 65-246 Zielona Gora, Poland
e-mail: k.galkowski@issi.uz.zgora.pl

controlled (filtering error) process, respectively, is stable along the pass and has prescribed disturbance attenuation performance as measured by $\mathscr{H}_\infty$ and ℓ_2–ℓ_∞ norms.

Keywords Control · Discrete time · Filtering · Lyapunov method · Repetitive process

1 Introduction

The operation of a repetitive process, i.e. a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length can lead to oscillations in the output sequence of pass profiles which increase in amplitude in the pass to pass direction. These are caused by the fact that the previous pass profile acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile and so on.

To introduce a formal definition, let $\alpha < +\infty$ denote the pass length (assumed constant). Then in a repetitive process the pass profile (or process output) $y_k(p),\ 0 \leq p \leq \alpha - 1$, generated on pass k acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile $y_{k+1}(p),\ 0 \leq p \leq \alpha - 1,\ k \geq 0$. The source of the unique control problem then appears (if at all) in the output sequence generated, i.e. the collection of pass profile vectors $\{y_k\}_k$. Note that for repetitive processes, as opposed to 2D systems, information propagation in one of the independent directions, along the pass, only occurs over a finite duration—the pass length. Also the boundary conditions are reset before the start of each new pass and the structure of these can be somewhat complex. For example, if they are an explicit function of points on the previous pass profile then these alone can destroy the most basic performance specification of stability.

Physical examples of repetitive processes include long-wall coal cutting and metal rolling operations (see, for example, the references cited in Rogers and Owens 1992). Also in recent years applications have arisen where adopting a repetitive process setting for analysis has distinct advantages over alternatives. Examples of these so-called algorithmic applications include classes of iterative learning control (ILC) schemes (see, for example, Moore et al. 2005) and iterative algorithms for solving nonlinear dynamic optimal control problems based on the maximum principle (Roberts 2002). In the case of iterative learning control for the linear dynamics case, the stability theory for differential (and discrete) linear repetitive processes is one method which can be used to undertake a stability/convergence analysis of a powerful class of such algorithms and thereby produce vital design information concerning the trade-offs required between convergence and transient performance.

In terms of control laws for repetitive processes, it is necessary to use feedback control action on the current pass and/or feedforward control from the previous pass (or passes). The critical role of the previous pass profile dynamics means that current pass feedback control alone is not enough and it must be augmented by feedforward control. This approach has been the subject of significant research effort and results have emerged on how to undertake

E. Rogers
School of Electronics and Computer Science, University of Southampton,
Southampton SO17 1BJ, UK
e-mail: etar@ecs.soton.ac.uk

A. Kummert
Faculty of Electrical, Information and Media Engineering, University of Wuppertal,
Rainer-Gruenter-Strasse 21, 42119 Wuppertal, Germany
e-mail: kummert@uni-wuppertal.de

control law design in the presence of uncertainty. For example, Paszke et al. (2006) give results on control law design in an $\mathscr{H}_\infty$ setting. The control laws used in some of this work are based on the use of feedback of the current state vector which, of course, requires that all entries in this vector can be measured to allow control law implementation. Often, however, this assumption is not valid for various reasons.

There are two commonly used methods to deal with the control design problem when all entries in the state vector cannot be measured. One is to design an observer to estimate the unmeasurable state vector entries and use this to implement an observer-based control law. The other is to design a controller, or control law, which is only activated by pass profile (or output) information where such controllers are usually classified as either static or dynamic, respectively.

Generally speaking, dynamic output feedback is the more flexible since the control law or controller introduces additional dynamics. Also it is known that the problem of designing such control laws can be formulated as a convex optimization problem over linear matrix inequalities (LMIs) (see, for example, Paszke et al. 2006) and hence the possibility of numerically reliable computation using numerical optimization packages. This work also shows that there are two complementary approaches to problem formulation. These are the well known variables elimination procedure and the use of linearizing variable transforms, respectively.

This latter approach provides a general framework to formulate control law synthesis as a convex optimization problem involving LMIs. It is based on applying specific invertible transforms of the controller parameters to achieve LMI conditions in terms of the new set of variables. When the resulting LMIs have a solution, the control law parameters can be computed by applying inverse transforms. This approach becomes less computationally effective as the number of decision variables increases and hence elimination of some of these can be still required, but this can only be achieved by application to specific structures within the underlying matrix inequalities. The known results on designing a so-called $\mathscr{H}_\infty$ dynamic pass profile controller are based on this approach, see Paszke et al. (2006).

Clearly, there is still much research which needs to be done on the development of alternative design algorithms based on linearizing variable transform methods, with the overall aim of providing a general set of control law/controller design tools for the designer to chose the one most appropriate to the particular application under consideration. In particular, to-date only $\mathscr{H}_\infty$ and $\mathscr{H}_2$ (and mixed $\mathscr{H}_2/\mathscr{H}_\infty$) settings have been addressed and also this analysis assumed full access to either the current pass state or pass profile vectors, an assumption which may not be particularly relevant to physical cases where the pass profile vector (the process output) is corrupted by noise etc. Here we develop significant new results with such a case in mind using $\mathscr{H}_\infty$ and ℓ_2–ℓ_∞ performance measures which can be split into two main parts.

The $\mathscr{H}_\infty$ and ℓ_2–ℓ_∞ (or $\mathscr{L}_2$–$\mathscr{L}_\infty$ for continuous-time systems) settings have been extensively used in optimal control/filtering for many classes of systems, see, for example, Du and Lam (2006), Wang et al. (2006), Wu et al. (2008, 2006, 2007). In particular, they are known to be particularly well suited to cases when noise is present whose stochastic content is not precisely known. In an $\mathscr{H}_\infty$ sense, the control (filter) minimizes the worst-case energy gain from the noise inputs to the controlled output (estimation error) (Wu et al. 2008, 2006, 2007); while in an ℓ_2–ℓ_∞ (or $\mathscr{L}_2$–$\mathscr{L}_\infty$) sense, the control (filter) minimizes the worst-case energy to peak gain from the noise inputs to the controlled output (estimation error) (Du and Lam 2006; Wu et al. 2006, 2007).

The first set of new results developed in this paper give control law (or controller) design algorithms to guarantee stability and disturbance rejection, as measured by $\mathscr{H}_\infty$ and ℓ_2–ℓ_∞ norms for one version of so-called discrete linear repetitive processes which arise in the representation of both physical and algorithmic examples. These results are the first for the latter performance measure and in both cases we do not assume full access to either the current pass state or pass profile vectors—in Paszke et al. (2006) this assumption was made for the $\mathscr{H}_\infty$ case.

There is clearly also a need to develop a filtering theory for these processes which can be (eventually) used to enable the implementation of control laws and/or enable (as one of many possible uses) reliable estimates of key signals to be obtained from measured data. The second set of major results here solves the underlying problem of the design of a full order filter which gives a stable filter error and has prescribed disturbance attenuation performance as measured by either an $\mathscr{H}_\infty$ or an ℓ_2–ℓ_∞ norm measure. This leads to the formulation of the filter existence problem in an LMI setting for each case and hence the corresponding design task as a convex optimization problem which can be computed using well known interior-point algorithms. Two numerical examples are given to highlight the potential offered by these new results.

Throughout this paper, the null matrix and the identity matrix with appropriate dimensions are denoted by 0 and I, respectively. Moreover, $M > 0$ (≥ 0) denotes a real symmetric positive definite (semi-definite) matrix. Similarly, $M < 0$ (≤ 0) denotes a real symmetric negative definite (semi-definite) matrix, and $*$ is used to denote transposed block entries in these matrices. We also require the signal space $\ell_2\{[0,\infty),[0,\infty)\}$, i.e. the space of square summable sequences on $\{[0,\infty),[0,\infty)\}$ with values in $\mathbb{R}^q$, written ℓ_2^q for short.

2 $\mathscr{H}_\infty$ And ℓ_2–ℓ_∞ performance

2.1 Process description and preliminaries

As essential background for the rest of this paper, this section defines what is meant by $\mathscr{H}_\infty$ and ℓ_2–ℓ_∞ performance for discrete linear repetitive processes described by the following state-space model over $0 \leq p \leq \alpha - 1,\ k \geq 0$,

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + B_0 y_k(p) + B_1 \omega_{k+1}(p) \\ y_{k+1}(p) &= Cx_{k+1}(p) + D_0 y_k(p) + D_1 \omega_{k+1}(p) \end{aligned} \tag{1}$$

where on pass k, $x_{k+1}(p) \in \mathbb{R}^n$ is the state vector; $y_k(p) \in \mathbb{R}^m$ is the pass profile vector; $\omega_{k+1}(p) \in \mathbb{R}^l$ is the disturbance vector which belongs to ℓ_2^l.

Remark 1 It is important to note that the pass-to-pass updating structure in this model is the simplest possible, i.e. at any point on the current pass the contribution from the previous pass is only from the same point, i.e. only $y_k(p)$ contributes to $x_{k+1}(p)$ and hence to $y_{k+1}(p)$ for any $0 \leq p \leq \alpha - 1$. It is also possible that (in the most general case) all points along the previous pass profile contribute to the state and pass profile vectors at any point on the current pass. Indeed this can arise in physical examples such as long-wall coal cutting where it is known as inter-pass smoothing (Rogers and Owens 1992). Given that no work has previously been reported on filtering problems for discrete linear repetitive processes we focus on the model above with extension to inter-pass smoothing, which could well be a non-trivial problem, left as a topic for further work (see also the conclusions section for more discussion of this general point).

Often in practical applications it cannot be assumed that the current pass state ($x_{k+1}(p)$) and pass profile ($y_k(p)$) vector are fully accessible. In the first case, this often arises physically as sum or all of the entries in this vector may not be available for measurement and hence control law implementation cannot be achieved unless it is possible to design a suitable observer structure to estimate the missing state variables.

The pass profile vector is the process output, but from the Roesser type 2D linear systems state-space model point of view, it has also the interpretation of the system vertical state sub-vector (and the state vector is the horizontally transmitted state sub-vector) and hence it could be the case that all elements in this vector are not available for measurement. More likely is the situation where measurements are corrupted by noise. In such cases, one option is to assume availability of a so-called measured output signal vector given by

$$z_{k+1}(p) = Ex_{k+1}(p) + F_0 y_k(p) + F_1 \omega_{k+1}(p) \tag{2}$$

where $z_{k+1}(p) \in \mathbb{R}^r$. The controlled output signal, or signal to be estimated, can be written as

$$v_{k+1}(p) = Gx_{k+1}(p) + H_0 y_k(p) \tag{3}$$

where $v_{k+1}(p) \in \mathbb{R}^q$.

To complete the process description, it is necessary to specify the boundary conditions, i.e. the state initial vector on each pass and the initial pass profile (i.e. on pass 0). Here we consider the case when

$$\begin{aligned} x_{k+1}(0) &= d_{k+1}, \quad k \geq 0 \\ y_0(p) &= f(p), \quad 0 \leq p \leq \alpha - 1 \end{aligned} \tag{4}$$

where $d_{k+1} \in \mathbb{R}^n$ has known constant entries and $f(p) \in \mathbb{R}^m$ is an vector whose entries are known functions of p over $[0, \alpha - 1]$. This overall state-space model description allows for disturbances which affect both the state and pass profile dynamics on each pass.

Remark 2 The boundary conditions assumed here are the simplest possible. In some applications, however, there is a need to consider boundary conditions where the state initial vector on each pass is an explicit function of points along the previous pass profile. An example here is the optimal control application (Rogers et al. 2007). Such boundary conditions are termed dynamic and they can have a very critical effect on the process dynamics. Indeed they alone can cause instability—see Rogers et al. (2007) and the relevant cited references for a complete treatment of this key point. Here we leave the problem of filtering in the presence of such boundary conditions as a topic for further work.

The stability theory (Rogers and Owens 1992) for linear repetitive processes such as those considered here is based on an abstract model in a Banach space setting which includes a wide range of such processes as special cases, including those described by (1) and (4). In terms of their dynamics it is the pass-to-pass coupling (noting again the unique control problem for them) which is critical. This is of the form $y_{k+1} = L_\alpha y_k$, where $y_k \in E_\alpha$ (E_α a Banach space with norm $||\cdot||$) and L_α is a bounded linear operator mapping E_α into itself. At least two distinct forms of stability can be defined and the first of these, known as asymptotic stability, holds if, and only if, there exist numbers $M_\alpha > 0$ and $\lambda_\alpha \in (0, 1)$ independent of α such that $||L_\alpha^k|| \leq M_\alpha \lambda_\alpha^k$, $k \geq 0$ (where $||\cdot||$ also denotes the induced operator norm) and can be interpreted as bounded-input bounded-output stability over the finite pass length.

If asymptotic stability holds then the sequence of pass profiles generated converge strongly to a so-called limit profile, i.e. after a sufficiently large number of passes have elapsed, the pass profiles converge in the k direction but the finite pass length means that there could be unacceptable along the pass dynamics. Stability along the pass prevents this from arising by, in effect, demanding the bounded-input bounded-output property for any possible value of the pass length. This holds if, and only if there exist numbers $M_\infty > 0$ and $\lambda_\infty \in (0, 1)$

independent of α such that $||L_{\alpha}^{k}|| \leq M_{\infty}\lambda_{\infty}^{k},\ k \geq 0$. (Note also that stability along the pass can be analyzed mathematically by letting $\alpha \to \infty$ and we make no further explicit reference to this fact for the remainder of this paper).

It is of interest to relate this theory to a physical example in the form of long-wall coal cutting where the pass profile is the thickness (relative to a fixed datum) of the coal left after the cutting machine has moved along the pass length, i.e. the coal face. The stability problem here is caused by the machine's weight as it rests on the previous pass profile during the cutting of the next pass profile. The undulations caused can be very severe and result in productive work having to stop to enable them to be removed.

For the processes considered here (see Rogers and Owens 1992 for the original analysis), asymptotic stability holds if, and only if, all eigenvalues of the matrix D_0 have modulus strictly less than unity, i.e. $\rho(D_0) < 1$ where $\rho(\cdot)$ denotes the spectral radius of its matrix argument. This condition is trivially checked and if it holds then the resulting limit profile is governed by a standard, or 1D, discrete linear systems state-space model with state matrix $A + B_0(I - D_0)^{-1}C$. It is, however, easy to find examples where asymptotic stability holds but the resulting limit profile is unstable as a 1D discrete linear system, i.e. the dynamics in the along the pass direction are bounded but not uniformly bounded (i.e. independent of the value of the pass length α). Stability along the pass prevents this from arising and the following is one set of necessary and sufficient conditions for this stronger property where we note that it is independent of the disturbance terms.

Theorem 1 (Rogers and Owens 1992) *A discrete linear repetitive process described by* (1) *and* (4) *is stable along the pass if, and only if,*

(i) $\rho\,(D_0) < 1$*;*
(ii) $\rho\,(A) < 1$*; and*
(iii) *all eigenvalues of* $G(z) = C\,(zI - A)^{-1}\,B_0 + D_0$ *for all* $|z| = 1$ *have modulus strictly less than unity.*

Note here that examples can be found which show that $\rho(A) < 1$ is also only a necessary condition for stability along the pass.

In terms of testing a particular example for stability along the pass, it is clearly the third condition here which is the most intensive computationally. Also this result has not proved to be a general purpose way to undertake control law design for stability along the pass or stability along the pass plus performance objectives. One alternative is to use LMIs for which the following is the basic result.

Lemma 1 (Galkowski et al. 2002) *A discrete linear repetitive process described by* (1) *and* (4) *with* $\omega_{k+1}(p) = 0$ *is stable along the pass if there exists a matrix* $W = \mathrm{diag}(W_1, W_2) > 0$ *such that the following LMI holds:*

$$\begin{bmatrix} -W & M^T W \\ * & -W \end{bmatrix} < 0 \tag{5}$$

where $M = \begin{bmatrix} A & B_0 \\ C & D_0 \end{bmatrix}$.

To assess performance using $\mathscr{H}_\infty$ and ℓ_2–ℓ_∞ measures, we introduce the following definition.

Definition 1 A discrete linear repetitive process described by (1) with zero boundary conditions is said to have $\mathscr{H}_\infty$ performance level $\gamma_{2,2} > 0$ if it is stable along the pass with $\omega_{k+1}(p) = 0$ and for all nonzero $\omega_{k+1}(p) \in \ell_2^l$

$$\|v_{k+1}(p)\|_{2,\alpha} < \gamma_{2,2} \|\omega_{k+1}(p)\|_{2,\alpha} \tag{6}$$

and ℓ_2–ℓ_∞ performance level $\gamma_{2,\infty} > 0$ if

$$\|v_{k+1}(p)\|_{\infty,\alpha} < \gamma_{2,\infty} \|\omega_{k+1}(p)\|_{2,\alpha} \tag{7}$$

for the $\mathscr{H}_\infty$ and ℓ_2–ℓ_∞ cases, respectively where

$$\|f_k(p)\|_{2,\alpha} \triangleq \sqrt{\sum_{k=0}^{\infty} \sum_{p=0}^{\alpha-1} f_k^T(p) f_k(p)}$$
$$\|f_k(p)\|_{\infty,\alpha} \triangleq \sqrt{\sup_{k\geq 0,\, p\in[0,\alpha-1]} f_k^T(p) f_k(p)}$$

Remark 3 Since we consider only linear dynamics, the process response consists (in the absence of control inputs) of two parts, one of which arises from the boundary conditions and the other from the disturbance terms. Since the performance measure above seeks to address the response to disturbances the boundary conditions are set to zero. In applications, of course, many aspects will be important and there is no attempt to achieve the maximum possible benefit over all measures. For some applications, disturbance rejection may be the major consideration and in such a case the most emphasis would be placed on this aspect of overall performance.

The ℓ_2–ℓ_∞ performance measure here is the minimization of the maximum peak amplitude amplification, measured by the ℓ_∞ norm for a signal with finite energy as measured by the ℓ_2 norm. In particular, the design task here is to find a control law which gives stability along the pass and also minimizes the worst case amplification effect of a finite energy disturbance on the controlled output (for further background on these norms and their use in other areas of systems theory see, for example, Gao and Wang 2003; Palhares and Peres 2000). The $\mathscr{H}_\infty$ performance measure is the ratio of the energy in the output signal to the energy in the disturbance signal as measured by the ℓ_2 norm. In all applications the pass length α is finite and this means that we are dealing with a subspace of the usual ℓ_2 space, but, for notational simplicity, the performance is still referred to as the $\mathscr{H}_\infty$ measure. Its main advantage is the fact that it is insensitive to the exact knowledge of the statistics of the disturbance signals. The relevance of these measures for discrete linear repetitive processes is well founded physically by noting the conditions in which physical examples have to operate, e.g. long-wall coal cutting and iterative learning control applications such as using a gantry robot to synchronously place objects on a chain conveyor (for details in this last case see the references listed in Rogers et al. 2007).

2.2 $\mathscr{H}_\infty$ Performance

The following result on $\mathscr{H}_\infty$ performance for discrete linear repetitive process can now be established.

Theorem 2 *A discrete linear repetitive process described by* (1) *with zero boundary conditions is stable along the pass with $\mathscr{H}_\infty$ performance level $\gamma_{2,2} > 0$ if there exist matrices $P > 0$ and $Q > 0$ such that the following LMI holds*:

$$\begin{bmatrix} -P & 0 & 0 & A^T P & C^T Q & G^T \\ * & -Q & 0 & B_0^T P & D_0^T Q & H_0^T \\ * & * & -\gamma_{2,2}^2 I & B_1^T P & D_1^T Q & 0 \\ * & * & * & -P & 0 & 0 \\ * & * & * & * & -Q & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{8}$$

Proof First, we establish the stability along the pass using the candidate Lyapunov function

$$\begin{aligned} V(k,p) &\triangleq V_1(p,k) + V_2(k,p), \\ V_1(p,k) &\triangleq x_{k+1}^T(p) P x_{k+1}(p), \\ V_2(k,p) &\triangleq y_k^T(p) Q y_k(p) \end{aligned} \tag{9}$$

where $P > 0$, $Q > 0$, with increment $\Delta V(k,p)$ defined by

$$\Delta V(k,p) \triangleq \Delta V_1(p,k) + \Delta V_2(k,p) \tag{10}$$

Hence

$$\begin{aligned} \Delta V_1(p,k) &= x_{k+1}^T(p+1) P x_{k+1}(p+1) - x_{k+1}^T(p) P x_{k+1}(p) \\ &= \left[A x_{k+1}(p) + B_0 y_k(p)\right]^T P \left[A x_{k+1}(p) + B_0 y_k(p)\right] \\ &\quad - x_{k+1}^T(p) P x_{k+1}(p) \end{aligned} \tag{11}$$

$$\begin{aligned} \Delta V_2(k,p) &= y_{k+1}^T(p) Q y_{k+1}(p) - y_k^T(p) Q y_k(p) \\ &= \left[C x_{k+1}(p) + D_0 y_k(p)\right]^T Q \left[C x_{k+1}(p) + D_0 y_k(p)\right] \\ &\quad - y_k^T(p) Q y_k(p) \end{aligned} \tag{12}$$

and it follows that

$$\Delta V(k,p) = \zeta_k^T(p) \left(\bar{A}^T \bar{P} \bar{A} + \bar{C}^T \bar{Q} \bar{C} - \bar{P} - \bar{Q}\right) \zeta_k(p) \triangleq \zeta_k^T(p) \Psi \zeta_k(p) \tag{13}$$

where

$$\zeta_k(p) \triangleq \begin{bmatrix} x_{k+1}(p) \\ y_k(p) \end{bmatrix}, \ \bar{A} \triangleq \begin{bmatrix} A & B_0 \\ 0 & 0 \end{bmatrix}, \ \bar{C} \triangleq \begin{bmatrix} 0 & 0 \\ C & D_0 \end{bmatrix}, \ \bar{P} \triangleq \begin{bmatrix} P & 0 \\ 0 & 0 \end{bmatrix}, \ \bar{Q} \triangleq \begin{bmatrix} 0 & 0 \\ 0 & Q \end{bmatrix}.$$

Application of the Schur's complement formula to the LMI (8) now leads immediately to $\Psi < 0$. Hence for any $\zeta_k(p) \neq 0$, we have $\Delta V(k,p) < 0$ and it follows immediately from results in Rogers et al. (2007) that stability along the pass holds.

We also have

$$\sum_{k=0}^{\infty} \sum_{p=0}^{\alpha-1} \Delta V(k,p) \triangleq \sum_{p=0}^{\alpha-1} \Delta V_1(p,k) + \sum_{k=0}^{\infty} \Delta V_2(k,p) \tag{14}$$

which will be used in establishing the $\mathscr{H}_\infty$ performance bound for a stable along the pass example.

Consider the associated performance index:

$$\mathscr{J} \triangleq \|v_{k+1}(p)\|_{2,\alpha}^2 - \gamma_{2,2}^2 \|\omega_{k+1}(p)\|_{2,\alpha}^2 \tag{15}$$

Then (on making use of (14))

$$
\begin{aligned}
\mathscr{J} &< \|v_{k+1}(p)\|_{2,\alpha}^2 - \gamma_{2,\infty}^2 \|\omega_{k+1}(p)\|_{2,\alpha}^2 + V(\infty,\alpha) - V(0,0) \\
&= \sum_{k=0}^{\infty}\sum_{p=0}^{\alpha-1}\left[v_{k+1}^T(p)v_{k+1}(p) - \gamma_{2,2}^2\omega_{k+1}^T(p)\omega_{k+1}(p)\right] + \sum_{p=0}^{\alpha-1}\Delta V_1(p,k) + \sum_{k=0}^{\infty}\Delta V_2(k,p) \\
&= \sum_{k=0}^{\infty}\sum_{p=0}^{\alpha-1}\left[v_{k+1}^T(p)v_{k+1}(p) - \gamma_{2,2}^2\omega_{k+1}^T(p)\omega_{k+1}(p) + \Delta V(k,p)\right] \\
&\triangleq \sum_{k=0}^{\infty}\sum_{p=0}^{\alpha-1}\eta_k^T(p)\Pi\eta_k(p)
\end{aligned}
\tag{16}
$$

where $\eta_k(p) \triangleq \begin{bmatrix} x_{k+1}^T(p) & y_k^T(p) & \omega_{k+1}^T(p) \end{bmatrix}^T$, $V(\infty,\alpha) \triangleq V_1(\alpha,k) + V_2(\infty,p)$, $V(0,0) \triangleq V_1(0,k) + V_2(0,p)$ and

$$
\Pi \triangleq \begin{bmatrix} -P & 0 & 0 \\ * & -Q & 0 \\ * & * & -\gamma_{2,2}^2 I \end{bmatrix} + \begin{bmatrix} A^T \\ B_0^T \\ B_1^T \end{bmatrix} P \begin{bmatrix} A^T \\ B_0^T \\ B_1^T \end{bmatrix}^T + \begin{bmatrix} C^T \\ D_0^T \\ D_1^T \end{bmatrix} Q \begin{bmatrix} C^T \\ D_0^T \\ D_1^T \end{bmatrix}^T + \begin{bmatrix} G^T \\ H_0^T \\ 0 \end{bmatrix}\begin{bmatrix} G^T \\ H_0^T \\ 0 \end{bmatrix}^T
$$

By the Schur's complement formula, (8) implies $\Pi < 0$ and hence for all $\eta_k(p) \neq 0$, we have $\mathscr{J} < 0$, i.e. $\|v_{k+1}(p)\|_{2,\alpha} < \gamma_{2,2}\|\omega_{k+1}(p)\|_{2,\alpha}$ for all nonzero $\omega_{k+1}(p) \in \ell_2^l$ and the proof is complete. □

2.3 ℓ_2–ℓ_∞ Performance

In the case of ℓ_2–ℓ_∞ performance, we have the following result.

Theorem 3 *A discrete linear repetitive process described by* (1) *is stable along the pass with ℓ_2–ℓ_∞ performance level $\gamma_{2,\infty} > 0$ if there exist matrices $P > 0$ and $Q > 0$ such that the following LMIs hold:*

$$
\begin{bmatrix} -P & 0 & 0 & A^TP & C^TQ \\ * & -Q & 0 & B_0^TP & D_0^TQ \\ * & * & -I & B_1^TP & D_1^TQ \\ * & * & * & -P & 0 \\ * & * & * & * & -Q \end{bmatrix} < 0 \tag{17}
$$

$$
\begin{bmatrix} P & 0 & G^T \\ * & Q & H_0^T \\ * & * & \gamma_{2,\infty}^2 I \end{bmatrix} > 0 \tag{18}
$$

Proof The proof of stability along the pass is identical to that in the previous result and hence the details are omitted here. To establish (noting again the assumption on the boundary conditions) the ℓ_2–ℓ_∞ performance bound for a stable along the pass process described by (1), consider the associated performance index:

$$
\mathscr{I} = V(k,p) - \sum_{s=0}^{k-1}\sum_{\beta=0}^{p-1}\omega_{s+1}^T(\beta)\omega_{s+1}(\beta) \tag{19}
$$

Then, we have

$$\begin{aligned}\mathscr{I} &= V(k,p) - V(0,0) - \sum_{s=0}^{k-1}\sum_{\beta=0}^{p-1} \omega_{s+1}^T(\beta)\omega_{s+1}(\beta)\\ &= \sum_{\beta=0}^{p-1} \Delta V_1(\beta,k) + \sum_{s=0}^{k-1}\Delta V_2(s,p) - \sum_{s=0}^{k-1}\sum_{\beta=0}^{p-1} \omega_{s+1}^T(\beta)\omega_{s+1}(\beta)\\ &< \sum_{s=0}^{k-1}\sum_{\beta=0}^{p-1}\left[\Delta V(s,\beta) - \omega_{s+1}^T(\beta)\omega_{s+1}(\beta)\right]\\ &\triangleq \sum_{s=0}^{k-1}\sum_{\beta=0}^{p-1} \eta_s^T(\beta)\Omega\eta_s(\beta) \end{aligned}\tag{20}$$

where $\eta_s(\beta) \triangleq \begin{bmatrix} x_{s+1}^T(\beta) & y_s^T(\beta) & \omega_{s+1}^T(\beta)\end{bmatrix}$, $V(0,0) \triangleq V_1(0,k) + V_2(0,p)$ and

$$\Omega \triangleq \begin{bmatrix} -P & 0 & 0\\ * & -Q & 0\\ * & * & -I\end{bmatrix} + \begin{bmatrix} A^T\\ B_0^T\\ B_1^T\end{bmatrix} P \begin{bmatrix} A^T\\ B_0^T\\ B_1^T\end{bmatrix}^T + \begin{bmatrix} C^T\\ D_0^T\\ D_1^T\end{bmatrix} Q \begin{bmatrix} C^T\\ D_0^T\\ D_1^T\end{bmatrix}^T$$

On applying the Schur's complement formula, the LMI of (17) implies that $\Omega < 0$. Hence, for all $\eta_k(p) \neq 0$, we have $\mathscr{I} < 0$, i.e

$$x_{k+1}^T(p)Px_{k+1}(p) + y_k^T(p)Qy_k(p) = V(k,p) < \sum_{s=0}^{k-1}\sum_{\beta=0}^{p-1}\omega_{s+1}^T(\beta)\omega_{s+1}(\beta) \tag{21}$$

Conversely, by applying the Schur's complement formula again, (18) is equivalent to

$$\begin{bmatrix} G^T\\ H_0^T\end{bmatrix}\begin{bmatrix} G & H_0\end{bmatrix} < \gamma_{2,\infty}^2 \begin{bmatrix} P & 0\\ 0 & Q\end{bmatrix} \tag{22}$$

Hence we can conclude from (3), (21) and (22) that for any $k > 0$ and $p \in [0, \alpha]$

$$\begin{aligned} v_{k+1}^T(p)v_{k+1}(p) &= \left[Gx_{k+1}(p) + H_0 y_k(p)\right]^T \left[Gx_{k+1}(p) + H_0 y_k(p)\right]\\ &< \gamma_{2,\infty}^2 \left[x_{k+1}^T(p)Px_{k+1}(p) + y_k^T(p)Qy_k(p)\right]\\ &< \gamma_{2,\infty}^2 \sum_{s=0}^{k-1}\sum_{\beta=0}^{p-1}\omega_{s+1}^T(\beta)\omega_{s+1}(\beta)\\ &< \gamma_{2,\infty}^2 \sum_{s=0}^{\infty}\sum_{\beta=0}^{\alpha-1}\omega_{s+1}^T(\beta)\omega_{s+1}(\beta)\end{aligned}\tag{23}$$

Taking the supremum over $k > 0$ and $p \in [0, \alpha - 1]$ yields (7) and the proof is complete. □

Remark 4 Repetitive processes are defined over the finite pass length α, and in practice the process will only complete a finite number of passes, say, N. Hence the corresponding cost function in this last result should be evaluated as given in (19). However, it is routine to argue that the signals involved can be extended from $[0, \alpha]$ to the infinite interval in such a way that projection of the infinite interval solution onto the finite interval is possible. Likewise from the infinite set to $[0, N]$, and hence we will work with (19).

3 Dynamic output feedback control

3.1 Problem formulation

The process state-space model is that of (1) augmented by control input terms, i.e.

$$\begin{aligned} x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + B_0 y_k(p) + B_1 \omega_{k+1}(p) \\ y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + D_0 y_k(p) + D_1 \omega_{k+1}(p) \end{aligned} \tag{24}$$

where on pass k, $u_{k+1}(p) \in \mathbb{R}^s$ is the control input vector.

Here, we seek to design a (full-order) dynamic output feedback controller of general structure described by

$$\begin{aligned} \varphi_{k+1}(p+1) &= A_c \varphi_{k+1}(p) + B_{0c} \phi_k(p) + B_c z_{k+1}(p) \\ \phi_{k+1}(p) &= C_c \varphi_{k+1}(p) + D_{0c} \phi_k(p) + D_c z_{k+1}(p) \\ u_{k+1}(p) &= G_c \varphi_{k+1}(p) + H_{0c} \phi_k(p) + H_c z_{k+1}(p) \end{aligned} \tag{25}$$

where $\varphi_{k+1}(p) \in \mathbb{R}^n$ and $\phi_k(p) \in \mathbb{R}^m$ are the controller state vectors in the along the pass and pass-to-pass directions, respectively, and $z_{k+1}(t)$ is the measured output vector defined by (2).

Remark 5 In the control design analysis in this paper we assume that a control law of the form considered can be found to give the required properties and characterize this in terms of LMI based sufficient conditions which if they hold lead immediately to the required numerical parameters. It would, of course, be much better to have necessary and sufficient conditions for the existence of a given control law, e.g. a result equivalent to that which states that controllability of 1D discrete linear systems state-space model is equivalent to a solution of the state feedback based pole placement problem. No such general result exists for discrete linear repetitive processes due, for example, to the fact that controllability for these processes and what a pole means are still relatively open questions—for progress on this area see Rogers et al. (2007).

Augmenting the model of (1) to include the states of dynamic output feedback controller (25) and using (2)–(3) gives the following state-space model for the controlled process

$$\begin{aligned} \xi_{k+1}(p+1) &= \tilde{A}\xi_{k+1}(p) + \tilde{B}_0 \zeta_k(p) + \tilde{B}_1 \omega_{k+1}(p) \\ \zeta_{k+1}(p) &= \tilde{C}\xi_{k+1}(p) + \tilde{D}_0 \zeta_k(p) + \tilde{D}_1 \omega_{k+1}(p) \\ v_{k+1}(p) &= \tilde{G}\xi_{k+1}(p) + \tilde{H}_0 \zeta_k(p) \end{aligned} \tag{26}$$

where $\xi_{k+1}(p) \triangleq \left[x_{k+1}^T(p)\ \varphi_{k+1}^T(p) \right]^T$, $\zeta_k(p) \triangleq \left[y_k^T(p)\ \phi_k^T(p) \right]^T$ and

$$\begin{aligned} &\tilde{A} \triangleq \begin{bmatrix} A + BH_c E & BG_c \\ B_c E & A_c \end{bmatrix}, \ \tilde{B}_0 \triangleq \begin{bmatrix} B_0 + BH_c F_0 & BH_{0c} \\ B_c F_0 & B_{0c} \end{bmatrix}, \ \tilde{B}_1 \triangleq \begin{bmatrix} B_1 + BH_c F_1 \\ B_c F_1 \end{bmatrix}, \\ &\tilde{C} \triangleq \begin{bmatrix} C + DH_c E & DG_c \\ D_c E & C_c \end{bmatrix}, \ \tilde{D}_0 \triangleq \begin{bmatrix} D_0 + DH_c F_0 & DH_{0c} \\ D_c F_0 & D_{0c} \end{bmatrix}, \ \tilde{D}_1 \triangleq \begin{bmatrix} D_1 + DH_c F_1 \\ D_c F_1 \end{bmatrix}, \\ &\tilde{G} \triangleq \left[G\ 0 \right], \ \tilde{H}_0 \triangleq \left[H_0\ 0 \right] \end{aligned} \tag{27}$$

The problem considered in this section is the design of a controller of the form (25), with either $\mathscr{H}_\infty$ or ℓ_2–ℓ_∞ performance, subject to the following two requirements:

1. The controlled process is stable along the pass.
2. The controlled process has disturbance attenuation level $\gamma_{2,2}$ in an $\mathscr{H}_\infty$ (or level $\gamma_{2,\infty}$ in an ℓ_2–ℓ_∞) sense. In particular, for all nonzero $\omega_{k+1}(p) \in \ell_2^l$, (6) holds for the $\mathscr{H}_\infty$ case and (7) for ℓ_2–ℓ_∞.

Remark 6 The controller defined in (25) uses local information at $(k+1, p)$, (k, p) to determine the control signal at $(k+1, p)$. It is possible to utilize information of the whole previous profile, namely, information from $(k, 0)$, $(k, 1)$, …, $(k, \alpha-1)$ to determine the control signal at $(k+1, p)$ but clearly this should only occur if the extra complexity involved produces clear performance advantages over the simpler structures considered here. A detailed investigation of this general point should be undertaken once the full potential of controllers, such as those considered here, which only make the minimum use of previous pass information has been established.

3.2 $\mathscr{H}_\infty$ Dynamic output feedback control design

First, we state the following preliminary result whose proof follows identical steps to that of Theorem 2 and is hence omitted here.

Theorem 4 *A discrete linear repetitive process with state-space model* (26) *is stable along the pass with $\mathscr{H}_\infty$ performance level $\gamma_{2,2} > 0$ if there exist matrices $P > 0$ and $Q > 0$ such that the following LMI holds:*

$$\begin{bmatrix} -P & 0 & 0 & \tilde{A}^T P & \tilde{C}^T Q & \tilde{G}^T \\ * & -Q & 0 & \tilde{B}_0^T P & \tilde{D}_0^T Q & \tilde{H}_0^T \\ * & * & -\gamma_{2,2}^2 I & \tilde{B}_1^T P & \tilde{D}_1^T Q & 0 \\ * & * & * & -P & 0 & 0 \\ * & * & * & * & -Q & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{28}$$

This result however does not allow us to achieve the controller required, but it provides the frame to solve effectively the $\mathscr{H}_\infty$ the dynamic output feedback control problem together with the design procedure.

Theorem 5 *Consider a discrete linear repetitive process described by* (1) *and let $\gamma_{2,2} > 0$ be a prescribed scalar. Then for such a process there exists a full-order dynamic output feedback controller of the form* (25) *for which the resulting controlled process* (26) *is stable along the pass and* (6) *holds if there exist matrices $\mathscr{P} > 0$, $\mathscr{R} > 0$, $\mathscr{Q} > 0$, $\mathscr{S} > 0$, $\mathscr{A}_c$, $\mathscr{B}_{0c}$, $\mathscr{B}_c$, $\mathscr{C}_c$, $\mathscr{D}_{0c}$, $\mathscr{D}_c$, $\mathscr{G}_c$, $\mathscr{H}_{0c}$ and $\mathscr{H}_c$ such that the following LMI holds:*

$$\begin{bmatrix} -\mathscr{P} & -I & 0 & 0 & 0 & \Psi_{16}^T & \Psi_{17}^T & \Psi_{18}^T & \Psi_{19}^T & G^T \\ * & -\mathscr{R} & 0 & 0 & 0 & \mathscr{A}_c^T & \Psi_{27}^T & \mathscr{C}_c^T & \Psi_{29}^T & \mathscr{R}G^T \\ * & * & -\mathscr{Q} & -I & 0 & \Psi_{36}^T & \Psi_{37}^T & \Psi_{38}^T & \Psi_{39}^T & H_0^T \\ * & * & * & -\mathscr{S} & 0 & \mathscr{B}_{0c}^T & \Psi_{47}^T & \mathscr{D}_{0c}^T & \Psi_{49}^T & \mathscr{S}H_0^T \\ * & * & * & * & -\gamma_{2,2}^2 I & \Psi_{56}^T & \Psi_{57}^T & \Psi_{58}^T & \Psi_{59}^T & 0 \\ * & * & * & * & * & -\mathscr{P} & -I & 0 & 0 & 0 \\ * & * & * & * & * & * & -\mathscr{R} & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\mathscr{Q} & -I & 0 \\ * & * & * & * & * & * & * & * & -\mathscr{S} & 0 \\ * & * & * & * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{29}$$

where

$$\Psi_{16} \triangleq \mathscr{P}A + \mathscr{B}_c E,\ \Psi_{36} \triangleq \mathscr{P}B_0 + \mathscr{B}_c F_0,\ \Psi_{56} \triangleq \mathscr{P}B_1 + \mathscr{B}_c F_1,\ \Psi_{17} \triangleq A + B\mathscr{H}_c E$$
$$\Psi_{27} \triangleq A\mathscr{R} + B\mathscr{G}_c,\ \Psi_{37} \triangleq B_0 + B\mathscr{H}_c F_0,\ \Psi_{47} \triangleq B_0\mathscr{S} + B\mathscr{H}_{0c},\ \Psi_{57} \triangleq B_1 + B\mathscr{H}_c F_1$$
$$\Psi_{18} \triangleq \mathscr{Q}C + \mathscr{D}_c E,\ \Psi_{38} \triangleq \mathscr{Q}D_0 + \mathscr{D}_c F_0,\ \Psi_{58} \triangleq \mathscr{Q}D_1 + \mathscr{D}_c F_1,\ \Psi_{19} \triangleq C + D\mathscr{H}_c E$$
$$\Psi_{29} \triangleq C\mathscr{R} + D\mathscr{G}_c,\ \Psi_{39} \triangleq D_0 + D\mathscr{H}_c F_0,\ \Psi_{49} \triangleq D_0\mathscr{S} + D\mathscr{H}_{0c},\ \Psi_{59} \triangleq D_1 + D\mathscr{H}_c F_1$$

Moreover, a desired $\mathscr{H}_\infty$ dynamic output feedback controller can be found by solving the following equations:

$$\begin{aligned}
\mathscr{H}_c &= H_c,\\
\mathscr{H}_{0c} &= H_c F_0 \mathscr{S} + H_{0c} S_{12}^T,\\
\mathscr{G}_c &= H_c E\mathscr{R} + G_c R_{12}^T,\\
\mathscr{D}_c &= \mathscr{Q}DH_c + Q_{12}D_c,\\
\mathscr{B}_c &= \mathscr{P}BH_c + P_{12}B_c,\\
\mathscr{D}_{0c} &= \mathscr{Q}(D_0 + DH_cF_0)\mathscr{S} + Q_{12}D_cF_0\mathscr{S} + \mathscr{Q}DH_{0c}S_{12}^T + Q_{12}D_{0c}S_{12}^T,\\
\mathscr{C}_c &= \mathscr{Q}(C + DH_cE)\mathscr{R} + Q_{12}D_cE\mathscr{R} + \mathscr{Q}DG_cR_{12}^T + Q_{12}C_cR_{12}^T,\\
\mathscr{B}_{0c} &= \mathscr{P}(B_0 + BH_cF_0)\mathscr{S} + P_{12}B_cF_0\mathscr{S} + \mathscr{P}BH_{0c}S_{12}^T + P_{12}B_{0c}S_{12}^T,\\
\mathscr{A}_c &= \mathscr{P}(A + BH_cE)\mathscr{R} + P_{12}B_cE\mathscr{R} + \mathscr{P}BG_cR_{12}^T + P_{12}A_cR_{12}^T
\end{aligned} \tag{30}$$

where P_{12}, R_{12}, Q_{12} and S_{12} are defined by any full rank factorization of $P_{12}R_{12}^T = I - \mathscr{P}\mathscr{R}$ and $Q_{12}S_{12}^T = I - \mathscr{Q}\mathscr{S}$, respectively (derived from $P_{11}R_{11} + P_{12}R_{12}^T = I$ and $Q_{11}S_{11} + Q_{12}S_{12}^T = I$, respectively).

Proof It follows immediately from Theorem 4 that the matrices P and Q are nonsingular if (28) holds since $P > 0$ and $Q > 0$. Also introduce $R = P^{-1}$, $Q = S^{-1}$, and partition P, R, Q and S as follows:

$$P \triangleq \begin{bmatrix} P_{11} & P_{12} \\ * & P_{22} \end{bmatrix},\ R = P^{-1} \triangleq \begin{bmatrix} R_{11} & R_{12} \\ * & R_{22} \end{bmatrix}$$
$$Q \triangleq \begin{bmatrix} Q_{11} & Q_{12} \\ * & Q_{22} \end{bmatrix},\ S = Q^{-1} \triangleq \begin{bmatrix} S_{11} & S_{12} \\ * & S_{22} \end{bmatrix}$$

Then since we are considering a full-order controller, P_{12} and R_{12} are square and without loss of generality we also assume that these matrices are nonsingular. (If this is not the case , P_{12} and R_{12} may be perturbed by matrices ΔP_{12} and ΔR_{12}, respectively with sufficiently small norms such that $P_{12} + \Delta P_{12}$ and $P_{12} + \Delta P_{12}$ are nonsingular and satisfy (28)). Similarly, we assume that Q_{12} and S_{12} are nonsingular and hence we can define the following nonsingular matrices

$$\Gamma_P \triangleq \begin{bmatrix} P_{11} & I \\ P_{12}^T & 0 \end{bmatrix},\ \Gamma_R \triangleq \begin{bmatrix} I & R_{11} \\ 0 & R_{12}^T \end{bmatrix},\ \Gamma_Q \triangleq \begin{bmatrix} Q_{11} & I \\ Q_{12}^T & 0 \end{bmatrix},\ \Gamma_S \triangleq \begin{bmatrix} I & S_{11} \\ 0 & S_{12}^T \end{bmatrix} \tag{31}$$

Note that

$$\begin{aligned}
&P\Gamma_R = \Gamma_P,\ R\Gamma_P = \Gamma_R,\ P_{11}R_{11} + P_{12}R_{12}^T = I,\\
&Q\Gamma_S = \Gamma_Q,\ S\Gamma_Q = \Gamma_S,\ \ Q_{11}S_{11} + Q_{12}S_{12}^T = I
\end{aligned} \tag{32}$$

and also pre- and post-multiplying (28) by the diagonal matrix diag$(\Gamma_R, \Gamma_S, I, \Gamma_R, \Gamma_S, I)$, gives

$$\begin{bmatrix} -\Gamma_R^T\Gamma_P & 0 & 0 & \Gamma_R^T\tilde{A}^T\Gamma_P & \Gamma_R^T\tilde{C}^T\Gamma_Q & \Gamma_R^T\tilde{G}^T \\ * & -\Gamma_S^T\Gamma_Q & 0 & \Gamma_S^T\tilde{B}_0^T\Gamma_P & \Gamma_S^T\tilde{D}_0^T\Gamma_Q & \Gamma_S^T\tilde{H}_0^T \\ * & * & -\gamma_{2,2}^2 I & \tilde{B}_1^T\Gamma_P & \tilde{D}_1^T\Gamma_Q & 0 \\ * & * & * & -\Gamma_R^T\Gamma_P & 0 & 0 \\ * & * & * & * & -\Gamma_S^T\Gamma_Q & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{33}$$

Now introduce $\mathscr{P} \triangleq P_{11}, \mathscr{R} \triangleq R_{11}, \mathscr{Q} \triangleq Q_{11}, \mathscr{S} \triangleq S_{11}$ and the following matrices:

$$\begin{aligned}
\mathscr{A}_c &\triangleq P_{11}(A + BH_cE)R_{11} + P_{12}B_cER_{11} + P_{11}BG_cR_{12}^T + P_{12}A_cR_{12}^T \\
\mathscr{B}_{0c} &\triangleq P_{11}(B_0 + BH_cF_0)S_{11} + P_{12}B_cF_0S_{11} + P_{11}BH_{0c}S_{12}^T + P_{12}B_{0c}S_{12}^T \\
\mathscr{C}_c &\triangleq Q_{11}(C + DH_cE)R_{11} + Q_{12}D_cER_{11} + Q_{11}DG_cR_{12}^T + Q_{12}C_cR_{12}^T \\
\mathscr{D}_{0c} &\triangleq Q_{11}(D_0 + DH_cF_0)S_{11} + Q_{12}D_cF_0S_{11} + Q_{11}DH_{0c}S_{12}^T + Q_{12}D_{0c}S_{12}^T \\
\mathscr{B}_c &\triangleq P_{11}BH_c + P_{12}B_c \\
\mathscr{D}_c &\triangleq Q_{11}DH_c + Q_{12}D_c \\
\mathscr{G}_c &\triangleq H_cER_{11} + G_cR_{12}^T \\
\mathscr{H}_{0c} &\triangleq H_cF_0S_{11} + H_{0c}S_{12}^T \\
\mathscr{H}_c &\triangleq H_c
\end{aligned} \tag{34}$$

Then, noting (27), we have the following in (33):

$$\begin{aligned}
&\Gamma_P^T\tilde{A}\Gamma_R \triangleq \begin{bmatrix} \mathscr{P}A + \mathscr{B}_cE & \mathscr{A}_c \\ A + B\mathscr{H}_cE & A\mathscr{R} + B\mathscr{G}_c \end{bmatrix}, \quad \Gamma_P^T\Gamma_R \triangleq \begin{bmatrix} \mathscr{P} & I \\ I & \mathscr{R} \end{bmatrix}, \\
&\Gamma_P^T\tilde{B}_0\Gamma_S \triangleq \begin{bmatrix} \mathscr{P}B_0 + \mathscr{B}_cF_0 & \mathscr{B}_{0c} \\ B_0 + B\mathscr{H}_cF_0 & B_0\mathscr{S} + B\mathscr{H}_{0c} \end{bmatrix}, \quad \Gamma_S^T\Gamma_Q \triangleq \begin{bmatrix} \mathscr{Q} & I \\ I & \mathscr{S} \end{bmatrix}, \\
&\Gamma_Q^T\tilde{C}\Gamma_R \triangleq \begin{bmatrix} \mathscr{Q}C + \mathscr{D}_cE & \mathscr{C}_c \\ C + D\mathscr{H}_cE & C\mathscr{R} + D\mathscr{G}_c \end{bmatrix}, \quad \Gamma_P^T\tilde{B}_1 \triangleq \begin{bmatrix} \mathscr{P}B_1 + \mathscr{B}_cF_1 \\ B_1 + B\mathscr{H}_cF_1 \end{bmatrix}, \\
&\Gamma_Q^T\tilde{D}_0\Gamma_S \triangleq \begin{bmatrix} \mathscr{Q}D_0 + \mathscr{D}_cF_0 & \mathscr{D}_{0c} \\ D_0 + D\mathscr{H}_cF_0 & D_0\mathscr{S} + D\mathscr{H}_{0c} \end{bmatrix}, \quad \Gamma_Q^T\tilde{D}_1 \triangleq \begin{bmatrix} \mathscr{Q}D_1 + \mathscr{D}_cF_1 \\ D_1 + D\mathscr{H}_cF_1 \end{bmatrix}, \\
&\tilde{G}\Gamma_R \triangleq \begin{bmatrix} G & G\mathscr{R} \end{bmatrix}, \quad \tilde{H}_0\Gamma_S \triangleq \begin{bmatrix} H_0 & H_0\mathscr{S} \end{bmatrix}
\end{aligned} \tag{35}$$

Substituting (35) into (33) now gives (29). Conversely, substituting $\mathscr{P} \triangleq P_{11}$, $\mathscr{R} \triangleq R_{11}$, $\mathscr{Q} \triangleq Q_{11}$ and $\mathscr{S} \triangleq S_{11}$ into (34) gives (30). Hence on applying Theorem 4 we have that the controlled process is stable along the pass with $\mathscr{H}_\infty$ performance level $\gamma_{2,2}$. □

Remark 7 Note that Theorem 5 gives a sufficient condition for solvability of $\mathscr{H}_\infty$ dynamic output feedback control problem for the discrete linear repetitive processes. Since the obtained condition is in LMI form, a desired controller can be determined by solving the following convex optimization problem:

$$\min\ \sigma_1 \qquad \textit{subject to}\ (29) \quad (\textit{where}\ \sigma_1 = \gamma_{2,2}^2) \tag{36}$$

3.3 ℓ_2–ℓ_∞ Dynamic output feedback control

In a similar manner to the $\mathscr{H}_\infty$ case, the following result can be established using, in effect, the arguments required in the proof of Theorem 3 and hence the details are omitted here.

Theorem 6 *A discrete linear repetitive process described by (26) is stable along the pass with ℓ_2–ℓ_∞ performance level $\gamma_{2,\infty} > 0$ if there exist matrices $P > 0$ and $Q > 0$ such that the following LMIs hold:*

$$\begin{bmatrix} -P & 0 & 0 & \tilde{A}^T P & \tilde{C}^T Q \\ * & -Q & 0 & \tilde{B}_0^T P & \tilde{D}_0^T Q \\ * & * & -I & \tilde{B}_1^T P & \tilde{D}_1^T Q \\ * & * & * & -P & 0 \\ * & * & * & * & -Q \end{bmatrix} < 0 \tag{37}$$

$$\begin{bmatrix} -P & 0 & \tilde{G}^T \\ * & -Q & \tilde{H}_0^T \\ * & * & -\gamma_{2,\infty}^2 I \end{bmatrix} < 0 \tag{38}$$

This result however again does not allow us to achieve the controller required, but it provides the setting to solve the ℓ_2–ℓ_∞ the dynamic output feedback control problem together with the controller design procedure.

Theorem 7 *Consider a discrete linear repetitive process described by* (1) *and let $\gamma_{2,\infty} > 0$ be a prescribed scalar. There exists a full-order dynamic output feedback controller of the form* (25) *such that the controlled process* (26) *is stable along the pass and* (7) *is satisfied if there exist matrices $\mathscr{P} > 0$, $\mathscr{R} > 0$, $\mathscr{Q} > 0$, $\mathscr{S} > 0$, $\mathscr{A}_c$, $\mathscr{B}_{0c}$, $\mathscr{B}_c$, $\mathscr{C}_c$, $\mathscr{D}_{0c}$, $\mathscr{D}_c$, $\mathscr{G}_c$, $\mathscr{H}_{0c}$ and $\mathscr{H}_c$ such that the following LMIs hold*:

$$\begin{bmatrix} -\mathscr{P} & -I & 0 & 0 & 0 & \Psi_{16}^T & \Psi_{17}^T & \Psi_{18}^T & \Psi_{19}^T \\ * & -\mathscr{R} & 0 & 0 & 0 & \mathscr{A}_c^T & \Psi_{27}^T & \mathscr{C}_c^T & \Psi_{29}^T \\ * & * & -\mathscr{Q} & -I & 0 & \Psi_{36}^T & \Psi_{37}^T & \Psi_{38}^T & \Psi_{39}^T \\ * & * & * & -\mathscr{S} & 0 & \mathscr{B}_{0c}^T & \Psi_{47}^T & \mathscr{D}_{0c}^T & \Psi_{49}^T \\ * & * & * & * & -I & \Psi_{56}^T & \Psi_{57}^T & \Psi_{58}^T & \Psi_{59}^T \\ * & * & * & * & * & -\mathscr{P} & -I & 0 & 0 \\ * & * & * & * & * & * & -\mathscr{R} & 0 & 0 \\ * & * & * & * & * & * & * & -\mathscr{Q} & -I \\ * & * & * & * & * & * & * & * & -\mathscr{S} \end{bmatrix} < 0 \tag{39}$$

$$\begin{bmatrix} -\mathscr{P} & -I & 0 & 0 & G^T \\ * & -\mathscr{R} & 0 & 0 & \mathscr{R}G^T \\ * & * & -\mathscr{Q} & -I & H_0^T \\ * & * & * & -\mathscr{S} & \mathscr{S}H_0^T \\ * & * & * & * & -\gamma_{2,\infty}^2 I \end{bmatrix} < 0 \tag{40}$$

where Ψ_{ij} $(i = 1, 2, 3, 4, 5;\ j = 6, 7, 8, 9)$ are defined in Theorem 5. *Moreover, a desired ℓ_2–ℓ_∞ dynamic output feedback controller can be computed from* (30).

Proof Defining Γ_P, Γ_R, Γ_Q and Γ_S as in (31) and then pre- and post-multiplying (37) and (38) by matrices diag $(\Gamma_R, \Gamma_S, I, \Gamma_R, \Gamma_S)$ and diag (Γ_R, Γ_S, I), respectively yield

$$\begin{bmatrix} -\Gamma_R^T\Gamma_P & 0 & 0 & \Gamma_R^T\tilde{A}^T\Gamma_P & \Gamma_R^T\tilde{C}^T\Gamma_Q \\ * & -\Gamma_S^T\Gamma_Q & 0 & \Gamma_S^T\tilde{B}_0^T\Gamma_P & \Gamma_S^T\tilde{D}_0^T\Gamma_Q \\ * & * & -I & \tilde{B}_1^T\Gamma_P & \tilde{D}_1^T\Gamma_Q \\ * & * & * & -\Gamma_R^T\Gamma_P & 0 \\ * & * & * & * & -\Gamma_S^T\Gamma_Q \end{bmatrix} < 0 \tag{41}$$

$$\begin{bmatrix} -\Gamma_R^T\Gamma_P & 0 & \Gamma_R^T\tilde{G}^T \\ * & -\Gamma_S^T\Gamma_Q & \Gamma_S^T\tilde{H}_0^T \\ * & * & -\gamma_{2,\infty}^2 I \end{bmatrix} < 0 \tag{42}$$

Substituting (35) into (41) and (42), we obtain (39) and (40), respectively. The second part of proof follows the same argument as the corresponding part in Theorem 5 and hence the details are omitted here. □

Remark 8 Theorem 7 provides a sufficient condition for solvability of ℓ_2–ℓ_∞ dynamic output feedback control problem for the discrete linear repetitive processes. As in the $\mathscr{H}_\infty$ case, a desired controller can be determined by solving the following convex optimization problem:

$$\min\ \sigma_2 \qquad \textit{subject to } (39) \textit{ and } (40) \quad (\textit{where } \sigma_2 = \gamma_{2,\infty}^2) \tag{43}$$

4 $\mathscr{H}_\infty$ and ℓ_2–ℓ_∞ filtering

Suppose that the example under consideration is stable along the pass. Then the problem considered in this section is the estimation of the signal $v_{k+1}(p) \in \mathbb{R}^q$ of (3) for a discrete linear repetitive process described by (1) based on the measured output vector $z_{k+1}(p) \in \mathbb{R}^r$ defined by (2). The aim is to construct a linear full-order dynamic filter

$$\begin{aligned} \varphi_{k+1}(p+1) &= A_f\varphi_{k+1}(p) + B_{0f}\phi_k(p) + B_f z_{k+1}(p) \\ \phi_{k+1}(p) &= C_f\varphi_{k+1}(p) + D_{0f}\phi_k(p) + D_f z_{k+1}(p) \\ \hat{v}_{k+1}(p) &= G_f\varphi_{k+1}(p) + H_{0f}\phi_k(p) + H_f z_{k+1}(p) \end{aligned} \tag{44}$$

where on pass k, $\varphi_{k+1}(p) \in \mathbb{R}^n$ and $\phi_k(p) \in \mathbb{R}^m$ are the state vector and the profile vector for the filter, respectively.

Augmenting (1) to include the states of filter (44) and using (2)–(3) gives the following description of the filtering error process

$$\begin{aligned} \xi_{k+1}(p+1) &= \tilde{A}\xi_{k+1}(p) + \tilde{B}_0\zeta_k(p) + \tilde{B}_1\omega_{k+1}(p) \\ \zeta_{k+1}(p) &= \tilde{C}\xi_{k+1}(p) + \tilde{D}_0\zeta_k(p) + \tilde{D}_1\omega_{k+1}(p) \\ e_{k+1}(p) &= \tilde{G}\xi_{k+1}(p) + \tilde{H}_0\zeta_k(p) + \tilde{H}_1\omega_{k+1}(p) \end{aligned} \tag{45}$$

where $\xi_{k+1}(p) \triangleq \left[x_{k+1}^T(p)\ \varphi_{k+1}^T(p) \right]^T$, $\zeta_k(p) \triangleq \left[y_k^T(p)\ \phi_k^T(p) \right]^T$, $e_{k+1}(p) \triangleq v_{k+1}(p) - \hat{v}_{k+1}(p)$ and

$$\begin{aligned} &\tilde{A} \triangleq \begin{bmatrix} A & 0 \\ B_f E & A_f \end{bmatrix}, \quad \tilde{B}_0 \triangleq \begin{bmatrix} B_0 & 0 \\ B_f F_0 & B_{0f} \end{bmatrix}, \quad \tilde{B}_1 \triangleq \begin{bmatrix} B_1 \\ B_f F_1 \end{bmatrix}, \\ &\tilde{C} \triangleq \begin{bmatrix} C & 0 \\ D_f E & C_f \end{bmatrix}, \quad \tilde{D}_0 \triangleq \begin{bmatrix} D_0 & 0 \\ D_f F_0 & D_{0f} \end{bmatrix}, \quad \tilde{D}_1 \triangleq \begin{bmatrix} D_1 \\ D_f F_1 \end{bmatrix}, \\ &\tilde{G} \triangleq \left[G - H_f E \ -G_f \right], \quad \tilde{H}_0 \triangleq \left[H_0 - H_f F_0 \ -H_{0f} \right], \quad \tilde{H}_1 \triangleq -H_F F_1 \end{aligned} \tag{46}$$

The problem now is to develop a full-order $\mathscr{H}_\infty$ (or ℓ_2–ℓ_∞) filter of the form (44) such that the resulting filtering error process (45) is stable along the pass with noise attenuation level $\gamma_{2,2}$ in an $\mathscr{H}_\infty$ (or $\gamma_{2,\infty}$ in an ℓ_2–ℓ_∞) sense. More specifically, under zero boundary conditions and for all nonzero $\omega_{k+1}(p) \in \ell_2^l$, we require that

$$\|e_{k+1}(p)\|_{2,\alpha} < \gamma_{2,2}\,\|\omega_{k+1}(p)\|_{2,\alpha} \tag{47}$$

for the $\mathscr{H}_\infty$ filtering problem, and

$$\|e_{k+1}(p)\|_{\infty,\alpha} < \gamma_{2,\infty}\,\|\omega_{k+1}(p)\|_{2,\alpha} \tag{48}$$

for the ℓ_2–ℓ_∞ filtering problem.

4.1 $\mathscr{H}_\infty$ Filtering

The following result is proved using identical steps to that of Theorem 2 and hence the details are omitted here.

Theorem 8 *The filtering error process described by (45) is stable along the pass with $\mathscr{H}_\infty$ performance level $\gamma_{2,2} > 0$ if there exist matrices $P_1 > 0$ and $P_2 > 0$ such that the following LMI holds*:

$$\begin{bmatrix} -P_1 & 0 & 0 & \tilde{A}^T P_1 & \tilde{C}^T P_2 & \tilde{G}^T \\ * & -P_2 & 0 & \tilde{B}_0^T P_1 & \tilde{D}_0^T P_2 & \tilde{H}_0^T \\ * & * & -\gamma_{2,2}^2 I & \tilde{B}_1^T P_1 & \tilde{D}_1^T P_2 & \tilde{H}_1^T \\ * & * & * & -P_1 & 0 & 0 \\ * & * & * & * & -P_2 & 0 \\ * & * & * & * & * & -I \end{bmatrix} < 0 \tag{49}$$

Similarly as in previous sections for the control problem, this result cannot be directly used for the respective filter design, but provides the setting for this the $\mathscr{H}_\infty$ case.

Theorem 9 *Consider a discrete linear repetitive process described by (1) and let $\gamma_{2,2} > 0$ be a prescribed scalar. Then a full-order filter of the form (44) can be designed such that the filtering error process (45) is stable along the pass and (47) is satisfied if there exist matrices $\mathscr{U}_1 > 0$, $\mathscr{V}_1 > 0$, $\mathscr{U}_2 > 0$, $\mathscr{V}_2 > 0$, $\mathscr{A}_f$, $\mathscr{B}_{0f}$, $\mathscr{B}_f$, $\mathscr{C}_f$, $\mathscr{D}_{0f}$, $\mathscr{D}_f$, $\mathscr{G}_f$, $\mathscr{H}_{0f}$ and $\mathscr{H}_f$ such that the following LMI holds:*

$$\begin{bmatrix} -\mathscr{U}_1 & -\mathscr{V}_1 & 0 & 0 & 0 & \Upsilon_{16}^T & \Upsilon_{17}^T & \Upsilon_{18}^T & \Upsilon_{19}^T & \Upsilon_{110}^T \\ * & -\mathscr{V}_1 & 0 & 0 & 0 & \mathscr{A}_f^T & \mathscr{A}_f^T & \mathscr{C}_f^T & \mathscr{C}_f^T & -\mathscr{G}_f^T \\ * & * & -\mathscr{U}_2 & -\mathscr{V}_2 & 0 & \Upsilon_{36}^T & \Upsilon_{37}^T & \Upsilon_{38}^T & \Upsilon_{39}^T & \Upsilon_{310}^T \\ * & * & * & -\mathscr{V}_2 & 0 & \mathscr{B}_{0f}^T & \mathscr{B}_{0f}^T & \mathscr{D}_{0f}^T & \mathscr{D}_{0f}^T & -\mathscr{H}_{0f}^T \\ * & * & * & * & -\gamma_{2,2}^2 I & \Upsilon_{56}^T & \Upsilon_{57}^T & \Upsilon_{58}^T & \Upsilon_{59}^T & -F_1^T \mathscr{H}_f^T \\ * & * & * & * & * & -\mathscr{U}_1 & -\mathscr{V}_1 & 0 & 0 & 0 \\ * & * & * & * & * & * & -\mathscr{V}_1 & 0 & 0 & 0 \\ * & * & * & * & * & * & * & -\mathscr{U}_2 & -\mathscr{V}_2 & 0 \\ * & * & * & * & * & * & * & * & -\mathscr{V}_2 & 0 \\ * & * & * & * & * & * & * & * & * & -I \end{bmatrix} < 0 \tag{50}$$

where

$$\begin{aligned}
&\Upsilon_{16} \triangleq \mathscr{U}_1 A + \mathscr{B}_f E, \quad \Upsilon_{36} \triangleq \mathscr{U}_1 B_0 + \mathscr{B}_f F_0, \quad \Upsilon_{56} \triangleq \mathscr{U}_1 B_1 + \mathscr{B}_f F_1,\\
&\Upsilon_{17} \triangleq \mathscr{V}_1 A + \mathscr{B}_f E, \quad \Upsilon_{37} \triangleq \mathscr{V}_1 B_0 + \mathscr{B}_f F_0, \quad \Upsilon_{57} \triangleq \mathscr{V}_1 B_1 + \mathscr{B}_f F_1,\\
&\Upsilon_{18} \triangleq \mathscr{U}_2 C + \mathscr{D}_f E, \quad \Upsilon_{38} \triangleq \mathscr{U}_2 D_0 + \mathscr{D}_f F_0, \quad \Upsilon_{58} \triangleq \mathscr{U}_2 D_1 + \mathscr{D}_f F_1,\\
&\Upsilon_{19} \triangleq \mathscr{V}_2 C + \mathscr{D}_f E, \quad \Upsilon_{39} \triangleq \mathscr{V}_2 D_0 + \mathscr{D}_f F_0, \quad \Upsilon_{59} \triangleq \mathscr{V}_2 D_1 + \mathscr{D}_f F_1,\\
&\Upsilon_{110} \triangleq G - \mathscr{H}_f E, \quad \Upsilon_{310} \triangleq H_0 - \mathscr{H}_f F_0
\end{aligned} \tag{51}$$

Moreover, the filter can be computed from

$$\begin{bmatrix} A_f & B_{0f} & B_f \\ C_f & D_{0f} & D_f \\ G_f & H_{0f} & H_f \end{bmatrix} = \begin{bmatrix} \mathscr{V}_1^{-1} & 0 & 0 \\ 0 & \mathscr{V}_2^{-1} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathscr{A}_f & \mathscr{B}_{0f} & \mathscr{B}_f \\ \mathscr{C}_f & \mathscr{D}_{0f} & \mathscr{D}_f \\ \mathscr{G}_f & \mathscr{H}_{0f} & \mathscr{H}_f \end{bmatrix} \tag{52}$$

Proof From Theorem 8, P_1 and P_2 are both nonsingular if (49) holds since $P_1 > 0$ and $P_2 > 0$. Now, compatibly partition these matrices as

$$P_1 \triangleq \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{13} \end{bmatrix}, \; P_2 \triangleq \begin{bmatrix} P_{21} & P_{22} \\ P_{22}^T & P_{23} \end{bmatrix} \tag{53}$$

Then since we are considering a full-order filter, P_{12} and P_{12} are square and without loss of generality we assume P_{12} and P_{22} are nonsingular (if not, P_{12} and P_{22} may be perturbed by matrices ΔP_{12} and ΔP_{22} with sufficiently small norms respectively such that $P_{12} + \Delta P_{12}$ and $P_{22} + \Delta P_{22}$ are nonsingular and satisfy (49)). Also introduce the following matrices:

$$\begin{aligned}
&\Gamma_1 \triangleq \begin{bmatrix} I & 0 \\ 0 & P_{13}^{-1} P_{12}^T \end{bmatrix}, \quad \Gamma_2 \triangleq \begin{bmatrix} I & 0 \\ 0 & P_{23}^{-1} P_{22}^T \end{bmatrix}\\
&\mathscr{U}_1 \triangleq P_{11}, \; \mathscr{V}_1 \triangleq P_{12} P_{13}^{-1} P_{12}^T, \; \mathscr{U}_2 \triangleq P_{21}, \; \mathscr{V}_2 \triangleq P_{22} P_{23}^{-1} P_{22}^T
\end{aligned} \tag{54}$$

and

$$\begin{bmatrix} \mathscr{A}_f & \mathscr{B}_{0f} & \mathscr{B}_f \\ \mathscr{C}_f & \mathscr{D}_{0f} & \mathscr{D}_f \\ \mathscr{G}_f & \mathscr{H}_{0f} & \mathscr{H}_f \end{bmatrix} \triangleq \begin{bmatrix} P_{12} & 0 & 0 \\ 0 & P_{22} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} A_f & B_{0f} & B_f \\ C_f & D_{0f} & D_f \\ G_f & H_{0f} & H_f \end{bmatrix} \begin{bmatrix} P_{13}^{-1} P_{12}^T & 0 & 0 \\ 0 & P_{23}^{-1} P_{22}^T & 0 \\ 0 & 0 & I \end{bmatrix} \tag{55}$$

Pre- and post-multiplying (49) by diag $(\Gamma_1, \Gamma_2, I, \Gamma_1, \Gamma_2, I)$ now gives

$$\begin{bmatrix}
-\Gamma_1^T P_1 \Gamma_1 & 0 & 0 & \Gamma_1^T \tilde{A}^T P_1 \Gamma_1 & \Gamma_1^T \tilde{C}^T P_2 \Gamma_2 & \Gamma_1^T \tilde{G}^T \\
* & -\Gamma_2^T P_2 \Gamma_2 & 0 & \Gamma_2^T \tilde{B}_0^T P_1 \Gamma_1 & \Gamma_2^T \tilde{D}_0^T P_2 \Gamma_2 & \Gamma_2^T \tilde{H}_0^T \\
* & * & -\gamma_{2,2}^2 I & \tilde{B}_1^T P_1 \Gamma_1 & \tilde{D}_1^T P_2 \Gamma_2 & \tilde{H}^T \\
* & * & * & -\Gamma_1^T P_1 \Gamma_1 & 0 & 0 \\
* & * & * & * & -\Gamma_2^T P_2 \Gamma_2 & 0 \\
* & * & * & * & * & -I
\end{bmatrix} < 0 \tag{56}$$

where

$$
\begin{aligned}
\Gamma_1^T P_1 \tilde{A} \Gamma_1 &\triangleq \begin{bmatrix} \mathscr{U}_1 A + \mathscr{B}_f E & \mathscr{A}_f \\ \mathscr{V}_1 A + \mathscr{B}_f E & \mathscr{A}_f \end{bmatrix}, \quad \Gamma_1^T P_1 \tilde{B}_0 \Gamma_2 \triangleq \begin{bmatrix} \mathscr{U}_1 B_0 + \mathscr{B}_f F_0 & \mathscr{B}_{0f} \\ \mathscr{V}_1 B_0 + \mathscr{B}_f F_0 & \mathscr{B}_{0f} \end{bmatrix}, \\
\Gamma_1^T P_1 \tilde{B}_1 &\triangleq \begin{bmatrix} \mathscr{U}_1 B_1 + \mathscr{B}_f F_1 \\ \mathscr{V}_1 B_1 + \mathscr{B}_f F_1 \end{bmatrix}, \quad \Gamma_2^T P_2 \tilde{C} \Gamma_1 \triangleq \begin{bmatrix} \mathscr{U}_2 C + \mathscr{D}_f E & \mathscr{C}_f \\ \mathscr{V}_2 C + \mathscr{D}_f E & \mathscr{C}_f \end{bmatrix}, \\
\Gamma_2^T P_2 \tilde{D}_0 \Gamma_2 &\triangleq \begin{bmatrix} \mathscr{U}_2 D_0 + \mathscr{D}_f F_0 & \mathscr{D}_{0f} \\ \mathscr{V}_2 D_0 + \mathscr{D}_f F_0 & \mathscr{D}_{0f} \end{bmatrix}, \quad \Gamma_2^T P_2 \tilde{D}_1 \triangleq \begin{bmatrix} \mathscr{U}_2 D_1 + \mathscr{D}_f F_1 \\ \mathscr{V}_2 D_1 + \mathscr{D}_f F_1 \end{bmatrix}, \\
\Gamma_1^T P_2 \Gamma_1 &\triangleq \begin{bmatrix} \mathscr{U}_1 & \mathscr{V}_1 \\ \mathscr{V}_1 & \mathscr{V}_1 \end{bmatrix}, \quad \Gamma_2^T P_2 \Gamma_2 \triangleq \begin{bmatrix} \mathscr{U}_2 & \mathscr{V}_2 \\ \mathscr{V}_2 & \mathscr{V}_2 \end{bmatrix}, \quad \tilde{H} \triangleq -\mathscr{H}_f F_1, \\
\tilde{G} \Gamma_1 &\triangleq \begin{bmatrix} G - \mathscr{H}_f E & -\mathscr{G}_f \end{bmatrix}, \quad \tilde{H}_0 \Gamma_2 \triangleq \begin{bmatrix} H_0 - \mathscr{H}_f F_0 & -\mathscr{H}_{0f} \end{bmatrix}
\end{aligned} \tag{57}
$$

Substituting (53)–(55) and (57) into (56) now yields (50). Conversely, (55) is equivalent to

$$
\begin{aligned}
\begin{bmatrix} A_f & B_{0f} & B_f \\ C_f & D_{0f} & D_f \\ G_f & H_{0f} & H_f \end{bmatrix} &\triangleq \begin{bmatrix} P_{12}^{-1} & 0 & 0 \\ 0 & P_{22}^{-1} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathscr{A}_f & \mathscr{B}_{0f} & \mathscr{B}_f \\ \mathscr{C}_f & \mathscr{D}_{0f} & \mathscr{D}_f \\ \mathscr{G}_f & \mathscr{H}_{0f} & \mathscr{H}_f \end{bmatrix} \begin{bmatrix} P_{12}^{-T} P_{13} & 0 & 0 \\ 0 & P_{22}^{-T} P_{23} & 0 \\ 0 & 0 & I \end{bmatrix} \\
&= \begin{bmatrix} \Lambda_1^{-1} \mathscr{V}_1^{-1} & 0 & 0 \\ 0 & \Lambda_2^{-1} \mathscr{V}_2^{-1} & 0 \\ 0 & 0 & I \end{bmatrix} \begin{bmatrix} \mathscr{A}_f & \mathscr{B}_{0f} & \mathscr{B}_f \\ \mathscr{C}_f & \mathscr{D}_{0f} & \mathscr{D}_f \\ \mathscr{G}_f & \mathscr{H}_{0f} & \mathscr{H}_f \end{bmatrix} \begin{bmatrix} \Lambda_1 & 0 & 0 \\ 0 & \Lambda_2 & 0 \\ 0 & 0 & I \end{bmatrix}
\end{aligned} \tag{58}
$$

where $\Lambda_1 \triangleq P_{12}^{-T} P_{13}$ and $\Lambda_2 \triangleq P_{22}^{-T} P_{23}$. Note also that the filter matrices of (44) can be written in the form of (58). This, in turn, implies that matrix diag $(\Lambda_1, \Lambda_2, I)$ can be viewed as a similarity transformation on the state-space realization of the filter and, as such, has no effect on the filter mapping from $z_{k+1}(p)$ to $\hat{v}_{k+1}(p)$. Without loss of generality, we can set $\Lambda_1 = \Lambda_2 = I$, thus obtain (52) and hence the filter in (44) can be constructed by (52). □

Remark 9 Note that Theorem 9 provides a sufficient condition for solvability of the $\mathscr{H}_\infty$ filter problem and, since the condition which must hold is in LMI form, a desired filter can be determined by solving the following convex optimization problem:

$$
\min \ \delta_1 \qquad \textit{subject to } (50) \quad (\textit{where } \delta_1 = \gamma_{2,2}^2) \tag{59}
$$

4.2 ℓ_2–ℓ_∞ Filtering

The ℓ_2–ℓ_∞ filter has the form of (44) with $H_f = 0$, which is different from the $\mathscr{H}_\infty$ case. The reason why H_f should be set to zero is that now the corresponding filtering error output $e_{k+1}(p)$ should be independent of the disturbance $\omega_{k+1}(p)$ which enables us to establish the ℓ_2–ℓ_∞ performance for the filtering error process. Hence, as in $\mathscr{H}_\infty$ filtering, we first analyze stability along the pass and ℓ_2–ℓ_∞ performance for filtering error process described above. The result is the next theorem, whose proof follows from identical steps to the ones given above and is hence omitted here, which gives a sufficient condition for (7) to hold.

Theorem 10 *The filtering error process of* (45) *is stable along the pass with ℓ_2–ℓ_∞ performance level $\gamma_{2,\infty} > 0$ if there exist matrices $P_1 > 0$ and $P_2 > 0$ such that the following*

LMIs hold:

$$\begin{bmatrix} -P_1 & 0 & 0 & \tilde{A}^T P_1 & \tilde{C}^T P_2 \\ * & -P_2 & 0 & \tilde{B}_0^T P_1 & \tilde{D}_0^T P_2 \\ * & * & -I & \tilde{B}_1^T P_1 & \tilde{D}_1^T P_2 \\ * & * & * & -P_1 & 0 \\ * & * & * & * & -P_2 \end{bmatrix} < 0 \tag{60}$$

$$\begin{bmatrix} -P_1 & 0 & \tilde{G}^T \\ * & -P_2 & \tilde{H}_0^T \\ * & * & -\gamma_{2,\infty}^2 I \end{bmatrix} < 0 \tag{61}$$

Now, we are in the position to give the final result for the ℓ_2–ℓ_∞ filtering problem together with the filter design procedure.

Theorem 11 *Consider a discrete linear repetitive process described by* (1) *and let* $\gamma_{2,\infty} > 0$ *be a prescribed scalar. Then a full-order filter of the form* (44) *can be designed such that the filtering error process* (45) *is stable along the pass and* (48) *is satisfied if there exist matrices* $\mathscr{U}_1 > 0$, $\mathscr{V}_1 > 0$, $\mathscr{U}_2 > 0$, $\mathscr{V}_2 > 0$, $\mathscr{A}_f$, $\mathscr{B}_{0f}$, $\mathscr{B}_f$, $\mathscr{C}_f$, $\mathscr{D}_{0f}$, $\mathscr{D}_f$, $\mathscr{G}_f$ *and* $\mathscr{H}_{0f}$ *such that*

(i) *the LMI obtained from* (50) *by removing last block row and column and setting* $\gamma_{2,\infty} = 1$,

(ii) *and*

$$\begin{bmatrix} -\mathscr{U}_1 & -\mathscr{V}_1 & 0 & 0 & G^T \\ * & -\mathscr{V}_1 & 0 & 0 & -\mathscr{G}_f^T \\ * & * & -\mathscr{U}_2 & -\mathscr{V}_2 & H_0^T \\ * & * & * & -\mathscr{V}_2 & -\mathscr{H}_{0f}^T \\ * & * & * & * & -\gamma_{2,\infty}^2 I \end{bmatrix} < 0 \tag{62}$$

hold.

Moreover, the filter can be computed from (52) *with* $\mathscr{H}_f = 0$ *and* $H_f = 0$.

Proof Pre- and post-multiplying (60) and (61) by diag $(\Gamma_1, \Gamma_2, I, \Gamma_1, \Gamma_2)$ and diag (Γ_1, Γ_2, I), respectively, gives

$$\begin{bmatrix} -\Gamma_1^T P_1 \Gamma_1 & 0 & 0 & \Gamma_1^T \tilde{A}^T P_1 \Gamma_1 & \Gamma_1^T \tilde{C}^T P_2 \Gamma_2 \\ * & -\Gamma_2^T P_2 \Gamma_2 & 0 & \Gamma_2^T \tilde{B}_0^T P_1 \Gamma_1 & \Gamma_2^T \tilde{D}_0^T P_2 \Gamma_2 \\ * & * & -I & \tilde{B}_1^T P_1 \Gamma_1 & \tilde{D}_1^T P_2 \Gamma_2 \\ * & * & * & -\Gamma_1^T P_1 \Gamma_1 & 0 \\ * & * & * & * & -\Gamma_2^T P_2 \Gamma_2 \end{bmatrix} < 0 \tag{63}$$

$$\begin{bmatrix} -\Gamma_1^T P_1 \Gamma_1 & 0 & \Gamma_1^T \tilde{G}^T \\ * & -\Gamma_2^T P_2 \Gamma_2 & \Gamma_2^T \tilde{H}_0^T \\ * & * & -\gamma_{2,\infty}^2 I \end{bmatrix} < 0 \tag{64}$$

Substituting (53)–(55) and (57) into (63)–(64) and setting $H_f = 0$ complete the first part of the proof. In the case of the second, this follows identical steps to that of Theorem 9 and hence the details are omitted here. □

Remark 10 Theorem 11 provides a sufficient condition for the solvability of the ℓ_2–ℓ_∞ filter problem. As in the $\mathscr{H}_\infty$ case, a desired filter can be determined by solving the following convex optimization problem:

$$\min \delta_2 \quad \textit{subject to (i) and } (62) \textit{ in Theorem } 11 \quad (\textit{where} \quad \delta_2 = \gamma_{2,\infty}^2) \tag{65}$$

5 Illustrative examples

In the remainder of this paper we provide two numerical examples which illustrate the control and filtering results respectively developed in this paper.

Example 1 (Control Problem) Consider the case of (24) when $\alpha = 20$, $k \geq 0$ and

$$A = \begin{bmatrix} -0.21 & -0.42 & 0.00 \\ 0.60 & 1.56 & -0.10 \\ 0.30 & 0.00 & 0.43 \end{bmatrix}, \ B_0 = \begin{bmatrix} 0.73 & 0.15 \\ -0.22 & 0.64 \\ 0.33 & 0.41 \end{bmatrix}, \ B = \begin{bmatrix} -0.43 & -0.13 \\ 0.23 & 0.48 \\ 0.21 & -0.18 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -0.40 \\ 0.24 \\ -0.21 \end{bmatrix}, \ C = \begin{bmatrix} -0.40 & -0.28 & 0.37 \\ 0.52 & 0.38 & -0.15 \end{bmatrix}$$

$$D_0 = \begin{bmatrix} 1.18 & 0.31 \\ 0.15 & 0.54 \end{bmatrix}, \ D_1 = \begin{bmatrix} 0.18 \\ 0.35 \end{bmatrix}$$

$$D = \begin{bmatrix} -0.24 & -0.52 \\ -0.11 & 0.32 \end{bmatrix}, \ E = \begin{bmatrix} -0.21 & 0.26 & 0.10 \\ 0.04 & 0.32 & 0.11 \end{bmatrix}, \ F_0 = \begin{bmatrix} -0.15 & 0.26 \\ 0.06 & 0.20 \end{bmatrix}$$

$$F_1 = \begin{bmatrix} -0.30 \\ -0.22 \end{bmatrix}, \ G = \begin{bmatrix} 0.25 & -0.20 & 0.61 \\ 0.18 & 0.12 & 0.40 \end{bmatrix}, \ H_0 = \begin{bmatrix} 0.15 & 0.30 \\ 0.42 & 0.35 \end{bmatrix}$$

This example is asymptotically unstable (and hence unstable along the pass) since $\rho(D_0) > 1$ (use Theorem 1). Hence the aim is to design an $\mathscr{H}_\infty$ (or ℓ_2–ℓ_∞) dynamic output feedback controller which will result in stability along the pass, where we first consider the $\mathscr{H}_\infty$ dynamic output feedback control problem.

Choose $R_{12} = I$ and $S_{12} = I$, solve LMI (29), we obtain the minimum $\gamma_{2,2}$ as $\gamma_{2,2}^* = 1.2826$ and the associated matrices for the $\mathscr{H}_\infty$ dynamic output feedback controller are given by

$$A_c = \begin{bmatrix} -30.6612 & -15.5813 & 21.9663 \\ 5.1333 & 3.6007 & -3.5220 \\ -38.4054 & -19.7032 & 27.4851 \end{bmatrix}, \ B_{0c} = \begin{bmatrix} -122.9863 & 103.6883 \\ 15.9588 & -13.4455 \\ -151.2636 & 127.5249 \end{bmatrix},$$

$$B_c = \begin{bmatrix} -57.9587 & 76.2177 \\ 8.9239 & -12.0102 \\ -72.0967 & 95.3568 \end{bmatrix}, \ C_c = \begin{bmatrix} -0.7358 & 3.0024 & 0.8312 \\ -1.0677 & 3.6472 & 1.1549 \end{bmatrix},$$

$$D_{0c} = \begin{bmatrix} 16.8350 & -14.0734 \\ 18.5725 & -15.5148 \end{bmatrix}, \ D_c = \begin{bmatrix} 4.1180 & -9.2435 \\ 4.6027 & -10.8674 \end{bmatrix},$$

$$G_c = \begin{bmatrix} -3.4422 & 0.5945 & 2.8307 \\ -12.6879 & 4.7883 & 10.8503 \end{bmatrix},$$

$$H_{0c} = \begin{bmatrix} -6.3555 & 5.3648 \\ -26.7491 & 22.5795 \end{bmatrix}, \ H_c = \begin{bmatrix} -8.4332 & 13.6340 \\ -9.1956 & 3.4202 \end{bmatrix} \tag{66}$$

Next, under the same conditions, we solve the ℓ_2–ℓ_∞ dynamic output feedback control problem. In this case, solving the LMIs (39) and (40) of Theorem 6, we obtain the minimum $\gamma_{2,\infty}$ as $\gamma_{2,\infty}^* = 0.8388$ and the associated matrices for the ℓ_2–ℓ_∞ dynamic output feedback controller are given by

$$A_c = \begin{bmatrix} 10.3064 & -11.2526 & -6.5672 \\ -2.8925 & 3.5481 & 1.9490 \\ 22.2281 & -24.9127 & -14.3440 \end{bmatrix}, \ B_{0c} = \begin{bmatrix} 11.4500 & -9.0488 \\ -4.7303 & 3.7363 \\ 27.2136 & -21.5076 \end{bmatrix},$$
$$B_c = \begin{bmatrix} 2.4153 & 0.9373 \\ -1.4211 & 0.6252 \\ 5.9460 & 1.7628 \end{bmatrix}, \ C_c = \begin{bmatrix} -2.1511 & 1.6965 & 1.2334 \\ -2.3232 & 1.8394 & 1.3425 \end{bmatrix},$$
$$D_{0c} = \begin{bmatrix} -3.0737 & 2.2795 \\ -4.0785 & 3.0259 \end{bmatrix}, \ D_c = \begin{bmatrix} -1.1915 & 1.2536 \\ -1.5960 & 1.6031 \end{bmatrix},$$
$$G_c = \begin{bmatrix} -3.2534 & 2.1712 & 1.7370 \\ -1.7699 & 0.4185 & 0.8167 \end{bmatrix},$$
$$H_{0c} = \begin{bmatrix} -0.7744 & 0.5946 \\ -0.5884 & 0.4523 \end{bmatrix}, \ H_c = \begin{bmatrix} -6.7261 & 9.7518 \\ -4.3705 & 1.9787 \end{bmatrix} \tag{67}$$

To illustrate the response of the controlled process, let the boundary conditions be

$$\begin{cases} x_{k+1}(0) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^T & k \geq 0 \\ y_0(p) = \begin{bmatrix} \sin\left(\frac{p}{20}\pi\right) & \sin\left(\frac{p}{20}\pi\right) \end{bmatrix}^T & 0 \leq p \leq 19 \end{cases}$$

and take the disturbance input vector $\omega_{k+1}(p)$ as

$$\omega_{k+1}(p) = \begin{cases} \vartheta(k,p), & 1 \leq k \leq 19, 1 \leq p \leq 19 \\ 0, & \text{otherwise} \end{cases} \tag{68}$$

where $\vartheta(k,p)$ is a random variable drawn from a normal distribution with zero mean and unit variance.

Figures 1(a)–(c) and 2(a)–(c) show the responses of the entries in the current pass state vector of the controlled process under the controllers (66) and (67), respectively. Figure 3(a) and (b) show the control input sequence in the 1st and 2nd channels respectively under the $\mathscr{H}_\infty$ controller and Fig. 3(c) and (d) the corresponding plots for the ℓ_2–ℓ_∞ controller.

Example 2 (Filtering Problem) Consider the case of (1) with $\alpha = 20$, $k \geq 0$ and

$$A = \begin{bmatrix} 0.25 & -0.12 \\ -0.51 & -0.15 \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0.37 \\ -0.50 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0.27 \\ 0.75 \end{bmatrix}$$
$$C = \begin{bmatrix} -0.30 & -1.09 \end{bmatrix}, \quad E = \begin{bmatrix} -0.80 & -0.80 \end{bmatrix}, \quad G = \begin{bmatrix} 0.49 & -0.95 \end{bmatrix}$$
$$D_0 = -0.20, \quad D_1 = -0.20, \quad F_0 = -0.50, \quad F_1 = 0.33, \quad H_0 = -0.20$$

From Lemma 1, the above process is stable along the pass and now consider the $\mathscr{H}_\infty$ filtering problem. Then on solving the convex optimization problem in (59) we obtain the minimum $\gamma_{2,2}$ as $\gamma_{2,2}^* = 0.9407$ and

$$A_f = \begin{bmatrix} -0.1435 & -0.4013 \\ 0.1605 & 0.5039 \end{bmatrix}, \ B_{0f} = \begin{bmatrix} 0.0015 \\ -0.0009 \end{bmatrix}, \ B_f = \begin{bmatrix} 0.4690 \\ -0.9030 \end{bmatrix}$$
$$C_f = 1.0 \times 10^{-3} \begin{bmatrix} 0.0443 & -0.3195 \end{bmatrix}, \ D_{0f} = -5.2874 \times 10^{-7}, \ D_f = -1.7681 \times 10^{-4}$$
$$G_f = \begin{bmatrix} -0.7316 & 0.6850 \end{bmatrix}, \ H_{0f} = 0.0012, \ H_f = 0.3915 \tag{69}$$

Now, under the same conditions, we consider the ℓ_2–ℓ_∞ filtering problem were solving the convex optimization problem in (65) gives the minimum $\gamma_{2,\infty}$ as $\gamma_{2,\infty}^* = 0.7037$, and

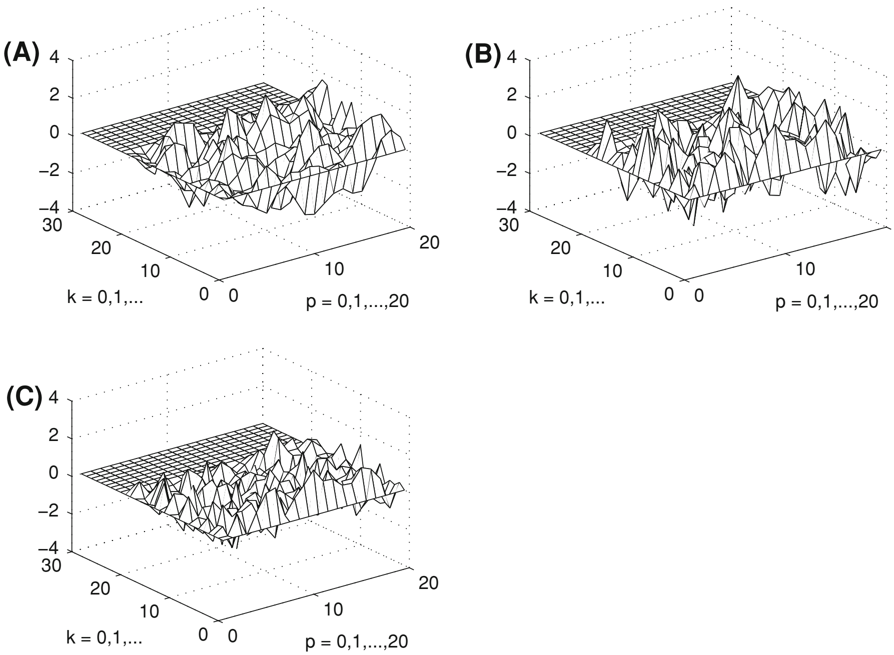

Fig. 1 States of the controlled process under $\mathscr{H}_\infty$ dynamic output feedback control

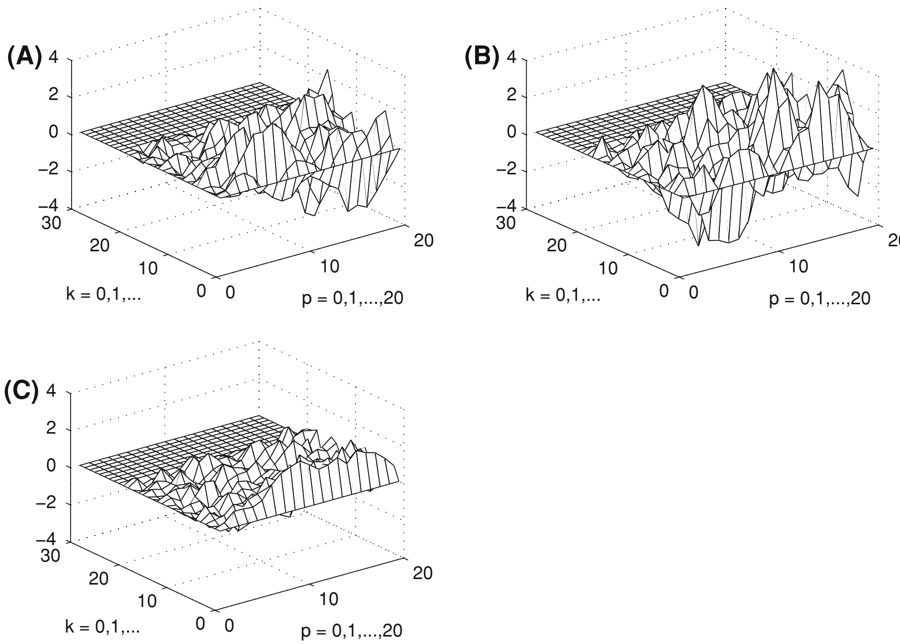

Fig. 2 States of the controlled process under ℓ_2–ℓ_∞ dynamic output feedback control

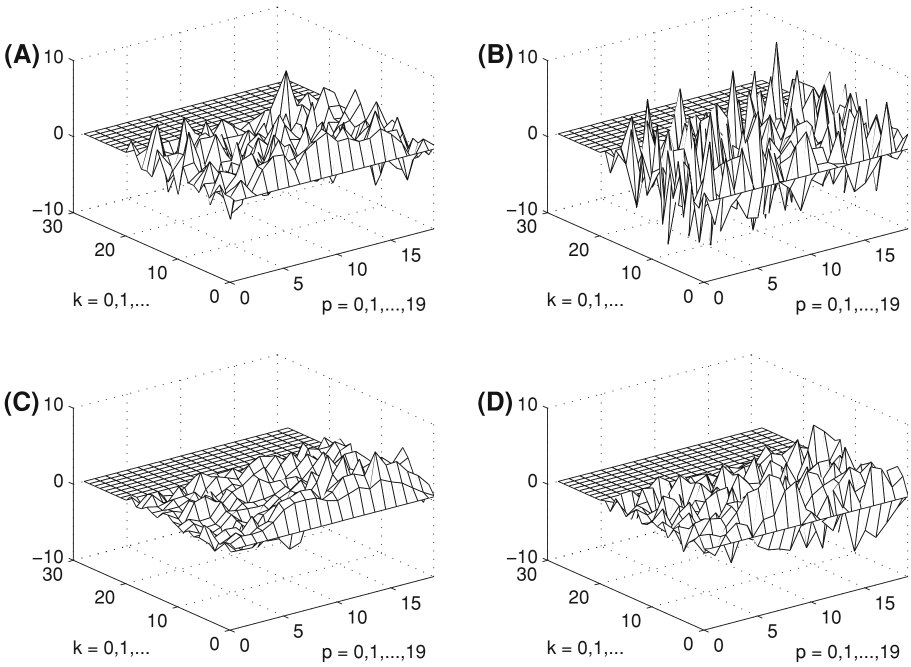

Fig. 3 Control inputs required for $\mathscr{H}_\infty$ and ℓ_2–ℓ_∞ dynamic output feedback control

the corresponding ℓ_2–ℓ_∞ filter parameter matrices are:

$$\begin{aligned} A_f &= \begin{bmatrix} 0.2158 & -0.5112 \\ -0.0848 & 0.4834 \end{bmatrix}, \ B_{0f} = \begin{bmatrix} 0.2485 \\ -0.0937 \end{bmatrix}, \ B_f = \begin{bmatrix} 0.6127 \\ -0.8754 \end{bmatrix} \\ C_f &= \begin{bmatrix} 0.0185 & -0.0414 \end{bmatrix}, \ D_{0f} = 0.0020, \ D_f = -0.0328 \\ G_f &= \begin{bmatrix} -0.5417 & 0.7466 \end{bmatrix}, \ H_{0f} = 0.1014 \end{aligned} \tag{70}$$

Consider now the case when the disturbance $\omega_{k+1}(p)$ is again given by (68), and assume zero boundary conditions (i.e. $x_{k+1}(0) = 0,\ k \geq 0$ and $y_0(p) = 0,\ 0 \leq p \leq \alpha - 1$). Then Fig. 4(a)–(c) show the responses generated by the first two entries in current pass state vector and the filtering error respectively for the $\mathscr{H}_\infty$ filter. Figure 5(a)–(c) the corresponding results for the ℓ_2–ℓ_∞ filter. This confirms that both filters guarantee that the error sequence generated converges to zero in both cases.

To compare the relative performance of these two filters, first define the following quantities:

- *Filtering error energy* ($\mathscr{E}$): $\mathscr{E} \triangleq \sqrt{\sum_{k=0}^{\infty} \sum_{p=0}^{19} e_{k+1}^T(p) e_{k+1}(p)}$
- *Filtering error peak* ($\mathscr{F}$): $\mathscr{F} \triangleq \sqrt{\sup_{\forall\, k \geq 0,\, p \in [0,19]} e_{k+1}^T(p) e_{k+1}(p)}$
- *Disturbance input energy* ($\mathscr{W}$): $\mathscr{W} \triangleq \sqrt{\sum_{k=0}^{\infty} \sum_{p=0}^{19} \omega_{k+1}(p) \omega_{k+1}(p)}$

We also use $\mathscr{X} \triangleq \frac{\mathscr{E}}{\mathscr{W}}$, $\mathscr{Y} \triangleq \frac{\mathscr{F}}{\mathscr{W}}$ as measures of the achieved $\mathscr{H}_\infty$ and ℓ_2–ℓ_∞ performance, respectively. Also we have constructed 50 test cases by using random seed numbers from 1 to 50 to generate (68). Figure 6 shows the actual filtering performance for these 50 random cases of disturbance signals with a filter obtained from minimizing the $\mathscr{H}_\infty$ performance

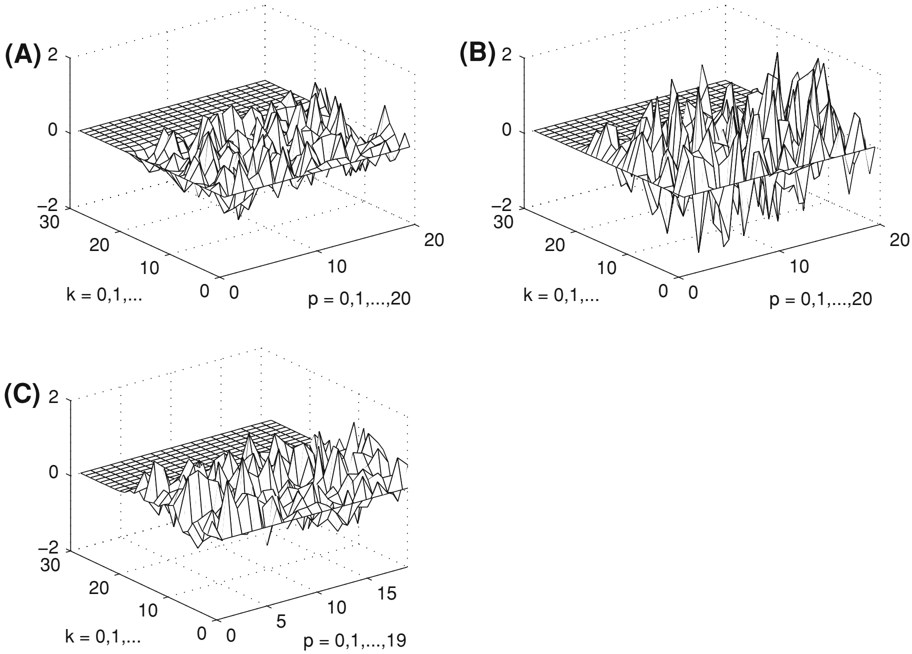

Fig. 4 States of the $\mathscr{H}_\infty$ filter and the filtering error

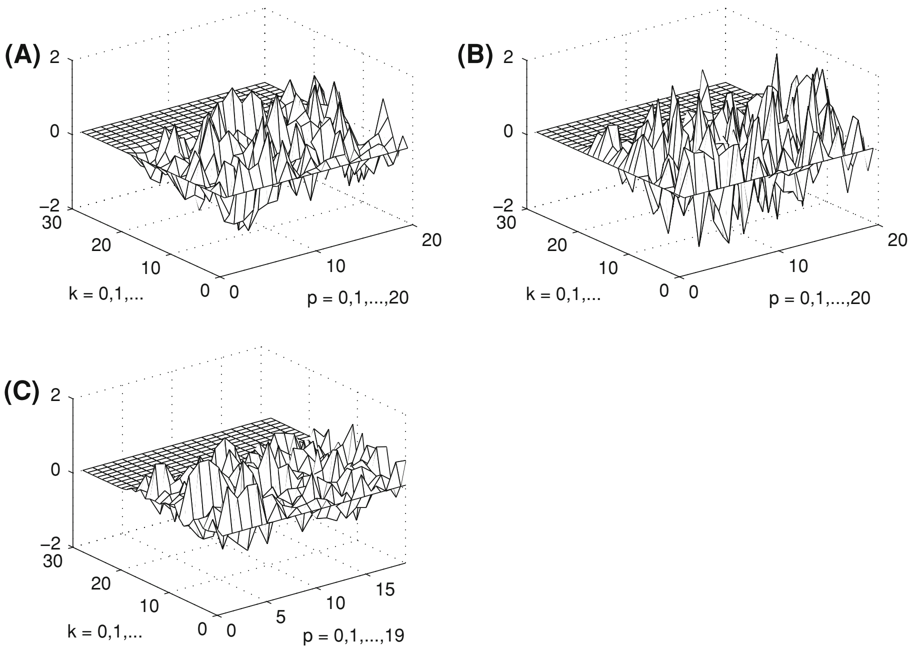

Fig. 5 States of the ℓ_2–ℓ_∞ filter and the filtering error

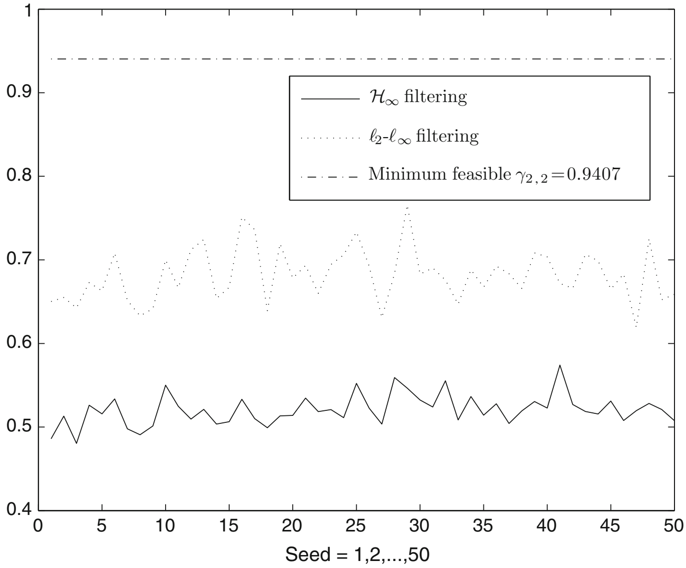

Fig. 6 Achieved $\mathscr{H}_\infty$ performance under $\mathscr{H}_\infty$ and ℓ_2–ℓ_∞ filtering

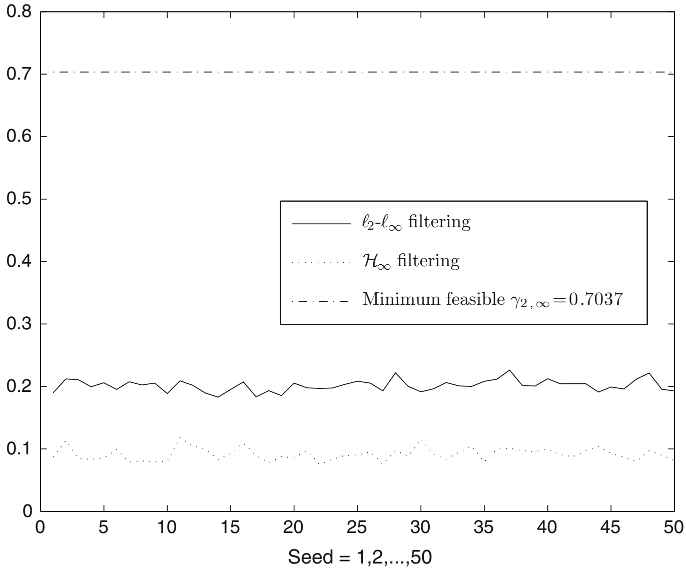

Fig. 7 Achieved ℓ_2–ℓ_∞ performance under $\mathscr{H}_\infty$ and ℓ_2–ℓ_∞ filtering

with $\gamma_{2,2}=0.9407$. Clearly, the $\mathscr{H}_\infty$ filter achieves the better performance under this measure. Figure 7 gives the corresponding comparison of the performances in the case of a filter designed under the ℓ_2–ℓ_∞ performance with achieved $\gamma_{2,\infty} = 0.7037$.

The results in this paper provide two performance measures which can, amongst others, be used in cases where the particular emphasis on, say, control versus filtering is to be decided by domain specific knowledge of the particular application under consideration. For example, if we are more concerned with the output (pass profile) energy rather than the peak value of the output then the $\mathscr{H}_\infty$ performance measure should be used. In this paper, the aim was to develop, to the level of computational algorithms, at least two performance criteria for the design engineer to select from.

Consider again Example 2 here. Then Figs. 6 and 7 show illustrate the different performance achievable. In particular, Figure 6 demonstrates that for this example a filter designed using the $\mathscr{H}_\infty$ performance measure has better performance. Figure 7 shows the opposite conclusion.

6 Conclusion

This paper has developed significant new results on filtering and control law (or controller) design for discrete linear repetitive processes using $\mathscr{H}_\infty$ and ℓ_2–ℓ_∞ settings. In the control case, the results given extend those previously reported to the case when full access to the pass profile vector (the output) is not available and all others for both filtering and control are new. Of course, these results invoke assumptions but it must be noted that physical applications in particular will require filtering of variables for successful control and the results in this paper should be interpreted as a first major step towards a general and applicable theory for onward translation into numerically reliable design algorithms for eventual experimental verification.

One of numerous areas for further research is to extend the results here to the case when inter-pass smoothing is present (see Remark 1) and dynamic boundary conditions (see Remark 2). Also it may be required to use weightings in the performance specifications. For example, if we wish to introduce a matrix weighting function between the disturbance vector (ω) and the signal to be estimated (v) then this could be achieved using the state-space model

$$\begin{aligned}\phi_{k+1}(p+1) &= A_w\phi_{k+1}(p) + B_{0w}\varphi_k(p)\\ \varphi_{k+1}(p) &= C_w\phi_{k+1}(p) + D_{0w}\varphi_k(p)\end{aligned}$$

where on pass $\phi_{k+1}(p)$ is the state vector and $\varphi_k(p)$ is the filter output (or pass profile) vector. The analysis of this case should then be straightforward extension of the results given here.

The results in this paper focus on the basic tools since these must be fully understood before effective transfer to applications. In iterative learning control it is already known that zero-phase filtering of the process response on any trial can be undertaken before the start of the next trial. One longer term application for the theory and algorithms developed here could be to provide another way of doing this to best advantage.

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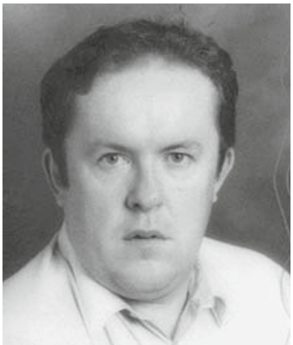
Author Biographies

Ligang Wu was born in Jiangxi Province, China, in 1977. He received a B.E. degree in Automation from the Harbin University of Science and Technology, Harbin, China, in 2001, and an M.S. degree and a Ph.D. degree in Control Science and Control Engineering from the Harbin Institute of Technology, Harbin, China, in 2003 and 2006, respectively. Dr. Wu was a Research Associate in the Department of Mechanical Engineering, University of Hong Kong, Hong Kong, from January 2006 to April 2007, and a Senior Research Associate in the Department of Mathematics, City University of Hong Kong, Hong Kong, from September 2007 to June 2008. He joined the Harbin Institute of Technology in March 2008, where he is currently an Associate Professor. His research interests include robust control/filter theory, time-delay systems, sliding mode control and multidimensional systems.

James Lam received a first class B.Sc. degree in Mechanical Engineering from the University of Manchester and was awarded the Ashbury Scholarship, the A.H. Gibson Prize and the H. Wright Baker Prize for his academic performance. From the University of Cambridge, he obtained the M.Phil. and Ph.D. degrees in control engineering. He is a Scholar and Fellow of the Croucher Foundation. Dr. Lam is now a Professor at the Department of Mechanical Engineering, the University of Hong Kong. Prior to that, he held faculty positions at now the City University of Hong Kong and the University of Melbourne. Professor Lam is a Chartered Mathematician and Chartered Scientist, a Fellow of the Institute of Mathematics and Its Applications, a Senior Member of the IEEE, a Member of the IEE. On the professional service side, Professor Lam is an Associate Editor of Asian Journal of Control, International Journal of Systems Science, Journal of Sound and Vibration, International Journal of Applied Mathematics and Computer Science, IEEE Transactions on Signal Processing, Journal of the Franklin Institute, Dynamics of Continuous, Discrete and Impulsive Systems (Series B: Applications and Algorithms), Automatica, the Conference Editorial Board of the IEEE Control Systems Society, an editorial member of IET Control Theory and Applications, Open Electrical and Electronic Engineering Journal, and Research Letters in Signal Processing. He was also an Editor-in-Chief of IEE Proceedings: Control Theory and Applications and the IFAC Technical Committee on Control Design. His research interests include reduced-order modelling, delay systems, descriptor systems, stochastic systems, multidimensional systems, robust control and filtering. He has published numerous research articles in these areas and co-authored a monograph entitled Robust Control and Filtering of Singular Systems (Springer: 2006).

Wojciech Paszke was born in Zielona Gora, Poland, in 1975. He received his M.Sc. and Ph.D. degrees, both in electrical engineering, from the Technical University of Zielona Gora, in 2000 and 2005, respectively. After holding a position of an Assistant Professor on the Faculty of Electrical Engineering, Computer Science and Telecommunications of the University of Zielona Gora for three years, he moved to Eindhoven University of Technology, where he holds currently a post-doctoral position. His general research interests include multidimensional (n-D) systems, repetitive processes and delayed systems with particular attention to computational aspects.

Krzysztof Gałkowski received the M.S., Ph.D. and Habilitation (D.Sc.) degrees in electronics/automatic control from Technical University of Wroc³aw, Poland in 1972, 1977 and 1994 respectively. In October 1996 he joined the Technical University of Zielona Góra (now the University of Zielona Góra), Poland where he holds the professor position, and he is a visiting professor in the School of Electronics and Computer Science, University of Southampton, UK. In 2002, he was awarded the degree "Professor of Technical Science" the highest scientific degree in Poland. He spent academic year 2004-2005 and 2006-2007 in The University of Wuppertal, Germany as and awardee of The Gerhard Mercator Guest Professor funded by DFG. He holds also the Professor position at The Nicolaus Copernicus University in Torun, Poland at The Department of Physics, Astarnomy and Computer Science. He is an associate editor of Int. J. of Multidimensional Systems and Signal Processing, International Journal of Control and Int. J. of Applied Mathematics and Computer Science. He has co-organised 4 International Workshops NDS 1998, Lagov, Poland, NDS 2000, Czocha Castle, Poland, the third as a Symposium within MTNS 2002 in Notre Dame IN, US, and the last one NDS 2005 in Wuppertal, Germany. In 2007 NDS 2007 has been organized in Aveiro, Portugal and in 2009 will be at Thessaloniki. In 2004, he obtained a Siemens Award for his research contributions. His research interests include multidimensional (nD) systems and repetitive processes—theory and applications, control and related numerical and symbolic algebra methods.



Eric Rogers was born in 1956 near Dungannon in Northern Ireland. He read Mechanical Engineering as an undergraduate in Queen's University, Belfast UK and was awarded his PhD degree by The University of Sheffield UK for a thesis in the area of multidimensional systems theory. Recently he has been awarded the D.Sc. degree by Queen's University Belfast for research in nD systems theory and applications. He has been with The University of Southampton UK since 1990 where he is currently Professor of Control Systems Theory and Design in The School of Electronics and Computer Science. His current major research interests include multidimensional systems theory and applications, with particular emphasis on behavioral systems theory approaches and systems with repetitive dynamics, iterative learning control, flow control, and active control of microvibrations. He is currently the editor of The International Journal of Control, an associate editor of Multidimensional Systems and Signal Processing, and a member of the editorial board of Applied Mathematics and Computer Science. In addition, he has served extensively on IEEE, IFAC and IEE technical committees and acted as a consultant to numerous companies and government agencies in the UK and abroad.

Anton Kummert was born in Amberg, Germany on April 6, 1959. He received the Dipl.-Ing. (FH) degree in electrical engineering from Fachhochschule Coburg, Coburg, Germany, in 1982, and the Dipl.-Ing. and Dr.-Ing. degrees from Ruhr-Universität Bochum, Bochum, Germany, in 1985 and 1988, respectively. From 1985 to 1991, he was a Research Assistant at the Department of Electrical Engineering of Ruhr-Universität Bochum. From 1991 to 1995, he was employed by STN Atlas Elektronik, Bremen, Germany. Since 1995, he has been a Professor for Communication Theory at University of Wuppertal, Wuppertal, Germany. Mr. Kummert is a recipient of the "Heinrich-Kost-Preis" (1989) and of the "Akademie-Preis 1990" of Rheinisch-Westfälische Akademie der Wissenschaften (1990). He is a member of Informationstechnische Gesellschaft, Germany (ITG) and Senior member of IEEE. He is and has been member of the editorial board of several international journals, organizer and chairman of numerous sessions at major conferences, and one of the General Chairs and organizers of the 4th Int. Workshop on Multidimensional (ND) Systems. His current research interests comprise multidimensional systems and circuit theory, image processing and generation, wireless ad hoc networks, and tomographic reconstruction methods.