Monoid based semantics for linear formulas

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Abstract

Each Girard quantale (i.e. commutative quantale with a selected dualizing element) provides a support for a semantics for linear propositional formulas (but not for linear derivations). Several constructions of Girard quantales are known. We give two more constructions, one using an arbitrary partially ordered monoid and one using a partially ordered group (both commutative). In both cases the semantics can be controlled be a relation between pairs of elements of the support and formulas. This gives us a neat way of handling duality.

Linear logic was introduced by Girard in [**2**]. Originally this was conceived as a way of organizing the properties of certain categories of domains, but it was soon recognized that it has a wider significance. Quantales have been around for much longer. Although their prehistory goes back to the 1930s, they were first studied systematically by Mulvey in the paper with a silly title [**4**] and later papers. Initially quantales were conceived as a way of producing a topological-like gadget in a situation where there doesn't seem to be enough points around to carry a classical topological space. But again it was soon recognized that quantales have a wider significance.

The two notions were brought together by Yetter [**9**] who showed that commutative quantales provide a standard semantics for linear formulas. (With hindsight we can see that there is some use of quantales in Girard's original paper, but not expressed in these terms.) Essentially, quantales are to linear logic what frames (complete Heyting algebras) are to intuitionistic logic and complete Boolean algebras are to classical logic. In the intuitionistic and classical cases these algebras can be specialized to the point based algebras of topologies and power sets, respectively. In fact, these point-sensitive algebras are more commonly used than the more general point-free algebras. In the linear case there is no such point-sensitive algebra, so only the point-free algebras, the quantales, can be used.

In short, quantales are an important tool for analysing certain aspects of linear logic, and also have a wider applicability.

The quantales most useful for linear logic are commutative with a selected dualizing element. These are often called Girard quantales. In this paper we give two new constructions of such quantales.

Each quantale Q is an infinitary structure (carried by a complete lattice), but can be generated by a finitary structure, a partially order monoid A. We prove two main results about a commutative partially ordered monoid A.

(S) If A carries a certain extra gadget then the free quantale generated by A is a Girard quantale

(M) The algebra of certain relations on A is a Girard quantale.

The starting point for these two results was the observation (by M) that any abelian group gives a semantics for linear formulas which handles duality in a simple way. Eventually we saw that if the abelian group is partially ordered then two slightly different semantics can be generated. Replacing the abelian group by a commutative monoid and pushing through the two constructions leads to the two results above.

The organization of the paper is as follows.

In Section 1 (the next section) we outline the basic facts we require about linear formulas. We are not concerned with linear derivations system so we don't need the details of these. However we do observe that all the semantics considered are sound for such systems.

In Section 2 we describe the two originating semantics using abelian groups. These are used later to illustrate our main results, so this section can be skipped to go straight to Section 3 which outlines the standard (and mostly well known) results on quantale semantics and the use of partially ordered monoids.

Section 4 contains the proof of result (S), and this is specialized in Section 5 to explain one of the group based semantics.

Section 6 contains the proof of result (M), and this immediately explains the other group base semantics.

Contents

1 Linear formulas and semantics **2**

2 Two group based semantics **4**

3 Poms, quantales, and the carried semantics **7**

4 Perp enriched poms **10**

5 A characterization of the S-semantics **14**

6 The general M-semantics **16**

1 Linear formulas and semantics

We are concerned here with linear propositional formulas without exponentials. Thus, for us, linear formulas are generated from a stock of variables and the four constants using the additive and multiplicative connectives and the implication. As pointed out by Seely [**7**], linear formulas can be interpreted in any category with finite limits and colimits and a suitable tensor structure, a linear category. The additive connectives are interpreted by products and sums, and the multiplicative connectives are interpreted using the tensor structure. Seely chose a notation to reflect this idea. Here we look at such interpretations where the underlying category is just a poset. The linear category becomes a partially ordered algebra (of a suitable kind). We will use a notation to reflect this idea.

Thus linear formulas are generated from

- a stock of variables P
- the additive constants $\top, \bot$
- the multiplicative constants $1, 0$

using

- the additive connectives $\sqcap, \sqcup$,

- the multiplicative connectives $\otimes, \oplus$
- the implication $\multimap$

where the five connectives are binary. For example

$$P \quad \top, \bot, 1, 0 \quad (\theta \sqcap \psi), (\theta \sqcup \psi), (\theta \otimes \psi), (\theta \oplus \psi), (\theta \multimap \psi)$$

are typical formulas where the first five are atomic and the second five are compound (built from smaller formulas θ, ψ). We may think of the constants $\top, \bot, 1, 0$ as the neutral elements for algebraic operations $\sqcap, \sqcup, \otimes, \oplus$, respectively.

For convenience we compare this notation with three other versions, all favoured to some degree by different authors.

	Additive				Multiplicative			
	Conj	Unit	Disj	Unit	Conj	Unit	Disj	Unit
Here	$\sqcap$	$\top$	$\sqcup$	$\bot$	$\otimes$	1	$\oplus$	0
Girard, [**2**]	&	$\top$	$\oplus$	0	$\otimes$	1	$\parr$	$\bot$
Troelstra, [**8**]	$\sqcap$	$\top$	$\sqcup$	$\bot$	$\star$	1	$+$	0
Seely, [**7**]	$\times$	1	$+$	0	$\otimes$	I	$\odot$	ϕ

A nice survey of linear logic (as it stood in 1995) can be found in [**3**].

One of the novelties of linear logic is that formulas carry a duality $(\cdot)^\perp$. For this we need to assume that the variables come in dual pairs

$$P \longleftrightarrow P^\perp$$

i.e. the dual of P is $P^\perp$ and the dual of this is $P^{\perp\perp} = P$. When we have such a dual pairing we may generate the formal dual $\phi^\perp$ of each formula ϕ.

$$\begin{array}{rclrcl rclrcl}
(\theta \sqcap \psi)^\perp &=& \theta^\perp \sqcup \psi^\perp & \top^\perp &=& \bot & (\theta \sqcup \psi)^\perp &=& \theta^\perp \sqcap \psi^\perp & \bot^\perp &=& \top \\
(\theta \otimes \psi)^\perp &=& \theta^\perp \oplus \psi^\perp & 1^\perp &=& 0 & (\theta \oplus \psi)^\perp &=& \theta^\perp \otimes \psi^\perp & 0^\perp &=& 1 \\
 & & & & & & (\theta \multimap \psi)^\perp &=& \theta \otimes \psi^\perp & & &
\end{array}$$

Note that for a $\{\multimap\}$-free formula ϕ

$$\phi^{\perp\perp} = \phi$$

holds. (The implication is often taken as a defined connective

$$\theta \multimap \psi ::= \theta^\perp \oplus \psi$$

but this is not good practice.)

There are several derivations system which structure families of linear formulas. All of these manipulate judgements

$$\Gamma \vdash \Sigma$$

where Γ and Σ are lists (or, more efficiently, bags) of linear formulas. The duality helps to bring out many of the symmetries inherent in such systems. We won't need the details of any of these, but we will allude to the soundness of a semantics relative to such a system.

There are several different kinds of semantics for linear logic (linear formulas including the exponentials and linear derivation systems). A nice survey is given in [**5**]. Here we restrict out attention to algebraic semantics.

An algebraic semantics for linear formulas (or a subset of formulas) can be obtained by first choosing an algebraic structure A and then attaching to each formula ϕ a value $[\![\phi]\!]$ in A. These values should be generated from an assumed assignment

$$P \longmapsto [\![P]\!]$$

(attaching a value to each variable) using clauses of the form

$$[\![\theta \circ \psi]\!] = [\![\theta]\!] \bullet [\![\psi]\!]$$

where $\circ$ is a connective and $\bullet$ is some binary operation carried by A. We also attach a value $[\![k]\!]$ to each constant k.

A suitable choice of algebraic structure A ensures that such a semantics is sound in the sense that the derivation system of interest can be use to predict which values occur. (A more general kind of semantics will distinguish between different derivations rather than merely the outcome of derivations, but we don't need that kind of sophistication here.) Commutative quantales seem to be the algebras best suited algebras for this kind of semantics.

As remarked earlier, we do not deal with the exponentials here. Each commutative quantale carries a canonical interior operation which provides an interpretation for '!'. For a Girard quantale the dualizing element can be used to convert the interior operation into an interpretation for '?'. Thus the semantics described here could be extended to handle exponentials, but we haven't anything to say about this beyond these routine remarks.

It can be argued that a quantale semantics doesn't tell us much about linear logic, for this kind of semantics attaches a value only to formulas. The import of linear logic is the way it can distinguish between different derivations of the same judgement, and an algebraic semantics fails to detect these differences. We don't dispute this. However, Girard quantales are interesting in their own right, we do have something to say about these, and linear formulas give us a convenient starting point.

At this juncture we can either go directly to the study of the general properties of quantales and the associated semantics, or we can look at the particular group based semantics from which the main results (S,M) were developed.

2 Two group based semantics

Let

$$(A, \leq, +, 0)$$

be an additively written, partially ordered abelian group. Thus

$$(A, \leq) \text{ is a poset} \qquad (A, +, 0) \text{ is an abelian group}$$

and the operation $+$ is monotone with respect to the comparison $\leq$, i.e.

$$x \leq a, y \leq b \Rightarrow x + y \leq a + b$$

holds for all $a, b, x, y \in A$. We say such a structure is discrete if the comparison $\leq$ is just equality (so that every such structure gives us a discrete structure by forgetting the comparison).

A separator on a partially ordered abelian group A is a binary relation S such that

$$aSb \Rightarrow pSq$$

holds for all $a, b, p, q \in A$ with $p \leq a$ and $b \leq q$, i.e. S is decreasing in the left argument and increasing in the right argument. For instance, the given comparison $\leq$ is itself a separator. For a different kind of example consider any cut L, U of $(A, \leq)$, i.e. let L be a lower section of A and let U be an upper section of A. Then

$$aSb \Leftrightarrow a \in L \text{ and } b \in U \qquad aSb \Leftrightarrow a \in L \text{ or } b \in U$$

defines two separators on A.

We wish to construct relations

$$(a,b) \Vdash \phi$$

between pairs (a,b) of elements of A (a given partially ordered abelian group) and linear formulas ϕ. For each such relation we start from a family of separators on A (indexed by the variables) and then generate the relation by recursion over ϕ. The constants and additive connectives are treated as follows.

$$\begin{array}{lll}
(a,b) \Vdash \theta \sqcap \psi \iff (a,b) \Vdash \theta \text{ and } (a,b) \Vdash \psi & & (a,b) \Vdash \top \iff \text{Always} \\
(a,b) \Vdash \theta \sqcup \psi \iff (a,b) \Vdash \theta \text{ or } (a,b) \Vdash \psi & & (a,b) \Vdash \bot \iff \text{Never} \\
& & (a,b) \Vdash 1 \iff a \leq 0 \\
& & (a,b) \Vdash 0 \iff b \not\leq 0
\end{array}$$

We can pass across the connectives $\otimes, \oplus \multimap$ on one of two ways. Thus each family of separators gives two relations.

1 DEFINITION. Let A be a partially ordered abelian group and for each variable P let (P) be a selected separator on A. The two relations

$$(a,b) \Vdash_{\mathsf{M}} \phi \qquad (a,b) \Vdash_{\mathsf{S}} \phi$$

are generated by recursion over the structure of ϕ using

$$(a,b) \Vdash P \iff a(P)b$$

for the base, the above clauses for the additive connectives and constants, and the clauses

$$\begin{array}{lllll}
(a,b) \Vdash_{\mathsf{M}} \theta \otimes \psi & \iff & (\exists x,y,\ x+y \geq a)[(x,b+y) \Vdash_{\mathsf{M}} \theta & \text{and} & (y,b+x) \Vdash_{\mathsf{M}} \psi] \\
(a,b) \Vdash_{\mathsf{M}} \theta \oplus \psi & \iff & (\forall x,y,\ x+y \geq b)\,[(a+y,x) \Vdash_{\mathsf{M}} \theta & \text{or} & (a+x,y) \Vdash_{\mathsf{M}} \psi] \\
(a,b) \Vdash_{\mathsf{M}} \theta \multimap \psi & \iff & (\forall x,y,\ x+y \geq b)\,[(x,a+y) \Vdash_{\mathsf{M}} \theta & \Rightarrow & (a+x,y) \Vdash_{\mathsf{M}} \psi] \\
\\
(a,b) \Vdash_{\mathsf{S}} \theta \otimes \psi & \iff & (\exists u,v,\ u+v \leq a)[(a-u,b+u) \Vdash_{\mathsf{S}} \theta & \text{and} & (a-v,b+v) \Vdash_{\mathsf{S}} \psi] \\
(a,b) \Vdash_{\mathsf{S}} \theta \oplus \psi & \iff & (\forall u,v,\ u+v \leq b)\,[(a+u,b-u) \Vdash_{\mathsf{S}} \theta & \text{or} & (a+v,b-v) \Vdash_{\mathsf{S}} \psi] \\
(a,b) \Vdash_{\mathsf{S}} \theta \multimap \psi & \iff & (\forall u,v,\ u+v \leq b)\,[(b-u,a+u) \Vdash_{\mathsf{S}} \theta & \Rightarrow & (a+v,b-v) \Vdash_{\mathsf{S}} \psi]
\end{array}$$

for the passage across the multiplicative connectives. ■

Our task is to understand these semantics and set them in the more general context of quantale semantics for linear formulas.

Notice that whereas the construction of $\Vdash_{\mathsf{S}}$ makes explicit mention of the subtraction of A, the construction of $\Vdash_{\mathsf{M}}$ does not. This means that the M-semantics can be constructed on a more general support (such as a partially ordered monoid). We look at this more general case later.

In the remainder of this section we outline some of the properties of these two semantics. We don't give full proofs for more general results are dealt with later.

2 LEMMA. *(a) If the abelian group A is discrete, then $\Vdash_{\mathsf{M}}$ and $\Vdash_{\mathsf{S}}$ agree.*
(b) In general $\Vdash_{\mathsf{M}}$ and $\Vdash_{\mathsf{S}}$ are different.

The proof of part (a) is easy. In the discrete case the restricted quantifiers become

$$(\exists x+y=a) \quad (\forall x+y=b) \qquad (\exists u+v=a) \quad (\forall u+v=b)$$

and there is an obvious way to pass between these.

For part (b) the following example illustrates the differences between the two relations. It is a modification of Girard's carcinogenic example.

3 EXAMPLE. Consider $\mathbb{Z}$ as linear ordered (additively written) abelian group. Fix $k, l \in \mathbb{Z}$ and suppose there is a variable L with associated separator $\pounds(k,l)$ given by

$$a\pounds(k,l)b \Leftrightarrow a \le k \text{ and } l \le b$$

(for $a, b \in \mathbb{Z}$). What does it mean to fulfil an implication $L \multimap \phi$ for an arbitrary formula ϕ?

For $a, b \in \mathbb{Z}$ let

$$m = \min(k, a + b - l)$$

so that $m < k$ if $a + b < k + l$. A few calculations show that

$$\begin{array}{lllll} (\mathrm{m}) & (a,b) \Vdash_{\mathsf{M}} L \multimap \phi & \Leftrightarrow & (a+k, b-m) \Vdash_{\mathsf{M}} \phi \\ (\mathrm{s}) & (a,b) \Vdash_{\mathsf{S}} L \multimap \phi & \Leftrightarrow & (a+m, b-m) \Vdash_{\mathsf{S}} \phi \end{array}$$

hold. In particular, the two relations differ when $a + b$ is sufficiently small. ■

Intuitively we may think of the relation

$$(a,b) \Vdash \phi$$

as measuring the effect of an environment on a formula ϕ. The environment has an external effect measured by a, and an internal effect measured by b. A change in the formula ϕ is reflected by a change in both the external and the internal environments, and some transfer between these.

A linear derivation system manipulates judgements

$$\Gamma \vdash \Sigma$$

where Γ and Σ are bags of formulas. It can be checked that both the semantics are sound for such systems. We do not need the details here.

When the variables are dually matched each formula ϕ has a dual $\phi^\perp$. The idea of using pairs of elements (a, b) is to give an easy way of handling such duals. Accordingly we say such a relation respects duality if

$$(a,b) \Vdash \phi^\perp \Leftrightarrow \neg[(b,a) \Vdash \phi]$$

holds for all pairs (a, b) and formulas ϕ. Notice how the pair is inverted as we move across the equivalence.

Of course, before this can hold the variables must be dually matched and the base relations (P) must satisfy

$$a(P^\perp)b \Leftrightarrow \neg[b(P)a] \tag{1}$$

(for all $a, b \in A$). For the two constructed semantics this necessary condition is also sufficient.

4 LEMMA. (Duality respecting) *Suppose the variables are dually matched and the base relations satisfy (1). Then both $\Vdash_{\mathsf{M}}$ and $\Vdash_{\mathsf{S}}$ respect duality.*

Proof. Both of these are proved by induction over formulas. Let's look at the proof for $\Vdash_{\mathsf{S}}$ (for later we prove a more general result for $\Vdash_{\mathsf{M}}$).

Most of the induction steps are immediate. Only the passage across $\otimes, \oplus, \multimap$ requires a bit of thought.

Consider $\phi = \theta \multimap \psi$. Then $\phi^\perp = \theta \otimes \psi^\perp$ and we have

$$\begin{array}{lll} (a,b) \Vdash_{\mathsf{S}} \phi^\perp & \Leftrightarrow & (\exists u + v \le a)[(a-u, b-u) \Vdash_{\mathsf{S}} \theta \text{ or } (a-v, b+v) \Vdash_{\mathsf{S}} \psi^\perp] \\ (b,a) \Vdash_{\mathsf{S}} \phi & \Leftrightarrow & (\forall u + v \le a)[(a-u, b-u) \Vdash_{\mathsf{S}} \theta \Rightarrow (b+v, a-v) \Vdash_{\mathsf{S}} \psi] \end{array}$$

so the required result follows by the induction hypothesis (on ψ) and the way negation passes through quantifiers. ■

Each formula gives us a pair of relations on A (one for S and one for M). When ϕ is a variable both of these are separators. What happens for a general ϕ?

5 LEMMA. *(m)* (Separation preserving) *The relation*

$$(a,b) \Vdash_{\mathsf{M}} \phi \Rightarrow (p,q) \Vdash_{\mathsf{M}} \phi$$

holds for all formulas ϕ and $a,b,p,q \in A$ with $p \leq a, b \leq q$.

(s) (Balanced separation preserving) *The relation*

$$(a,b) \Vdash_{\mathsf{S}} \phi \Rightarrow (p,q) \Vdash_{\mathsf{S}} \phi$$

holds for all formulas ϕ and $a,b,p,q \in A$ with $p \leq a, b \leq q$ and $a+b=p+q$.

We need not prove this here for each part is a simple consequence of later results. Notice that (m) says that each formula ϕ gives a separator on A. This will be the clue to understanding the M-semantics. Also the extra balancing restriction is important for the S-semantics.

Our job is to relate the M- and S-semantics to the general quantale semantics. To do this we first need to review this standard material.

3 Poms, quantales, and the carried semantics

We need some information about quantales. A nice account of these is given in [**6**]. Here we start with a more primitive notion.

A Partially Ordered Monoid (or pom for short) is a structure

$$(A, \leq, \cdot, 1)$$

where

$$(A,\leq) \text{ is a poset} \qquad (A,\cdot,1) \text{ is a monoid}$$

and the multiplication '$\cdot$' of A is monotone with respect to the comparison $\leq$, i.e.

$$x \leq a, y \leq b \Rightarrow xy \leq ab$$

holds for all $a,b,x,y \in A$.

We say such a structure is discrete if the comparison $\leq$ is just equality (so that every pom gives us a discrete pom – a monoid – by forgetting the comparison). Each pom A (as above) gives a second example

$$(A, \leq^{op}, \cdot, 1)$$

called the opposite, obtained by inverting the comparison. We write A^{op} for this pom. In general $A^{op} \neq A$ (unless the parent pom is discrete).

Here we are mostly concerned with commutative poms. Each partially ordered abelian group is such a commutative pom (with the operation written additively).

Poms form the objects of a category Pom where the arrows are those functions between poms which are simultaneously monotone and monoid morphisms.

We need the idea of a residuation and a co-residuation on a commutative pom. These are dual notions and can be defined in parallel.

Thus, for a commutative pom A and $i \in A$, we say A is

residuated co-residuated

at i if there is an operation

$$(\cdot)_i \qquad (\cdot)^i$$

on A such that

$$x \le y_i \Leftrightarrow xy \le i \qquad y^i \le x \Leftrightarrow i \le xy$$

holds for all $sx, y \in A$. Thus A is residuated at i exactly when the opposite A^{op} is co-residuated at i.

A quantale is a structure

$$(Q, \bigvee, \perp, \cdot, 1)$$

where

$$(Q, \bigvee, \perp) \text{ is a } \bigvee\text{-semilattice} \qquad (Q, \cdot, 1) \text{ is a monoid}$$

and the multiplication '$\cdot$' is monotone and satisfies the two quantale distributive laws

$$a(\bigvee X) = \bigvee\{ax \mid x \in X\} \qquad (\bigvee X)a = \bigvee\{xa \mid x \in X\}$$

for all $a \in Q$ and $X \subseteq Q$. Here we are mostly concerned with commutative quantales where the two distributive laws coincide.

Quantales form the objects of a category Qnl where the arrows are those functions between quantales which are simultaneously $\bigvee$-preserving and a monoid morphism.

Each quantale Q has an underlying pom (obtained by forgetting the $\bigvee$-structure). When Q is commutative, this pom is residuated. For $a, b \in Q$ let

$$X = \{x \in Q \mid bx \le a\}$$

and set

$$b \multimap a = \bigvee X$$

to produce an operation $\cdot \multimap \cdot$ on Q. The quantale distributive law ensures that

$$x \le b \multimap q \Leftrightarrow bx \le a$$

holds (for all $a, b, x \in Q$), and hence $\cdot \multimap a$ is a residuation at a. The binary operation $\cdot \multimap \cdot$ is often called the implication of Q. (A similar construction can be carried out on any quantale, but without commutativity we need to distinguish between a right residuation and a left residuation.)

As we have just observed, there is an obvious forgetful functor

$$\mathsf{Pom} \longleftarrow \mathsf{Qnl}$$

(which forgets the $\bigvee$-structure). This functor has a left adjoint, i.e. each pom A can be reflected into a quantale, the quantale freely generated by A.

A lower section of A is a subset X such that

$$y \le x \in X \Rightarrow y \in X$$

holds (for all $x, y \in A$). For instance, for each $a \in A$

$$\downarrow a = \{x \in A \mid x \le a\}$$

is the principal lower section generated by a. More generally, each subset H generates a lower section $\downarrow H$ given by

$$x \in \downarrow H \Leftrightarrow (\exists a \in H)[x \le a]$$

(so that $\downarrow a = \downarrow\{a\}$).

Let $\mathcal{L}A$ be the set of lower sections of A. Thus $\mathcal{L}A$ is a complete lattice (with set theoretic operations). For $X, Y \in \mathcal{L}A$ we let

$$XY = \downarrow\{xy \mid x \in X, y \in Y\}$$

to produce a 'multiplication' on $\mathcal{L}A$. It is routine to check that this converts $\mathcal{L}A$ into a quantale (with neutral element $\downarrow 1$). This construction $\mathcal{L}$ is functorial and the map

$$\begin{array}{rcl} A & \longrightarrow & \mathcal{L}A \\ a & \longmapsto & \downarrow a \end{array}$$

is a pom morphism which is the unit of the reflection from Pom to Qnl with reflector $\mathcal{L}$.

Observe that when A is commutative, then so is $\mathcal{L}A$, and implication operation on $\mathcal{L}A$ is given by

$$z \in X \multimap Y \Leftrightarrow (\forall x)[x \in X \Rightarrow zx \in Y]$$

(for all $X, Y \in \mathcal{L}A$ and $z \in A$).

The distinguished and carried attributes of a commutative quantale Q make such a structure a suitable support for a semantics for a fragment of the linear formulas. A Q-$\mathsf{assignment}$ is a map

$$P \longmapsto [\![P]\!]$$

which attaches to each variable P an element $[\![P]\!]$ of Q. Such an assignment can be lifted to produce the Q-$\mathsf{valuation}$

$$\phi \longmapsto [\![\phi]\!]$$

for $\{\oplus, 0\}$-free formulas ϕ. The value $[\![\phi]\!]$ is generated by recursion on the structure of ϕ using the following clauses.

$$\begin{array}{rclrclrclrcl} [\![\theta \sqcap \psi]\!] & = & [\![\theta]\!] \wedge [\![\psi]\!] & [\![\top]\!] & = & \top & [\![\theta \sqcup \psi]\!] & = & [\![\theta]\!] \vee [\![\psi]\!] & [\![\bot]\!] & = & \bot \\ [\![\theta \otimes \psi]\!] & = & [\![\theta]\!][\![\psi]\!] & [\![1]\!] & = & 1 \\ [\![\theta \multimap \psi]\!] & = & [\![\theta]\!] \multimap [\![\psi]\!] \end{array}$$

On the left of the identities, $\bot, \top$ are the bottom and top of the quantale, $\vee, \wedge$ are the binary lattice operations, and the multiplication is indicated by concatenation. This correspondence explains our slightly unusual choice of symbols for the linear connectives and constants.

It is routine to show, and well known, that such a quantale semantics is sound for the standard linear derivation system (restricted to $\{\oplus, 0\}$-free formulas).

When the quantale Q is $\mathcal{L}A$ for a commutative pom A we can rephrase this semantics in terms of A. We set up a $\mathsf{fulfilment\ relation}$

$$a \Vdash \phi$$

between elements $a \in A$ and $\{\oplus, 0\}$-free formulas ϕ. For want of a better name we call this the $\mathsf{screening\ semantics}$.

Starting with a given assignment

$$P \longmapsto (P)$$

from variables P to $\mathcal{L}A$, we let

$$a \Vdash P \Leftrightarrow a \in [\![P]\!]$$

and then generate the relations $a \Vdash \phi$ by recursion over ϕ using the following clauses.

$$\begin{array}{rcllrcl} a \Vdash \theta \sqcap \psi & \Longleftrightarrow & a \Vdash \theta \text{ and } a \Vdash \psi & & a \Vdash \top & \Longleftrightarrow & \text{Always} \\ a \Vdash \theta \sqcup \psi & \Longleftrightarrow & a \Vdash \theta \text{ or } a \Vdash \psi & & a \Vdash \bot & \Longleftrightarrow & \text{Never} \\ a \Vdash \theta \otimes \psi & \Longleftrightarrow & (\exists xy \geq a)[x \Vdash \theta \text{ and } y \Vdash \psi] & & a \Vdash 1 & \Longleftrightarrow & a \leq 0 \\ a \Vdash \theta \multimap \psi & \Longleftrightarrow & (\forall x)[x \Vdash \theta \Rightarrow ax \Vdash \psi] \end{array}$$

It is routine to show that

$$a \Vdash \phi \Leftrightarrow a \in [\![\phi]\!]$$

holds for all $a \in A$ and all $\{\oplus, 0\}$-free formulas ϕ. This gives us a more amenable semantics, for it is easier to calculate in A than in $\mathcal{L}A$. This is put to good effect by Ambler in [**1**].

This kind of semantics only handles $\{\oplus, 0\}$-free formulas ϕ. To handle all formulas we need an extra gadget.

For each $d \in Q$ (the commutative quantale) we have a residuation $(\cdot)_d$ which we can iterate to produce a closure operation

$$x \longmapsto x_{dd} = (x \multimap d) \multimap d$$

on Q. We say d is dualizing if $(\cdot)_{dd}$ is just the identity map on Q (i.e. if $(\cdot)_d$ is an involution).

When d is dualizing we can set

$$[\![\theta \oplus \psi]\!] = ([\![\theta]\!]_d [\![\psi]\!]_d)_d \qquad [\![0]\!] = d$$

to produce a Q-value $[\![\phi]\!]$ for all linear formulas ϕ. A commutative quantale with a selected dualizing element is called a Girard quantale and the associated semantics is the Girard semantics. This kind of semantics ought to be the standard against which all other algebraic semantics should be tested.

An arbitrary element $d \in Q$ need not be dualizing, but becomes dualizing in a certain quotient Q_d of Q. (The details of the construction of the quotient morphism $Q \longrightarrow Q_d$ are not needed here.) We thus have a general method of producing an algebraic semantics for all linear formulas.

We start from an arbitrary commutative pom A and first move to the free quantale $\mathcal{L}A$. We then choose some $D \in \mathcal{L}A$ and move to the quotient $(\mathcal{L}A)_D$. In essence this is Girard's original phase semantics [**2**]. (Strictly speaking Girard started from a discrete pom, but that is not necessary.)

The second step, taking the quotient, can be quite messy, so it is interesting to determine properties of a pom A which ensure that the free quantale $\mathcal{L}A$ already has a dualizing element. When this happens there is a chance that the screening semantics can be extended to handle all linear formulas. Thus we arrive at our next task.

4 Perp enriched poms

How can we furnish a commutative pom A with extra gadgets to ensure that the free quantale $\mathcal{L}A$ has a dualizing element? The answer to this is related to a standard technique with ordered structures.

Recall that the comparison $\leq$ of an ordered structure A is often characterized using a 'positive segment' U, and a 'negation' operation $(\cdot)^*$ by an equivalence

$$y \leq x \Leftrightarrow xy^* \in U$$

(for elements x, y). When this happens for a commutative pom A we say the comparison $\leq$ is controlled by U via $(\cdot)^*$. Our first main result is a characterization of this behaviour.

6 THEOREM. *Let $(\cdot)^*$ be an operation on the commutative pom A. The following are equivalent.*

(i) There is a upper section U of A which controls the comparison via $(\cdot)^*$.

(ii) There is some $i \in A$ such that A is co-residuated at i, this co-residuation is involutary, and $(\cdot)^* = (\cdot)^i$.

(iii) The operation $(\cdot)^*$ *is an involution and*

$$z \le xy^* \Leftrightarrow y \le xz^*$$

holds (for all $x, y, z \in A$*).*

Proof. We show the implications

$$(i) \Rightarrow (ii) \qquad (ii) \Rightarrow (iii) \qquad (iii) \Rightarrow (i)$$

in turn. Of these the first takes the longest.

(*(i)⇒(ii)*) Assuming (i) we first show that

$$\begin{array}{llllll} (1) & y \le x^* \Leftrightarrow x \le y^* & (2) & x \le x^{**} & & \\ (3) & y \le x \Rightarrow x^* \le y^* & (4) & x^{***} = x^* & (5) & x^{**} = x \end{array}$$

hold for all $x, y \in A$.

Since A is commutative we have

$$y \le x^* \Leftrightarrow x^*y^* \in U \Leftrightarrow y^*x^* \in U \Leftrightarrow x \le y^*$$

to give (1). Setting $y = x^*$ we have $y \le x^*$, so that (1) gives (2). Next using (2) and (1), we have

$$y \le x \Rightarrow y \le x^{**} \Rightarrow x^* \le y^*$$

to give (3). From (2) we have $x^* \le x^{***}$ and $x \le x^{**}$, so that (3) gives $x^{***} \le x^*$, to give (4). Finally, since $x \le x$ the conditions (4) and (i) give

$$xx^{***} = xx^* \in U$$

so that another use of (i) gives $x^{**} \le x$, and this with (2) gives (5).

The condition (5) shows that $(\cdot)^*$ is an involution. Now let $i = 1^*$. Then, using (i) and (5),

$$i \le x \Leftrightarrow xi^* \in U \Leftrightarrow x \in U$$

so that $U = \uparrow i$. This with (i) and (5) gives

$$y^* \le x \Leftrightarrow xy^{**} \in U \Leftrightarrow i \le xy$$

to show that A is co-residuated at i with $(\cdot)^* = (\cdot)^i$, as requires.

(*(ii)⇒(iii)*) Condition (ii) gives

$$y^* \le x \Leftrightarrow i \le xy$$

(for all $x, y \in A$) and $(\cdot)^*$ is an involution. Thus, for all $x, y, z \in A$,

$$\begin{array}{rcl} z \le xy^* & \Leftrightarrow & z^{**} \le xy^* \\ & \Leftrightarrow & i \le xy^*z^* \\ & \Leftrightarrow & i \le xz^*y^* \\ & \Leftrightarrow & y^{**} \le xz^* \quad \Leftrightarrow \quad y \le xz^* \end{array}$$

to give (iii), as required.

(*(iii)⇒(i)*) Assuming (iii) set $U = \uparrow 1^*$. Then, since $1^{**} = 1$, we have

$$y \le x \Leftrightarrow y \le x1^{**} \Leftrightarrow 1^* \le xy^* \Leftrightarrow xy^* \in U$$

to give (i), as required. ■

This kind of enrichment of a pom deserves a name.

7 DEFINITION. A perp enriched pom, or pep for short, is a commutative pom A together with an involution $(\cdot)^*$ on A which satisfies the equivalent conditions of Theorem 6. ■

The equivalent *(iii)* seems to be the most useful property of a pep. But we always have the element and upper section

$$i = 1^* \qquad U = \uparrow i$$

to hand, and either of these determine the enrichment $(\cdot)^*$.

8 EXAMPLES. (a) Let A be a partially ordered abelian group (written additively). For each $i \in A$ the operation

$$x \longmapsto x^i = i - x$$

is an involutary co-residuation. Furthermore, every involutary co-residuation on A is obtained in this way. Thus will be the trick which enables us to characterize the S-semantics.

(b) Consider a $\vee$-semilattice $(A, \vee, \perp)$ as a pom. Suppose A is boolean, i.e. is the $\vee$-structure of a boolean algebra. Then A carries a complementation operation $(\cdot)'$ which is an involutary co-residuation.

(c) When analysing properties of linear formulas it is often useful to see what happens in the classical case (where the additive and the multiplicative properties coincide). Here this means that the pom A is a $\wedge$-semilattice (and the generated quantale $\mathcal{L}A$ is a frame). What kind of involutary co-residuation can such a $\wedge$-semilattice carry? A simple argument shows that if A carries any such operation then A must be trivial, i.e. the 1-element lattice. This means that whatever the present analysis tells us about linear logic, it is entirely invisible in the classical case. ■

How does a pep $(A, (\cdot)^*)$ give us a free Girard quantale? For each $X \in \mathcal{L}A$

$$y \in X^* \Leftrightarrow y^* \notin X$$

defines a subset X^* of A. Note that

$$x \leq y \in X^* \Longrightarrow y^* \leq x^*,\ y^* \notin X \Longrightarrow x^* \notin X \Longrightarrow x \in X^*$$

so that $X^* \in \mathcal{L}A$. We continue with this notation in the next result.

9 THEOREM. *Let $(\cdot)^*$ be a perp enrichment of a commutative pom A. With $i = 1^*$ the lower section*

$$D = (\uparrow i)' = U'$$

is dualizing in $\mathcal{L}A$ and

$$X \multimap D = X^*$$

holds for all $X \in \mathcal{L}A$. (U' denotes the complement of U)

Proof. First of all note that since $X \in \mathcal{L}A$ we have

$$\begin{array}{rcll} y \in X \multimap D & \Leftrightarrow & (\forall x \in X)[xy \in D] & \\ & \Leftrightarrow & (\forall x)[i \leq xy \Rightarrow x \notin X] & \\ & \Leftrightarrow & (\forall x)[y^* \leq x \Rightarrow x \notin X] & \\ & \Leftrightarrow & y^* \notin X & \Leftrightarrow \quad y \in X^* \end{array}$$

to give

$$X \multimap D = X^*$$

as required. We use this description to show that D is dualizing.

Thus

$$(X \multimap D) \multimap D = X^{**}$$

and then

$$y \in X^{**} \Leftrightarrow y^* \notin X^* \Leftrightarrow y^{**} \in X \Leftrightarrow y \in X$$

to give $X^{**} = X$, as required. ■

This shows that a dualizing element of $\mathcal{L}A$, and hence a full linear $\mathcal{L}A$-semantics, can be obtained from any perp enrichment of A. We know the the partial $\mathcal{L}A$-semantics (for $\{\oplus, 0\}$-free formulas) is controlled by the screening semantics on A, so an extended version of this should give the full semantics.

10 DEFINITION. (The enriched screening semantics) Let $(\cdot)^*$ be a perp enrichment of the commutative pom A. Let

$$P \longmapsto [\![P]\!]$$

be an assignment which attaches a lower section $[\![P]\!] \in \mathcal{L}A$ to each variable P. The fulfilment relation

$$a \Vdash^* \phi$$

between elements a of A and all linear formulas ϕ is generated by recursion on ϕ using the clauses of the unenriched screening semantics for $\{\oplus, 0\}$-free formulas and the clauses

$$a \Vdash^* \theta \oplus \psi \iff (\forall x, y,\, x^*y^* \geq a^*)[x \Vdash^* \theta \text{ or } y \Vdash^* \psi] \qquad a \Vdash^* 0 \iff 1^* \not\leq a$$

to pass across $\oplus$ and 0. ■

Each pep $(A, (\cdot)^*)$ gives us a free Girard quantale $(\mathcal{L}A, D)$ (where $D = (\uparrow 1^*)'$) and this can be used to give values to all formulas ϕ (relative to a base assignment). Definition 10 is intended as a finitary version of this semantics.

11 THEOREM. (Compatibility) *Let $(\cdot)^*$ be a perp enrichment of the commutative pom A. Let*

$$P \longmapsto [\![P]\!]$$

be an $\mathcal{L}A$-assignment (from variables to values). Then the enriched screening semantics and the $(\mathcal{L}A, D)$-semantics agree, i.e.

$$a \Vdash^* \phi \Leftrightarrow a \in [\![\phi]\!]$$

holds for all $a \in A$ and formulas ϕ.

Proof. This is proved by induction on ϕ. Most of the steps have been done already (in the unenriched version) so here it will be enough to consider the passage across $\phi = \theta \oplus \psi$.

In this step we need to consider pairs of elements x, y with $x^*y^* \geq a^*$. It is convenient to set $u = x^*, v = y^*$, so that $u^* = x, y^* = y$. Using this convention we have

$$\begin{array}{rcll}
\neg[a \Vdash^* \phi] & \Leftrightarrow & \neg(\forall x^*y^* \geq a^*)[x \Vdash^* \theta \text{ or } y \Vdash^* \psi] & \\
& \Leftrightarrow & (\exists uv \geq a^*)[u^* \nVdash^* \theta \text{ and } v^* \nVdash^* \psi] & \\
& \Leftrightarrow & (\exists uv \geq a^*)[u^* \notin [\![\theta]\!] \text{ and } v^* \notin [\![\psi]\!]] & \\
& \Leftrightarrow & (\exists uv \geq a^*)[u \in [\![\theta]\!]^* \text{ and } v \in [\![\psi]\!]^*] & \\
& \Leftrightarrow & a^* \in [\![\theta]\!]^*[\![\psi]\!]^* & \Leftrightarrow \quad a \notin ([\![\theta]\!]^*[\![\psi]\!]^*)^* = [\![\phi]\!]
\end{array}$$

to give the required result. The third step uses the induction hypothesis. ■

Observe how this shows that

$$p \leq a \Vdash^* \phi \Rightarrow p \Vdash^* \phi$$

holds, i.e. $\Vdash^*$ is order respecting.

When the variables are dually matched we have a formal duality $(\cdot)^\perp$ on all formulas. Any decent semantics ought to respect this duality, and the enriched screening semantics does.

12 THEOREM. (Duality respecting) *Let $(\cdot)^*$ be a perp enrichment of the commutative pom A. Suppose the variables are dually matched*

$$P \longleftrightarrow P^\perp$$

and the assignment $P \longmapsto [\![P]\!]$ satisfies

$$[\![P^\perp]\!] = [\![P]\!]^*$$

for all variables P. Then the enriched screening semantics respects duality in the sense that

$$a \Vdash^* \phi^\perp \Leftrightarrow \neg[a^* \Vdash^* \phi]$$

holds for all $a \in A$ and formulas ϕ.

Proof. This is proved by induction over ϕ.

For the base case, where ϕ is a variable, we have

$$a \Vdash^* P^\perp \Leftrightarrow a \in [\![P^\perp]\!] = [\![P]\!]^* \Leftrightarrow a^* \notin [\![P]\!] \Leftrightarrow \neg[a^* \Vdash^* P]$$

using the given property of the base assignment.

The rest of the proof is the steps for the passage across each connectives (and the constants). Let's look at one of these cases.

Consider $\phi = \theta \otimes \psi$, so that $\phi^\perp = \theta^\perp \oplus \psi^\perp$. We need to consider pairs of elements x, y with $x^* y^* \geq a^*$. It is convenient to set $u = x^*, v = y^*$, so that $u^* = x, y^* = y$. Using this convention we have

$$\begin{array}{rcl} a \Vdash^* \phi^\perp & \Leftrightarrow & (\forall x^* y^* \geq a^*)[x \Vdash^* \theta^\perp \text{ or } y \Vdash^* \psi^\perp] \\ & \Leftrightarrow & (\forall uv \geq a^*)[u \nVdash^* \theta \text{ or } v \nVdash^* \psi] \\ & \Leftrightarrow & \neg(\exists uv \geq a^*)[u \Vdash^* \theta \text{ and } v \Vdash^* \psi^\perp] \quad \Leftrightarrow \quad \neg[a^* \Vdash^* \phi] \end{array}$$

as required.

The other steps are similar. ■

We now specialize these results to the abelian group case.

5 A characterization of the S-semantics

Let A be a fixed additively written partially ordered abelian group. Let

$$((P) \mid P \text{ a variable})$$

be a fixed family of separators on A. This family produces a semantic relation

$$(a, b) \Vdash_{\mathsf{S}} \phi$$

on A (as in Definition 1).

For each $i \in A$ let

$$y^i = i - y$$

so that, for $x \in A$

$$y^i \leq x \Leftrightarrow i \leq x + y$$

i.e. $(\cdot)^i$ is a co-residuation on A. For each variable P and $i \in A$ let $[\![P]\!]_i$ be the subset of A given by

$$a \in [\![P]\!]_i \Leftrightarrow a(P)a^i \Leftrightarrow a(P)b \text{ where } a + b = i$$

(for $a \in A$). It is easy to see that $[\![P]\!]_i \in \mathcal{L}A$, so we have an enriched screening semantics

$$a \Vdash^i \phi$$

on A. We are going to vary this parameter i, so let us write

$$[i, a] \Vdash \phi$$

for this fulfilment relation.

With this preparation we can relate the two semantics.

13 THEOREM. *Using the above correspondence at the base level, the equivalence*

$$(a, b) \Vdash_{\mathsf{S}} \phi \Leftrightarrow [i, a] \Vdash \phi$$

holds for all formulas ϕ and $a, b, i \in A$ with $a + b = i$.

Proof. This is proved by induction over ϕ. For a variable P we have

$$(a, b) \Vdash_{\mathsf{S}} P \Leftrightarrow a(P)b \Leftrightarrow a \in [\![P]\!]_i \Leftrightarrow [i, a] \Vdash P$$

where the central equivalence uses the construction of $[\![P]\!]_i$ from (P). The induction steps across the constants and the additive connectives are immediate. Only the passage across $\otimes, \oplus, \multimap$ needs some effort. The three cases are similar; let's look at the step across $\oplus$.

Suppose first that

$$(a, b) \Vdash_{\mathsf{S}} \theta \oplus \psi$$

where $i = a + b$. To show

$$[i, a] \Vdash \theta \oplus \psi$$

consider any x, y with $x^i + y^i \geq a^i$. Setting

$$u = x - a \qquad v = y - a$$

a simple calculation gives $u + v \leq b$ and hence either

$$(a + u, b - u) \Vdash_{\mathsf{S}} \theta \quad \text{or} \quad (a + v, b - v) \Vdash_{\mathsf{S}} \psi$$

hold. The induction hypothesis now gives

$$[i, x] \Vdash \theta \quad \text{or} \quad [i, y] \Vdash_{\mathsf{S}} \psi$$

as required.

Conversely suppose that

$$[i, a] \Vdash \theta \oplus \psi$$

where $i = a + b$. To show

$$(a, b) \Vdash_{\mathsf{S}} \theta \oplus \psi$$

consider any u, v with $u+v \leq b$. With

$$x = u + a \qquad y = v + a$$

we have

$$x^i = b - u \qquad y^i = b - v$$

so that

$$x^i + y^i \geq b = a^i$$

and hence either

$$[i, x] \Vdash \theta \quad \text{or} \quad [i, y] \Vdash_{\mathsf{S}} \psi$$

hold. The induction hypothesis now gives

$$(x, i - x) \Vdash_{\mathsf{S}} \theta \quad \text{or} \quad (y, i - y) \Vdash_{\mathsf{S}} \psi$$

i.e.

$$(a + u, b - u) \Vdash_{\mathsf{S}} \theta \quad \text{or} \quad (a + v, b - v) \Vdash_{\mathsf{S}} \psi$$

as required. ■

Notice how the group A plays two different roles. It carries the semantics and indexes the family of perp enrichments. It would be interesting to see a construction where these two roles are played by different algebras.

6 The general M-semantics

The M-semantics of Definition 1 is based on a partially ordered abelian group. However, the recursion clauses make no references to the inverse operation of the group and so the whole construction can be transferred to an arbitrary commutative pom. Of course, there is no guarantee that such a transfer will produce a successful semantics, for the group properties may be involved in the required verifications for that semantics. In this section we show that such a transfer is successful and, more importantly, leads to a new construction of a Girard quantale.

Let A be an arbitrary commutative pom. We fix this throughout the section.

A separator on a A is a binary relation S such that

$$p \leq aSb \leq q \Rightarrow pSq$$

holds for all $a, b, p, q \in A$.

We use these notions to define fulfilment relations

$$(a, b) \Vdash \phi$$

between pairs (a, b) of elements of A and linear formulas ϕ.

14 DEFINITION. Let A be a commutative pom and for each variable P let (P) be a selected separator on A. The relation

$$(a, b) \Vdash \phi$$

is generated by recursion over the structure of ϕ using

$$(a, b) \Vdash P \iff a(P)b$$

for the base, the clauses

$$\begin{array}{llll} (a,b) \Vdash \theta \sqcap \psi \iff (a,b) \Vdash \theta & \text{and} & (a,b) \Vdash \psi & \qquad (a,b) \Vdash \top \iff \text{Always} \\ (a,b) \Vdash \theta \sqcup \psi \iff (a,b) \Vdash \theta & \text{or} & (a,b) \Vdash \psi & \qquad (a,b) \Vdash \bot \iff \text{Never} \end{array}$$

for the additive connectives and constants, and the clauses

$$\begin{array}{lcl}(a,b) \Vdash \theta \otimes \psi & \Longleftrightarrow & (\exists x, y,\ xy \geq a)[(x, by) \Vdash \theta \quad \text{and} \quad (y, bx) \Vdash \psi] \\ (a,b) \Vdash \theta \oplus \psi & \Longleftrightarrow & (\forall u, v,\ uv \geq b)\,[(av, u) \Vdash \theta \quad \text{or} \quad (au, v) \Vdash \psi] \\ (a,b) \Vdash \theta \multimap \psi & \Longleftrightarrow & (\forall x, v,\ xv \geq b)\,[(x, av) \Vdash \theta \quad \Rightarrow \quad (ax, v) \Vdash \psi]\end{array}$$

$$(a,b) \Vdash 1 \Longleftrightarrow a \leq 1 \qquad (a,b) \Vdash 0 \Longleftrightarrow b \not\leq 1$$

for the multiplicative connectives and constants. ■

Observe that when A is an additively written abelian group this construction is exactly that of the M-semantics of Definition 1. We will show that for an arbitrary commutative pom A this construction gives a semantics for linear formulas. In particular, the fulfilment relation is separation preserving, i.e. the implication

$$(a,b) \Vdash \phi \Rightarrow (p,q) \Vdash \phi$$

holds for all formulas ϕ and $a, b, p, q \in A$ with $p \leq a, b \leq q$. Furthermore, when the variables are dually matched and the selected base relations (P) satisfy

$$a(P^{\perp})b \Leftrightarrow \neg[b(P)a]$$

(for all $a, b \in A$), then the fulfilment relation respects duality, i.e.

$$(a,b) \Vdash \phi^{\perp} \Leftrightarrow (b,a) \nVdash \phi$$

holds for all pairs (a, b) and formulas ϕ. We could verify these properties directly but it is better to view these in a different way. We construct a Girard quantale $\mathcal{S}A$ and then observe that the recursion clauses of definition 14 are merely the standard clauses which set up an $\mathcal{S}A$-valued semantics.

Let $\mathcal{S}A$ be the set of all separators on A. We will show that $\mathcal{S}A$ carries a natural Girard quantale structure. Furthermore, the selected base separators (P) give a base assignment $P \longmapsto [\![P]\!]$ from variables to $\mathcal{S}A$ (as a quantale) and this lifts to produce a value

$$[\![\phi]\!] \in \mathcal{S}A$$

for each formula ϕ such that

$$(a,b) \Vdash \phi \Longleftrightarrow a[\![\phi]\!]b$$

holds for all $a, b \in A$.

Each separator on A is a binary relation, and so these can be compared by inclusion. In particular

$$R \subseteq S \Longleftrightarrow (\forall x, y : A)[xRy \Rightarrow xSy]$$

holds for all $R, S \in \mathcal{S}A$, and hence $\mathcal{S}A$ is a poset (under inclusion). Families $\mathcal{R}$ of relations are closed under arbitrary unions and intersections

$$a(\bigcup \mathcal{R})b \Longleftrightarrow (\exists R \in \mathcal{R})[aRb] \qquad a(\bigcap \mathcal{R})b \Longleftrightarrow (\forall R \in \mathcal{R})[aRb]$$

and it is easy to check that if $\mathcal{R} \subseteq \mathcal{S}A$ then both $\bigcup \mathcal{R}$ and $\bigcap \mathcal{R}$ belong to $\mathcal{S}A$. Thus $\mathcal{S}A$ is a complete lattice with set theoretic operations. Notice that the bottom $\perp$ and top $\top$ are given by

$$a\perp b \Longleftrightarrow \text{Never} \qquad a\top b \Longleftrightarrow \text{Always}$$

and these are separators.

Given a pair R, S of separators we define three binary relations

$$R \otimes S \qquad R \oplus S \qquad R \multimap S$$

by

$$\begin{array}{lcllcl} a(R \otimes S)b & \Longleftrightarrow & (\exists x, y,\ xy \geq a)[xRby & \text{and} & ySbx] \\ a(R \oplus S)b & \Longleftrightarrow & (\forall u, v,\ uv \geq b)\,[vaRu & \text{or} & uaSv] \\ a(R \multimap S)b & \Longleftrightarrow & (\forall x, v,\ xv \geq b)\,[xRav & \Rightarrow & xaSv] \end{array}$$

(for $a, b \in A$). Compare these equivalences with the multiplicative clauses of Definition 14. The crucial observation is that these are also separators (and so the generated semantics is separating).

15 LEMMA. *For each pair $R, S \in \mathcal{S}A$ the relations*

$$R \otimes S \qquad R \oplus S \qquad R \multimap S$$

are separators on A.

Proof. We look at the three constructions in turn. In each case we consider elements

$$p \leq a \qquad b \leq q$$

where (a, b) is assumed to satisfy the constructed relation, and we show that (p, q) also satisfies the relation.

($\otimes$) Assuming

$$a(R \otimes S)b$$

we have both

$$xRby \text{ and } ySbx$$

for some pair x, y with $xy \geq a$. Since $b \leq q$ we have $by \leq qy$ and $bx \leq qx$ and then

$$xRqy \text{ and } ySqx$$

by the given separation properties of R, S. Since $xy \geq a \geq p$ these give

$$p(R \otimes S)q$$

as required.

($\oplus$) Suppose

$$a(R \oplus S)b$$

and consider any $uv \geq q$. Since $q \geq b$ this gives either

$$vaRu \quad \text{or} \quad uaSv$$

(by the definition of $\oplus$). But $p \leq a$ so $vp \leq va$ and $ua \leq up$ to give

$$vpRu \quad \text{or} \quad upSv$$

and hence

$$p(R \oplus S)q$$

by the choice of u, v.

($\multimap$) Suppose

$$a(R \multimap S)b$$

consider any $xv \geq q$, and suppose

$$xRpv$$

holds. Since $p \leq a$ we have $pv \leq av$ so that

$$xRav$$

and hence

$$xaSv$$

(by the definition of $\multimap$). But $p \leq a$ so that $xp \leq xa$ to give

$$xpSv$$

and hence

$$p(R \multimap S)q$$

by the choice of u, v.

This completes the three required verifications. ■

Let $\mathsf{1}$ be the relation on A given by

$$a\mathsf{1}b \Leftrightarrow a \leq 1$$

(for $a, b \in A$). Trivially, this is a separator.

16 LEMMA. *The operation $\otimes$ on $\mathcal{S}A$ is commutative and associated with $\mathsf{1}$ as a neutral element. Thus $(\mathcal{S}A, \otimes, \mathsf{1})$ is a commutative monoid.*

Proof. The commutativity of $\otimes$ is almost immediate. (We interchange the roles of the witnesses x and y.)

Given $R, S, T \in \mathcal{S}A$ and $a, b \in A$, both versions of

$$a(R \otimes S \otimes T)b$$

unravel to

$$(\exists x, y, z,\ xyz \geq a)[xRbyz \text{ and } ySbzx \text{ and } zTbxy]$$

to show that $\otimes$ is associated.

Finally, a simple calculation gives

$$R \otimes \mathsf{1} = R$$

(for $R \in \mathcal{S}A$), as required. ■

It can be shown directly that $\otimes$ is monotone and the quantale distributive laws hold (and hence $\mathcal{S}A$ is a quantale). This is not necessary for these are consequences of the following more useful result.

17 LEMMA. *The operation $\cdot \multimap \cdot$ is a residuation on $\mathcal{S}A$, i.e.*

$$R \otimes S \subseteq T \Longleftrightarrow R \subseteq S \multimap T$$

holds for all $R, S, T \in \mathcal{S}A$.

Proof. ($\Rightarrow$) Assuming

$$(1) \quad R \otimes S \subseteq T \qquad (2) \quad aRb \qquad (3) \quad pq \geq b \qquad (4) \quad pSaq$$

we show

$$pa(R \otimes S)q$$

so that (1) gives

$$paTq$$

which is what is required.

We want x, y with

$$xy \geq pa \qquad xRqy \qquad ySqx$$

holding. Let $x = a, y = p$ so that the first is immediate and (2,3) give the second. The third is just (4).

($\Leftarrow$) Suppose

$$(1) \quad R \subseteq S \multimap T \qquad (2) \quad a(R \otimes S)b$$

for $a, b \in A$. From (2) we have x, y with

$$(3) \quad xy \geq a \qquad (4) \quad xRby \qquad (5) \quad ySbx$$

holding. From (1,4) we have

$$x(S \multimap T)by$$

which, since $yb \geq by$ gives

$$ySxb \Rightarrow yxTb$$

and hence (5) with commutativity gives

$$xyTb$$

so that

$$aTb$$

holds using (3). ■

This shows that $\mathcal{S}A$ is a commutative quantale. We now want to show it is a Girard quantale. To do that we exhibit the dualizing element and the associated duality.

Each relation R on A has a converse $R^{\smile}$ given by

$$aR^{\smile}b \Leftrightarrow bRa$$

(for $a, b \in A$). When R is a separator this converse $R^{\smile}$ need not be, but its complement

$$R^{\perp} = R^{\smile\prime}$$

given by

$$aR^{\perp}b \Longleftrightarrow \neg[bRa]$$

is. (Try this!) Note that this operation $(\cdot)^{\perp}$ on $\mathcal{S}A$ is an involution. Let

$$\mathsf{0} = \mathsf{1}^{\perp}$$

i.e. consider the separator given by

$$a\mathsf{0}b \Leftrightarrow \neg[b\mathsf{1}a] \Leftrightarrow b \not\leq 1$$

(for $a, b \in A$).

18 LEMMA. *For each separator R*

$$R^{\perp} = R \multimap 0$$

holds.

Proof. For each $a, b \in A$ we have

$$\begin{array}{rcll} a(R \multimap 0)b & \Leftrightarrow & (\forall xv \geq b)[xRav \Rightarrow xa0v] & \\ & \Leftrightarrow & (\forall xv \geq b)[xRav \Rightarrow v \not\leq 1] & \\ & \Leftrightarrow & (\forall xv \geq b)[v \leq 1 \Rightarrow \neg[xRav]] & \Rightarrow \quad \neg[bRa] \end{array}$$

using $x = b, v = 1$. Conversely, if $xv \geq b$ with $v \leq 1$, then $b \leq x$ and $av \leq a$ so that

$$xRav \Rightarrow bRa$$

to give the required result. ■

For the record let us set down the observation of this section.

19 THEOREM. *For each commutative pom A the lattice theoretic algebra $(\mathcal{S}A,\ \otimes, 1)$ is a commutative quantale, with residuation $\cdot \multimap \cdot$, a dualizing element 0, and with $\oplus$ as the dual operation of $\otimes$.*

Proof. The only thing left to check is that $\oplus$ is dual to $\otimes$, i.e. that

$$(R \oplus S)^{\perp} = R^{\perp} \otimes S^{\perp}$$

holds. This is a routine quantifier manipulation. ■

At first sight the construction of $\mathcal{S}A$ can be confused with two other constructions.

The opposite A^{op} of A is also a commutative pom, hence so is the product $A \times A^{op}$. Thus the algebra $\mathcal{L}(A \times A^{op})$ of lower sections is a freely generated commutative quantale. These lower sections are nothing more than the separators on A. Thus, as sets

$$\mathcal{S}A = \mathcal{L}(A \times A^{op})$$

but the carried structures are different.

The Chu construction can be used to convert any commutative quantale into a Girard quantale. In particular, the Chu construction can be applied to the free quantale $\mathcal{L}A$, and the resulting operations look vaguely like those of $\mathcal{S}A$. However, the Chu construction furnishes

$$\mathcal{C}A = \mathcal{L}A \times \mathcal{L}A^{op}$$

as a quantale, and this is a proper subset of $\mathcal{S}A$. For instance, the given comparison $\leq$ is a separator but not in $\mathcal{C}A$. Furthermore, the insertion of $\mathcal{C}A$ into $\mathcal{S}A$ is not a quantale morphism.

To conclude. The original motivation for the work here came from considering models for linear logic which give some account of resource allocation. This is achieved by interpreting formulae as binary relations, intuitively these represent relations between an internal and external environment. A formula then represents a description of how resources are to be moved between these environments. The second consideration for such models is to provide a sensible way of handling $\perp$.

The given constructions fit into the wider quantale framework (theorem 19) and are separator preserving (lemma 5). In the case of group based semantics there is an interesting characterisation theorem (theorem 13) for the S semantics.

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