Intersection Points Of Self Transverse Immersions

William.P.R.Mitchell

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1 Introduction

Let ζ be a real k- dimensional vector bundle. A vector bundle β has a ζ structure if there is a bundle map $\beta \longrightarrow \zeta$. In [2] it is shown that the bordism group of immersions $M^{nk} \xrightarrow{i} \mathbb{R}^{n(k+1)}$ with a ζ structure on the normal bundle of the immersion is isomorphic to the stable homotopy group $\pi^s_{n(k+1)}(M\zeta)$ where $M\zeta$ is the Thom complex of ζ. Also in [2] there is defined a homomorphism

$$\theta_{k+1} : \pi^s_{n(k+1)}(M\zeta) \longrightarrow \pi^s_{n(k+1)}(D_{k+1}M\zeta)$$

where $D_{k+1}X$ is the $k+1$ -adic construction on X. $\pi^s_{n(k+1)}(D_{k+1}M\zeta)$ is always either $\mathbb{Z}$ or $\mathbb{Z}/2$.

Let θ'_{k+1} be θ_{k+1} composed with the projection map onto $\mathbb{Z}/2$. Let $M^{nk} \xrightarrow{i} \mathbb{R}^{n(k+1)}$ be a self transverse immersion with ζ structure then $\theta'_{k+1}([i]) =$ the number of $k+1$-fold intersection points mod 2 of i.

If we put $M\zeta = MO(n)$ then $\theta_{k+1} = \theta'_{k+1}$. Hughes [4] has considered the case when $k = 2$ and has shown that θ_3 is onto $\forall\ n \geq\ 1$.

When we put $M\zeta = MU(n)$ we have a homomorphism

$$\theta_{k+1} : \pi^s_{2n(k+1)}(MU(n)) \longrightarrow \mathbb{Z}$$

Let $M^{2nk} \xrightarrow{i} \mathbb{R}^{2n(k+1)}\ \epsilon \pi^s_{2n(k+1)}(MU(n))$ be self transverse. In particular this immersion is oriented,this orientation gives a sign to each $k+1$-fold intersection point. $\theta_{k+1}([i])$ is the sum of these signs.

It is still an open problem to find the minimum strictly positive integer l for which there is a β with $\theta_{k+1}(\beta) = l$. This paper attempts to give a necessary condition on l. Let $m\epsilon\mathbb{Z}$ and $m = \sum \alpha_i(k+1)^i$ where $0 \leq \alpha_i \leq k$. Define $\alpha(m) = \sum \alpha_i$. Further define

$$\delta(m) = \min\{s\epsilon\mathbb{N} \mid \alpha(km+s) \leq (k+1)s\}$$

The main theorem of this paper is

Theorem 1
For any $\beta \,\epsilon\, \pi^s_{2n(k+1)}(MU(n))$ and any $k \geq 1$, $\theta_{k+1}(\beta)$ is divisible by

$$(k+1)^{\delta(n)-1}$$

It follows directly from Rees and Thomas [8] that when $k=1$ and $n \geq 1$ there is in fact some $\alpha \epsilon \pi^s_{4n}(MU(n))$ such that $\theta_2(\alpha) = 2^{\delta(n)-1}$. In [7] it is shown that if n is not a power of 2 then there is an $\alpha \epsilon \pi^s_{4n}(MSU(n))$ such that $\theta_2(\alpha) = 2^{\delta(n)-1}$. In [3] it is proved that when $n \geq 3$ the image of θ_3 is $3^r\mathbb{Z}$ for some $r \geq \delta(n)-1$. This says, in particular, that $\forall\ n \geq 3$ there is a self transverse immersion $M^{4n} \xrightarrow{i} \mathbb{R}^{6n}$ with a complex structure which has an odd number of triple points.

2 Spherical Elements in Homology

Recall that $MU(n)$ is homotopy equivalent to $BU(n)/BU(n-1)$ so that we can identify $H^*(MU(n))$ with $c_n\mathbb{Z}[c_1,\ldots,c_n]$ where c_i is the i'th Chern class. The following theorem is from [3].

Proposition 1 *Let $h^s : \pi^s_*(MU(n)) \longrightarrow H_*(MU(n),\mathbb{Z})$ be the stable Hurewicz homomorphism. Let $\alpha \,\epsilon \pi^s_{2n(k+1)}(MU(n))$ then*

$$\langle c_n^{k+1}, h^s(\alpha)\rangle = (-1)^k(k+1)\theta_{k+1}(\alpha)$$

where $\langle\ ,\ \rangle$ denotes the Kronecker product.

Let MU denote the representing spectrum for complex bordism. Recall that $H^*(MU) = \mathbb{Z}[c_1, c_2 \ldots]$ and $H_*(MU) = \mathbb{Z}[\beta_1,\beta_2,\ldots]$ where β_i has degree $2i$. Also there is an inclusion $i : MU(n) \longrightarrow \sum^{2n} MU$ such that $i^*(c_{i_1}c_{i_2}\ldots c_{i_m}) = 0$ if $i_r > n$ for any i_r and $= c_{i_1}c_{i_2}\ldots c_{i_m}c_n$ otherwise. Thus for $\alpha \,\epsilon \pi^s_{2n(k+1)}(MU(n))$

$$\langle c_n^k, h^s(i_*(n)(\alpha))\rangle = (-1)^k(k+1)\theta_{k+1}(\alpha).$$

Let h be the Hurewicz homomorphism. In [8] it is shown that

$$\langle c_n, h(\pi^s_*(MU)\rangle = 2^{\delta(n)}\mathbb{Z}$$

for $n \geq 1$. Since i gives an isomorphism of homotopy groups in dimension $4n$ it follows that $\theta_{k+1}(\pi^s_{4n}(MU(n)) = 2^{\delta(n)-1}\mathbb{Z}$. Put $c_I = c_1^{i_1}c_2^{i_2}\ldots c_l^{i_l}$ where $I = (i_1,\ldots,i_l)$. The c_I form a basis for $H^*(MU)$. Let $\{\overline{c}_I\}$ denote the dual basis. Then there is a unique polynomial P_I such that $\overline{c}_I = P_I(\beta_1,\beta_2,\ldots,\beta_N)$ where $\Sigma_r\ ri_r = N$. This polynomial is explicitly described in [1] page 11. Given this description of $\overline{c}_I$ it is not too difficult to prove that if $a \,\epsilon H_{2n(k+1)}(MU(n))$ with $\langle c_n^{k+1}, a\rangle = \lambda$ then $i_*(a) = \lambda\beta_k^n+$(other terms) when written in terms of monomials in the β_i.

Definition 2 *Define*

$$\theta_{k+1}^{MU} = \min\{\ s\epsilon\mathbb{Z} \mid s \geq 1 \text{ and } \exists\, ((k+1)s\beta_k^n + (\text{otherterms}))\ \epsilon h\pi_*(MU)\}$$

From what we have just said it follows that

Proposition 3 *For all $\alpha\ \epsilon\pi_{2n(k+1)}^s(MU(n))\ \ \theta_{k+1}(\alpha)$ is divisible by θ_{k+1}^{MU}.*

In order to prove Theorem 1 we now only need to prove that θ_{k+1}^{MU} is divisible by $(k+1)^{\delta(n)-1}$. This we do in the following section.

3 Kozma's spherical generators

Definition 4 *(1) For $l\epsilon\mathbb{Z}$ let $\nu_{k+1}(l)$ be the maximum integer b such that $(k+1)^b$ divides l .*
(2) Let $x = l\beta_k^n +$ (other terms) $\epsilon H_*(MU)$. Define $\gamma(x) = \nu_{k+1}(l)$.
(3) Let $m_n = \frac{1}{n+1}h(\mathbb{CP}^n)\ \epsilon H_*(MU,\mathbb{Z})$.

$h(\mathbb{CP}^n)$ is divisible by $n+1$ by [1] page 81.

Proposition 5 *([1] Propn 7.5)*

$$m_n = \frac{1}{n+1}\left(\sum_{i\geq 0}\beta_i T^i\right)^{-n-1}_{T^n}$$

where for a polynomial $p(T), p(T)_{T^n}$ means the coefficient of the T^n term.

Corollary 6 $m_{kn} = \frac{(-1)^n}{kn+1}\begin{pmatrix} (k+1)n \\ n \end{pmatrix}\beta_k^n +$*(other terms).*

The following facts are easily proved.

Proposition 7 *(1)*

$$\nu_{k+1}(n) = \frac{\alpha(n-1)-\alpha(n)+1}{k}$$

(2)

$$\nu_{k+1}(n!) = \frac{n-\alpha(n)}{k}$$

(3)

$$\nu_{k+1}\left(\begin{pmatrix} m+n \\ n \end{pmatrix}\right) = \frac{\alpha(n)+\alpha(m)-\alpha(m+n)}{k}$$

.

Proposition 8 *$\gamma(m_{kn}) \geq \delta(n)$ provided $kn \neq (k+1)^l - 1$ for any l.*

Proof. By corollary 6

$$\gamma(m_{kn}) = \nu - k + 1\left(\frac{(-1)^n}{kn+1}\left(\begin{array}{c}(k+1)n \\ n\end{array}\right)\right) = \frac{\alpha(kn+1)-1}{k}$$

using theorem 7. Note this is 0 iff $\alpha(kn+1)-1 = 0$ iff $\exists l.\ kn+1 = (k+1)^l$. Now

$$\begin{aligned}
\alpha\left(kn + \tfrac{\alpha(kn+1)-1}{k}\right) &= \alpha\left(kn+1+\tfrac{\alpha(kn+1)-(k+1)}{k}\right) \\
&\leq \alpha(kn+1) + \alpha\left(\tfrac{\alpha(kn+1)-(k+1)}{k}\right) \\
&\quad \text{(provided } \forall l.\ kn+1 \neq (k+1)^l\text{)} \\
&\leq \alpha(kn+1) + \tfrac{\alpha(kn+1)-(k+1)}{k} \\
&= (k+1)\left(\tfrac{\alpha(kn+1)-1}{k}\right)
\end{aligned}$$

Hence by the defn of δ the result is true$\bullet$

Lemma 9 *(1) Let $b = \sum b_i \,\epsilon \mathbb{Z}$. Then $\sum \delta(b_i) \geq \delta(b)$.*
(2) For $b \neq 0$ $(k+1)\delta(b) \geq \delta((k+1)b+1)$.

Proof
Part (1) is easy so we prove only part (2).
Consider $\alpha(k((k+1)b+1)+(k+1)\delta(b)) = \alpha((k+1)(kb+\delta(b))+k)$
$= \alpha(kb+\delta(b))+k \leq (k+1)\delta(b)+k \leq 2(k+1)\delta(b) \leq (k+1)((k+1)\delta(b))$.
Thus by the definition of δ the result is true $\bullet$

Definition 10 *[6] Let $T_{q,i} \,\epsilon\, H_{2(qi-1)}(MU,\mathbb{Z})$ be defined by*

$$T_{q,i} = q\, m_{qi-1} - \sum_{sd=i, s\neq 1} m_{s-1}(T_{q,d})^s$$

whenever q is a prime.

Proposition 11 *[6] There are polynomial generators of the image of $\pi_*(MU)$ in $H_*(MU)$ given by linear combinations of the $T_{q,i}$.*

Definition 12 *Let q be a prime, $i \epsilon \mathbb{N}$ and suppose $qi = kr+1$ for some r. Define $\delta(T_{q,i}) = \delta(r)$.*

From now on assume $qi = kr+1$.

Proposition 13

$$\delta(T_{q,i}) \leq \gamma(T_{q,i}) \ \ \forall\ q, i.$$

Proof. We prove the result by induction on i. The result needs dividing into cases depending on the values of q and i.

case (1) $qi = (k+1)^l$

In this case clearly $\delta(r) = 1$ ($qi = kr + 1$). Using induction we can say that $\gamma(m_{s-1}(T_{q,d})^s) > 1$ for $s \neq 1$ and $sd = i$. Thus from definition 10 we have that

$$\gamma(T_{q,i}) = \gamma(q\, m_{qi-1}) = 1$$

case (2) $qi \neq (k+1)^l\ \forall l$

Since $qi \neq (k+1)^l$ we have that $\gamma(m_{qi-1}) \geq \delta(m_{qi-1})$ by proposition 8. Now consider $\gamma(m_{s-1}(T_{q,d})^s)$ where $sd = i, i \neq 1$. First suppose $s = (k+1)^l$ and $qd = kr' + 1$. Then by lemma 9 $(k+1)\delta(r') \geq \delta((k+1)r' + 1)$ i.e $(k+1)\delta(T_{q,d}) \geq \delta(T_{q,(k+1)d})\ \ -(*)$. Hence by lemma 9 $\gamma((T_{q,d})^s) = (k+1)^l \gamma(T_{q,d}) \geq (k+1)^l \delta(T_{q,d})$ (by inductive hypothesis as $d < i$) $\geq \delta(T_{q,i})$ by repeated application of $(*)$. So for $s = (k+1)^l$ $\gamma(m_{s-1}(T_{q,d})^s) \geq \delta(T_{q,i})$. Now suppose $s \neq (k+1)^l\ \forall\ l$. Let $s = kr_1 + 1$, $qd = kr_2 + 1$. Then $\gamma(m_{s-1}) \geq \delta(r_1)$ by proposition 8. Also $\gamma((T_{q,d})^s) = s\gamma(T_{q,d}) \geq s\delta(T_{q,d})$ by the inductive hypothesis. $s\delta(T_{q,d}) = s\delta(r_2) \geq \delta(sr_2)$ by lemma 9. Hence

$$\gamma(m_{s-1}(T_{q,d})^s) \geq \delta(sr_2 + r_1) = \delta(T_{q,i})$$

by lemma 9. This completes the inductive step of the proof and the base case is done by proposition 8 •

Proposition 14 *θ^{MU}_{k+1} is divisible by $(k+1)^{\delta(n)-1}\ \forall\ k \geq 1$.*

Proof. Since all spherical elements in $H_*(MU)$ are polynomials in the $T_{q,i}$ it follows that $\nu_{k+1}(\theta^{MU}_{k+1})$ is the minimum integer $s \geq 0$ such that for some polynomial $M = T_{q_1,i_1} \ldots T_{q_N,i_N}$ we have $\gamma(M) = s + 1$. By lemma 9 we have $\gamma(M) = \sum_r \gamma(T_{q_r,i_r}) \geq \sum_r \delta(T_{q_r,i_r})$ (by theorem 13) $\geq \delta(n)$ by lemma 9. Hence the result is true •

Thus we have proved Theorem 1.

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Department of Computer Science
University of Manchester
Manchester M13 9PL