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APPLICATION OF THE FRONT-FIXING METHOD FOR NUMERICAL MODELLING OF A THERMISTOR PROBLEM

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***Abstract* - Application of a finite difference front fixing method to a thermistor problem with strongly non-linear material properties is discussed. Advantages and implementation problems of the method are highlighted. Particular attention is given to conservation properties of the algorithm and accurate solutions close to the moving transition region. The algorithm is tested using a well-known solution of the plane diffusion problem with complex conditions at the moving interface.**

I. INTRODUCTION

A thermistor is a circuit device with very non-linear dependence of electric conductivity σ on temperature T [1]. The changes in σ may be rapid, typical σ variations of four orders of magnitude with T increasing from 100°C to 200°C have been recorded [2] (Figure 1). The transition region for a given point between temperatures T_1 and T_2, which corresponds to coordinates s_1 and s_2, is typically small and a coupled treatment of electric and thermal fields is complex. It is quite common to ignore the transition and consider a sharp interface with a step behaviour in σ. But it was noted [2] that the step function is not the most realistic model for σ and that a more complex σ(T) should be used for accurate predictions.

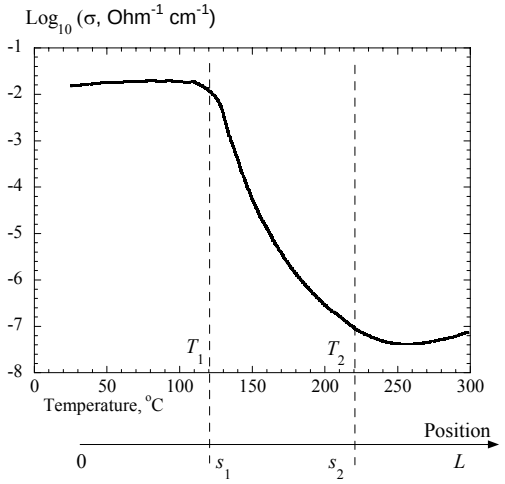

Fig. 1. Typical variation of conductivity with temperature for a thermistor.

This paper deals with modelling of coupled electric current and heat flow in a thermistor with a particular attention paid to a moving transient region. The motion is assumed to be a complex function of the solution itself. Many numerical methods developed previously, such as front tracking or re-meshing techniques, are often not able to cope effectively with such a strong coupling. On the other hand, the front-fixing transformations [3] introduce a co-ordinate system in which all of the spatial region boundaries are fixed to s_1 and s_2. One advantage of discretising in the transformed space is that the meshes automatically adjust themselves to accommodate the moving interface position. It is therefore possible to impose irregular meshes with fine resolution in regions where large temperature and field gradients are expected, while using larger space steps elsewhere.

The main challenges are the implementation of conservation laws at the moving boundaries and an effective mesh refinement. These issues are addressed in the paper.

II. FORMULATION OF THE PROBLEM

The electric current flow in the system and associated Joule heating are described by Ohm's law:

$$\nabla\mathbf{J} = 0,\ \mathbf{J} = \sigma\nabla\varphi,\ Q_{el} = \sigma^{-1}\mathbf{J}^2 \equiv \nabla\left(\sigma\varphi\nabla\varphi\right), \qquad (1)$$

where **J** is the current density, ϕ is an electric potential, Q_{el} is a Joule heat. Heat transport is governed by the heat diffusion equation:

$$C\rho\frac{\partial T}{\partial t} = \nabla\left(\kappa(T)\nabla T + \sigma(T)\phi\nabla\phi\right), \qquad (2)$$

where C is a specific heat, ρ is a density, κ is a thermal conductivity. Appropriate initial and boundary conditions for (1), (2) are discussed elsewhere [2].

III. FRONT-FIXING METHOD

Landau transformation utilises new positional variables (one for each domain). In the plane case, an introduction of $u=x/s_1(t)$ fixes the extent of high conductivity region to the domain $0\le u\le 1$, while $w=(x-s_2)/(L-s_2)$, and $v=(x-s_1)/(s_2-s_1)$, fix the extent of the other domains to $0\le w\le 1$ and $0\le v\le 1$. As an example, the divergent form of Equations (1) and (2) for u and interface equations at s_1 may be written as [3]:

$$\frac{\partial\left(C\rho s_1 T\right)}{\partial t} = \frac{ds_1}{dt}\frac{\partial(C\rho u T)}{\partial u} + \frac{1}{s_1}\frac{\partial}{\partial u}\left(\kappa\frac{\partial T}{\partial u} + \sigma\phi\frac{\partial\phi}{\partial u}\right), \qquad (3)$$

$$\text{for } u\in(0,1)\text{, with } T\big|_{u=1} = T\big|_{v=0} = T_1 \qquad (4)$$

$$\frac{1}{s_1}\left(\kappa\frac{\partial T}{\partial u}\right)\bigg|_{u=1} = \frac{1}{s_2-s_1}\left(\kappa\frac{\partial T}{\partial v}\right)\bigg|_{v=0} \qquad (5)$$

$$\frac{\partial}{\partial u}\left(\sigma\frac{\partial\phi}{\partial u}\right) = 0,\ \text{for } u\in(0,1) \qquad (6)$$

$$\frac{1}{s_1}\left(\sigma\frac{\partial\phi}{\partial u}\right)\bigg|_{u=1} = \frac{1}{s_2-s_1}\left(\sigma\frac{\partial\phi}{\partial v}\right)\bigg|_{v=0}. \qquad (7)$$

Equations similar to (3)-(7) can be written for *v* and *w*. In fact the Landau transformation introduces a co-ordinate system in which all of the spatial boundaries are fixed to 0 or 1. Under the transformation, the new computational domains remain the same with an additional advection term in (3) and non-linear equation for the boundary motion, Equation (5). This allows keeping nodes close to the interface independent of the motion, which gives higher accuracy for the same number of nodes used. A divergent form of Equations (3) and (6) ensures that there are no artificial energy sources. The thermal energy balance at the interfaces is fulfilled by the variation in interface positions s_1 and s_2 to satisfy Equation (5). Equations (3) to (5) represent a typical Stefan problem with the so-called implicit moving boundary and strongly non-linear Joule heat source.

Numerical scheme. The set of simultaneous equations involves the unknown future temperature field and potentials together with the future interface position. Since all of the equations are coupled, if the implicit scheme is to conserve energy, the entire system must be solved simultaneously. But the fact that they form a non-linear system means that this is potentially very demanding in terms of computing times. It is interesting to note, however, that Equations (3) and (5)-(7) are only weakly coupled; thus, if the future interface positions were known, the diffusion problem (3) would become quasi-linear. With a known temperature profile the potential could be found from (6), (7) using standard algorithms. Conversely, if future temperature and potentials were known, the future interface position could be calculated from Equation (5) relatively easily. It is possible to implement an efficient algorithm based on de-coupling the problem in this way. Normally it requires only few iterations to reach a consistent solution.

IV. VALIDATION

The quality of the solution is assessed using a simplified system with a known analytical solution. A diffusion-controlled phase change is an example of a non-linear, coupled diffusion-motion problem with a discontinuity at the moving interface, i.e. sharp front. The well known set of coupled non-linear differential equations may be used to model the system [3]. It can be described by variation of an abstract potential (ϕ) with position (x). Potential profiles depend on time (t). Diffusive processes occur simultaneously in two distinct domains (A & B), but the potential of one domain in contact with the other is fixed by a thermodynamic constraint (equilibrium potentials ϕ_A and ϕ_B). The rates at which potential diffuses towards the interface through A and is removed into B are not necessarily equal. In order to conserve energy the interface between the two domains must move. Writing the interface position as $s=s(t)$, the following set of coupled non-linear differential equations may be used to model the system

$$\frac{\partial \phi}{\partial t} = \frac{\partial}{\partial x}\left(D\big(\phi(x,t)\big)\frac{\partial \phi(x,t)}{\partial x}\right),\ 0 < x < L,\ x \neq s(t) \qquad (8)$$

$$D_A \left.\frac{\partial \phi}{\partial x}\right|_{s(t)^-} - D_B \left.\frac{\partial \phi}{\partial x}\right|_{s(t)^+} = \big(\phi_B - \phi_A\big)\frac{ds(t)}{dt},\ x = s(t) \qquad (9)$$

Landau transformation of Equations (8) and (9) is similar to Equations (3)-(5) and further discretisation of the problem is discussed in [4]. The iterative algorithm based on de-coupling of the problem was implemented, with only 3 to 4 iterations needed to reach a consistent solution in this case.

It is confirmed that the numerical scheme has conservative properties. The question of whether the solution is accurate remains. For particular initial and boundary conditions, analytical solutions are available [5]. This case is presented in Figure 2 (for geometry where L=1, D=1), the results correspond to calculations completed using both irregular and regular meshes. It is clear that, for a given number of discretisation points, it is possible to find more accurate solutions by using irregular meshes. In this way errors can be reduced without requiring extra computational effort.

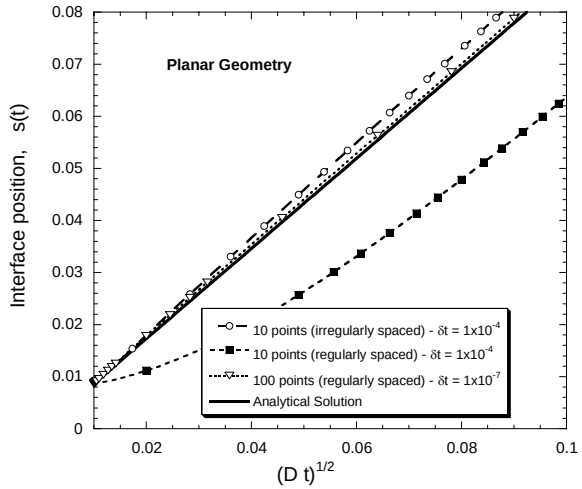

Fig. 2. Predictions for interface position: mesh size and time step effects [4].

V. CONCLUSIONS

The use of a front-fixing method for modelling non-linear coupled thermo-electric processes is demonstrated. Potential problems with implementation of conservation laws and complex boundary conditions are considered and solutions suggested. It is shown that high accuracy can be achieved on a coarse irregular mesh since the interface is fixed in new coordinates. The finite difference method is utilized in the paper as an example; the finite element technique can be also used for successful discretisation of space and time in the transformed equations.

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