Practical aspects of modal propagation through water filled pipes

N.R. Harris a, M. Hill b,*, J.D. Turner b

a Department of Electronics and Computer Science, University of Southampton, Highfield, Southampton, SO17 1BJ, UK
b Department of Mechanical Engineering, University of Southampton, Highfield, Southampton, SO17 1BJ, UK

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Abstract

The use of ultrasound for sensing and communication is growing. Many measurement applications (such as flow metering and level sensing) involve the use of guided acoustic energy, for example constrained within a liquid filled pipe. This paper discusses the problem of axisymmetric sound propagation in pipes, with particular reference to ultrasound, and extends the theory to allow prediction of the amplitudes of propagating modes for a given source, receiver and pipe geometry. An experimental confirmation of the theory has also been obtained and is described.

Keywords: Ultrasound; Propagation; Pipe; Waveguide; Acoustic

1. Introduction

Interest is growing in the use of guided sound waves in several measuring environments. One example of this is for level sensing using ultrasound, and another is that of flowmetering, in particular plane-wave flowmeters.

The propagation of sound waves in ducts and pipes has received much attention over the years, but much confusion is still apparent outside of some specialist areas. Many of the standard texts are too general for the purpose of analysing real systems (Morse [3], Kinsler and Frey [4]), while other authors can be very specialised (Fuller and Fahy [1], Lin [2]).

This paper describes an analytical approach to predicting the distribution of sound in a water-filled pipe, making practical approximations and assumptions. Finally, an experiment is described which confirms the results.

It is hoped that this work will aid a potential user to select the position, frequency and geometry of acoustic transducers for transmitting and receiving signals inside a liquid filled cylindrical pipe.

The following discussion applies for continuous wave transmission and serves to illustrate the effect of the parameters discussed above, without complicating the subject by trying to include transient or pulse effects.

2. Theoretical discussion

The three dimensional acoustic wave equation expressed in cylindrical coordinates is:

$$P^2 p = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial p}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2} = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2},$$

where $p$ is acoustic pressure, $t$ is time, $c$ is the speed of sound and $r$, $z$ and $\theta$ are respectively radial, longitudinal and angular variables.

Defining $p$ as $P(r, \theta, z, t)$, i.e. as a function of $r$, $\theta$, $z$ and $t$, and assuming a variable separable solution, then

$$P(r, \theta, z, t) = R(r) \Theta(\theta) Z(z) T(t),$$

where $R(t)$ denotes a function dependent on $r$, etc.

For the problem being considered here, it is possible to simplify the solution by assuming that a symmetry
of rotation can be applied to the axis of the pipe, so that all sources and receivers are constrained to be on the axis of the cylinder, and be either circular or cylindrical. In this way the variation of pressure with angle can be made constant, and therefore any differentials with respect to θ are zero. This simplifies the problem so that the solution is dependent on two variables rather than three.

The above equation for cylindrical symmetry can be solved to give a general solution

\[ P(r,z,t) = A_m J_0(k_m r) e^{i\omega t - k_z z}, \]  

which can be visualised as a travelling wave in the z direction, and a standing wave function across the radius of the pipe. The function \( J_0(k_m r) \) is a Bessel function of zero order, where \( k_m \) is a constant, and \( k_z \) in the exponential is a propagation constant known as the wave number.

3. Boundary conditions

In order to evaluate Eq. (3) further, it is necessary to evaluate these constants. This is done by applying boundary conditions. For a pipe of radius \( a \), a boundary condition can be applied at \( r = a \), i.e. at the pipe wall. The boundary can take many forms ranging from a rigid wall to a free surface. In this case a rigid wall will be used. In practice such a boundary is difficult to achieve, as a small amount of wall movement is inevitable. An analysis of the effects of non-rigid boundaries has been presented by Lin and Morgan [7], and it is shown that water filled metal pipes give a good approximation to the ideal rigid boundary.

The other extreme is a completely flexible boundary – a free surface. This can be approximated by a thin rubber wall, and has been modelled by a mercury/steel boundary [6].

In the case of a rigid walled pipe, the condition to be applied is that the particle velocity normal to the wall is zero.

\[ \frac{\partial P}{\partial r} = 0, \]  

which gives the boundary conditions as

\[ -J_1(k_m a) = 0. \]  

The zeros of \( J_1 \) are available from tables (e.g. Ref. [4], p. 452) and give \( k_m a = 3.83, 7.02, 10.17, 13.32, ... \). These we will denote by \( j_{1m} \).

Thus there are many solutions to the radial function (modes), and so the overall pressure Eq. (3) has to be modified to incorporate these solutions:

\[ P(r,z,t) = \sum_{m=0} A_m J_0 \left( \frac{j_{1m} r}{a} \right) e^{i\omega t - k_{mz} z}. \]  

Note that for each value of \( j_{1m} \) there is an associated value of \( k_m \), the wave number. This has been recognised by substituting the symbol \( k_{mz} \) for \( k_m \).

4. Cut-off frequency

From consideration of Eq. (6) above, it can be seen that there are two factors that are still unknown, and these are \( A_m \) and \( k_{mz} \). \( A_m \) will be considered in the next section, and \( k_{mz} \) will be considered here.

From the boundary conditions and the separation of variables solution, it can be shown that

\[ k_{mz} = \sqrt{\omega^2/c^2 - k_{mz}^2}. \]

For propagation to occur \( k_{mz} \) must be positive in order to give a real velocity for that mode. As \( k_{mz} \) approaches 0, the propagation vector for that mode approaches 90° to the axis of the pipe, and no propagation occurs down the pipe. Therefore cut-off occurs when \( k_{mz} = 0 \), i.e. when

\[ f_{cm} = \frac{k_{mz} c}{2\pi}, \]

where \( f_{cm} \) is the cut off frequency for each mode.

Thus each mode has its own cut-off frequency (similar to electromagnetic waveguides), below which that mode does not propagate. It should be noted that mode 0 is allowed to propagate at all frequencies, which is a result of practical importance. This mode can be likened to a plane wave travelling straight down the pipe, and is a consequence of the rigid boundary. If the boundary is free, then it can be shown that no plane wave mode is allowed to propagate. It should also be noted that as the frequency is increased, more modes are allowed to propagate. However, modes are only detectable if they are generated by the source in the first place. The modal pattern generated by a source is dictated by the response of the source. This manifests itself in the coefficients \( A_m \) in Eq. (6).

5. Evaluation of the relative constants

In order to predict the acoustic pressure at any point in the pipe it is necessary to be able to calculate the relative amplitudes of the propagating modes. The analysis must be undertaken in two parts:

1. \( m > 0 \).
2. \( m = 0 \).
5.1. Analysis for \( m > 0 \)

For the condition \( m > 0 \), and \( z = 0 \), i.e. at the source, Eq. (6) can be rewritten

\[
P(r,0,t) = \sum_{m=1}^{\infty} A_m J_0 \left( j_{1m} \frac{r}{a} \right) e^{i \omega t}. \tag{9}\]

To find the values of the coefficients \( A_m \) it is necessary to exploit the orthogonality principle. Multiply both sides by

\[
r J_0 \left( j_{1m} \frac{r}{a} \right) \]

and then integrate between 0 and \( a \) (the exponent can be ignored for the present as, being a function of time only, it is independent of \( r \)):

\[
\int_0^a P(r,0,t)r J_0 \left( j_{1m} \frac{r}{a} \right) dr = \sum_{m=1}^{\infty} A_m J_0 \left( j_{1m} \frac{r}{a} \right) J_0 \left( j_{1m} \frac{r}{a} \right) r dr. \tag{10}\]

The right hand side of this equation can now be rearranged to provide solutions by standard form, as in Ref. [5], giving

\[
\int_0^a P(r,0,t)r J_0 \left( j_{1m} \frac{r}{a} \right) dr = \frac{A_m a^2 J_0^2(j_{1m})}{2}. \tag{11}\]

To evaluate this further it is necessary to define a source, i.e. \( P(r,0,t) \) in the above expression. As an example assume a source that generates a constant pressure profile over a radius \( a_1 \), and zero between \( a_1 \) and \( a \), i.e. a piston type source with a diameter less than that of the pipe.

In this case the limits change to 0 and \( a_1 \) in Eq. (12).

Again, utilizing a standard result from Ref. [5], it can be shown that

\[
A_m = \frac{2a_1 J_1(j_{1m} a_1/a)}{j_{1m} a_2 J_0^2(j_{1m})}. \tag{13}\]

5.2. Evaluation of \( A_0 \)

The full general solution established earlier, Eq. (6), can be written

\[
P(r,z,t) = A_0 + \sum_{m=1}^{\infty} A_m J_0 \left( j_{1m} \frac{r}{a} \right) e^{i \omega t}. \tag{14}\]

because the solution for mode 0 is a special case. This can be seen by substituting \( j_{10} \) (= 0) into the solution for \( A_m \). The expression contains a singularity. This can be resolved by reviewing the boundary conditions as given by Eq. (5) and noting that

\[
k_{00} a = 0, \tag{15}\]

which defines the boundary condition as a constant, rather than a function, i.e.

\[
J_0(k_{00} a) \equiv J_0(0) = 1. \tag{16}\]

Therefore

\[
A_0 J_0(j_{10} r/a) = A_0 J_0(0) = A_0 \text{ for all } r. \tag{17}\]

The same method as above can now be followed to give the simple solution for the amplitude of \( A_0 \),

\[
A_0 = a^2 / a^2. \tag{18}\]

6. Finite receiver

The previous solution only allows for a point receiver. Practical receivers have a finite active area, and the electrical output is proportional to the pressure received over the whole area. For the co-ordinate system being used the simplest case is to assume that the active area is a disc on the axis of the pipe.

Given a circular active area of radius \( a_2 \), the total pressure is given by:

\[
P_T = \int_0^{a_2} P(r,z,t) 2\pi r dr. \tag{19}\]

Thus the expression for the total pressure received becomes

\[
P_T = A_0 \pi a_2^2 e^{i \omega t - k_2 z} + \sum_{m=1}^{\infty} \frac{2\pi A_m a_2^2}{j_{1m}} J_1 \left( j_{1m} \frac{a_2}{a} \right) e^{i \omega (z - k_2 z)}. \tag{20}\]

Taking the magnitude of the sum of this expression gives the value of the peak pressure experienced. The terms \( e^{i z} \) at a fixed distance \( z \) represent a fixed phase shift, and when all the sums are added and the resulting magnitude taken, this represents the peak of the harmonic function. This assumes that the receiver has established a steady state response.

7. Effect of frequency on pressure

The importance of the source dimensions with regard to the effect of higher order modes can be seen in Figs. 1 and 2. This represents the pressure detected by a hydrophone located a fixed distance away from the source, as the frequency is increased from 5 to 30 kHz. Fig. 1 is for a source of radius 4.5 cm, and Fig. 2 is for a source of 11 cm, i.e. close to the diameter of the pipe. As the frequency increases above the cut-off frequencies for the various modes, their influence is clearly
visible, with mode 2 in particular introducing some deep nulls, in this configuration. Thus care must be taken in positioning receivers when several modes are present, as signal strength can be severely compromised. However, the effects of higher order modes are much reduced for the larger radius source.

8. Experiment

A length of steel water main was used in the verification of the preceding theory. This pipe was about 7 m long and a commercially available sonar source was fitted at one end (see Fig. 3).

An arrangement whereby a B&K hydrophone could be remotely positioned at a known distance was devised and installed. The hydrophone was mounted on a trolley, which was moved by a brass rod. This rod extended through a gland in the end of the rig, thus allowing the operator to move the trolley without any leaking.

The parameters of the rig are as shown in Table 1, allowing seven modes to propagate at 42 kHz.

A synthesised frequency generator was used in its pulse mode to give a known number of complete cycles, and the received hydrophone signal was fed via a low noise amplifier to an oscilloscope.

9. Experimental method

The signal source was set to 42 kHz, and a known number of cycles fed to the transducer. The actual
Table 1
Parameters used

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Description</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a$</td>
<td>Radius of pipe</td>
<td>0.128 m</td>
</tr>
<tr>
<td>$a_1$</td>
<td>Radius of source</td>
<td>4.5 cm</td>
</tr>
<tr>
<td>$a_2$</td>
<td>Radius of hydrophone</td>
<td>1.8 cm</td>
</tr>
<tr>
<td>$z$</td>
<td>Distance from source</td>
<td>3.15–6.4 m</td>
</tr>
<tr>
<td>$c$</td>
<td>Speed of sound</td>
<td>1490 m/s</td>
</tr>
</tbody>
</table>

Note: Hydrophone cable is free to pass through the plastic pipe without water spillage as the opening (A) of the plastic pipe is above water level.

Fig. 3. Schematic of test rig.

10. Definition of parameters

In order to compare the theoretical expression developed earlier with the experimental results, the values shown in Table 1 were used in the simulation.

The theoretical expression was evaluated numerically using Mathcad software, and for the frequencies of interest, the following propagating modes and associated values of amplitude coefficients $A_m$ for the source were calculated (Table 2).

From this table it can be seen that the amplitudes of higher modes can be extremely significant, and it should

<table>
<thead>
<tr>
<th>Mode</th>
<th>$A_m$</th>
</tr>
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<tbody>
<tr>
<td>0</td>
<td>0.13</td>
</tr>
<tr>
<td>1</td>
<td>0.61</td>
</tr>
<tr>
<td>2</td>
<td>0.56</td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
</tr>
<tr>
<td>4</td>
<td>-0.32</td>
</tr>
<tr>
<td>5</td>
<td>-0.34</td>
</tr>
<tr>
<td>6</td>
<td>-0.02</td>
</tr>
</tbody>
</table>

This data is shown in Fig. 4. The effects of the various modes interfering can be seen in the cyclical nature of the results.

Fig. 4. Comparison of experimental and theoretical data at 42 kHz.
also be noted that mode 0, the plane wave mode, is not dominant for this configuration of source. The plane wave becomes dominant as the radius of the source becomes comparable to the radius of the pipe. Indeed, for a pistonic source, when the source radius equals the radius of the pipe, only the plane wave is generated in a rigid walled pipe.

11. Conclusions

As can be seen from Fig. 4, the experimental data generally follows the theoretical prediction, but the measured depth of the 'nulls' is not as great as the predicted depth. This is likely to be due to the assumptions made about the active area of the hydrophone. It has been assumed that the hydrophone has a circular active area and is mounted concentrically on the axis of symmetry whereas the active area of the hydrophone is cylindrical and non-concentric on the axis of symmetry, thus presenting an elongated area to the source. However, the periodic nature caused by the modes is demonstrated, showing that care is needed when trying to propagate sound down pipes above the cut off frequency of mode 1. If the source and receiver are a fixed distance apart, the strength of the received signal becomes dependant on frequency. Changing the frequency even slightly can move the receiver from a peak to a null. This is likely to be further complicated if flow is present in the pipe. It can also be seen that the dimensions of the source play an important part in determining the amplitudes of the modes, and an important implication is that it should be possible to design a source that only allows certain modes to propagate in a given pipe at a range of frequencies, as the amplitude coefficients $A_m$ are not defined by frequency. Being able to design a source for maximising certain modes at the expense of others has obvious attractions, as it is not always practical for a full radius piston source to be utilised. This is the only kind of source which can give 100% plane wave in a rigid walled guide, but other configurations may be able to emphasis both low and high order modes at the expense of mid order modes. By careful utilisation of the pipe's cut-off frequency a predominantly low order source could be realised which would be beneficial in many measurement applications.

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References