Closed-form approximation of MIMO capacity

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A closed-form expression is provided for the calculation of the minimum SNR required to achieve a target data-rate using a generic MIMO-aided M-QAM transceiver. The computationally efficient technique proposed facilitates the convenient characterisation of MIMO-assisted wireless systems.

Introduction: The capacity of communication systems is routinely utilised in both theoretic [1] as well as applied [2] performance studies. Yet, the computationally-efficient method of calculating the capacity of a generic multiple input multiple output (MIMO)-aided system remains an open problem.

Consider the generic communication system model $\boldsymbol{r} = \boldsymbol{Hs} + \boldsymbol{n}$, where the vectors $\boldsymbol{r}$, $\boldsymbol{s}$ and $\boldsymbol{n}$ denote the received and transmitted signal, as well as the Gaussian noise sample vectors, while the matrix $\boldsymbol{H}$ constitutes the linear transformation describing the impact of the communication channel. As demonstrated in [3], an indication of the maximum throughput achievable by a transceiver employing a particular modulation scheme is constituted by the *mutual information* between the transmitted signal $\boldsymbol{s}$ and the received signal $\boldsymbol{r}$, which may be expressed as follows

$$R = \mathcal{H}(\boldsymbol{r}) - \mathcal{H}(\boldsymbol{r}|\boldsymbol{s}) = \mathcal{H}(\boldsymbol{s}) - \mathcal{H}(\boldsymbol{s}|\boldsymbol{r}) \qquad (1)$$

where $\mathcal{H}(x) = -\mathrm{E}\log p(x)$ denotes the entropy function. A general $(n_t \times n_r)$-element M-QAM MIMO system communicating over a Rayleigh fading channel was considered in [2], where it was shown that

$$\mathcal{H}(\boldsymbol{r}|\boldsymbol{s}) = n_r \log_2(\pi e/\gamma_s) \qquad (2)$$

while

$$\mathcal{H}(\boldsymbol{r}) = -\mathrm{E}\log_2\left(\frac{\gamma_s^{n_r}}{M^{n_t}\pi^{n_r}}\sum_{\check{\boldsymbol{s}}} e^{-\gamma_s\|\boldsymbol{r}-\boldsymbol{H}\check{\boldsymbol{s}}\|^2}\right) \qquad (3)$$

where γ_s denotes the average SNR. The expression in (3) is typically computed numerically, which involves a Monte-Carlo-based averaging over the three sources of randomness in the choice of $\boldsymbol{s}$, $\boldsymbol{H}$ and $\boldsymbol{n}$ as well as the summation over 2^{Mn_t} possible M-QAM constellation values $\check{\boldsymbol{s}}$ for each combination of $\boldsymbol{s}$, $\boldsymbol{H}$ and $\boldsymbol{n}$. It is evident, however, that the computational complexity associated with this method of computation becomes excessive for high-order M-QAM schemes as well as for a high number of transmit antennas. Against this background, the novel contribution of this Letter is that we derive computationally efficient approximate expressions for the characterisation of practical MIMO-aided systems.

Mutual information of MIMO-aided M-QAM systems: First, let us elaborate on (3) as follows

$$\mathcal{H}(\boldsymbol{r}) = n_t \log_2 M - n_r \log_2(\pi/\gamma_s) - \mathrm{E}\log_2 \sum_{\check{\boldsymbol{s}}} e^{-\gamma_s\|\boldsymbol{r}-\boldsymbol{H}\check{\boldsymbol{s}}\|^2} \qquad (4)$$

Substituting (2) and (4) into (1) yields

$$R = n_t \log M - \mathrm{E}\log_2\left(e^{n_r}\sum_{\check{\boldsymbol{s}}} e^{-\gamma_s\|\boldsymbol{r}-\boldsymbol{H}\check{\boldsymbol{s}}\|^2}\right) \qquad (5)$$

where the first term of (5) may be identified as the entropy $\mathcal{H}(\boldsymbol{s})$ of the transmitted signal $\boldsymbol{s}$. Consequently, from (1) we may conclude that the second term of (5) constitutes the *ambiguity*, or loss of information $\mathcal{H}(\boldsymbol{s}|\boldsymbol{r})$ introduced by the transmission process. Let us therefore rewrite (5) in the following form

$$R = n_t \log_2 M[1 - \Phi(\gamma_s)] \qquad (6)$$

where we defined the ambiguity-related quantity $\Phi(\gamma_s)$ as

$$\Phi(\gamma_s) = \frac{1}{n_t \log_2 M}\mathrm{E}\log_2 \sum_{\check{\boldsymbol{s}}} e^{-\gamma_s(\|\boldsymbol{r}-\boldsymbol{H}\check{\boldsymbol{s}}\|^2 - \|w\|^2)} \qquad (7)$$

In practice the effective throughput R is typically achieved by invoking channel coding. Specifically, let us assume a rate-r_c coded MIMO-aided M-QAM system exhibiting an effective throughput of $R = n_t \log_2 M r_c$. Correspondingly, we may express the maximum coding rate reliably attainable at the SNR of γ_s as $r_c = 1 - \Phi(\gamma_s)$.

It may be empirically demonstrated that the quantity $\Phi(\gamma_s)$ can be accurately approximated using a parametric function of the form

$$\Phi(\gamma_s) \simeq e^{a - b\sqrt{c+\gamma_s}} \qquad (8)$$

where the fitting parameters a, b and c may be calculated separately for every $(n_t \times n_r)$-element M-QAM scenario. For instance, the capacity of the baseline (1×1)-element QPSK Gaussian as well as Rayleigh channel scenarios may be approximated using the parameters of $\{a, b, c\} = \{10.49, 3.78, 7.7\}$ and $\{0.33, 0.79, 0.16\}$, respectively, resulting in an SNR error of less than 0.2 dB. In our experiments, we have found that the accuracy of 0.2 dB may be attained for any $(n_t \times n_r)$-element M-QAM scenario.

In this Letter, however, we discuss a more generic solution, which does not require the coefficient fitting of (8) for each individual $\{n_t, n_r, M\}$ scenario. More specifically, using empirical evidence we demonstrate that an approximate expression describing the minimum SNR required to achieve a specific throughput R using a generic MIMO-aided M-QAM system may be derived using an approximation of the baseline SISO QPSK scenario and a simple SNR-scaling factor.

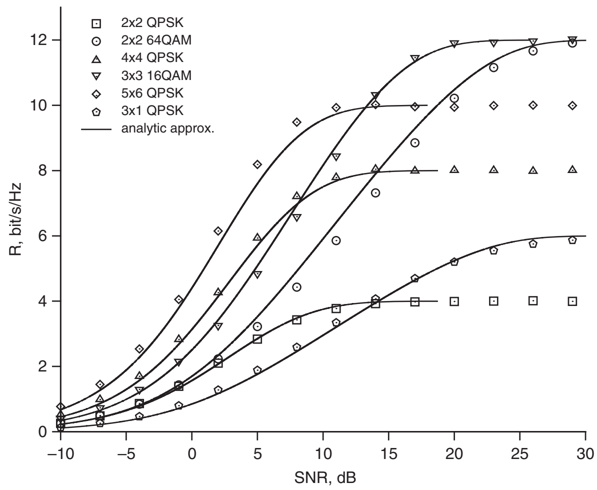

Fig. 1 *Comparison between approximate expression of (13) describing mutual information attainable by M-QAM MIMO system communicating over Rayleigh channel and corresponding exact values obtained using Monte-Carlo-based evaluation of (3).*

Let us recall the fundamental capacity upper-bound of a communication system quantified by the Shannon-Hartley theorem [3] of $R = B\,\log_2(1 + \gamma_s)$, where B denotes the number of complex-valued degrees of freedom per channel use available for communication. In systems exhibiting time, frequency or spatial spreading [4], Shannon-Hartley theorem becomes

$$R = \frac{B}{K}\log_2(1 + K\gamma_s) \qquad (9)$$

where K denotes the overall spreading factor. Inverting (9) yields $\gamma_s = (2^{R/B} - 1)/K$. Subsequently, we conjecture that the minimum SNR values γ_s and γ_s' required to achieve reliable communications at the specific throughputs of R and R', while invoking a coding scheme of rate r_c, approximately obey

$$\frac{\gamma_s}{\gamma_s'} = \frac{K'(2^{R/B} - 1)}{K(2^{R'/B'} - 1)} \qquad (10)$$

Let us now consider the following simple approximate expression in the form of (8) corresponding to the baseline SISO QPSK scenario

$$\Phi \simeq e^{1.2(1-\sqrt{1+\gamma_s'})} \qquad (11)$$

The inverse function of (11) yields the SNR required

$$\gamma_s' \simeq [1 - 0.83\log\Phi]^2 - 1 \qquad (12)$$

Combining the approximate SNR of the baseline scenario described by (12) and the scaling factor of (10) we may devise an approximate expression quantifying the minimum SNR required for reliable communications using an $(n_t \times n_r)$-element M-QAM system characterised by the effective throughput of $R = n_t\,\log_2 M r_c$, the number of complex-valued degrees of freedom $B = \min(n_t, n_r)$ as well as the spatial spreading factor of $K = n_r/B$, yielding

$$\gamma_s \simeq \frac{\min(n_t, n_r)(2^{n_t \log_2 M \cdot r_c/\min(n_t,n_r)} - 1)}{n_r(2^{2r_c} - 1)} \times ([1 - 0.83 \log(1 - r_c)]^2 - 1) \quad (13)$$

The comparison between the approximate expression of (13) and the corresponding values obtained using Monte-Carlo-based evaluation of (3) are illustrated in Fig. 1. In our experiments the approximate model of (13) exhibits an accuracy of 0.8 dB for QPSK MIMO systems as well as 1.2 dB in the case of high-order 64-QAM MIMO systems

Conclusion: We have derived computationally efficient approximate expressions for large-scale physical- as well as network-layer communication system analysis.

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References

1 Wiechman, G., and Sason, I.: 'An improved sphere-packing bound for finite-length codes over symmetric memoryless channels', *IEEE Trans. Inf. Theory*, 2008, **54**, (5), pp. 1962–1990

2 Hochwald, B., and ten Brink, S.: 'Achieving near-capacity on a multiple-antenna channel', *IEEE Trans. Commun.*, 2003, **51**, (3), pp. 389–399

3 Shannon, C.E.: 'A mathematical theory of communication', *Bell Syst. Tech. J.*, June and October 1948, **27**, pp. 379–423 and 623–656 [Online]. Available: http://plan9.bell-labs.com/cm/ms/what/shannonday/shannon1948.pdf

4 Veeravalli, V.V., and Mantravadi, A.: 'The coding-spreading tradeoff in CDMA systems', *IEEE J. Sel. Areas Commun.*, 2002, **20**, (2), pp. 396–408