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Calculation of inducted currents using edge elements and T–T_0 formulation

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Abstract: Methods of calculating induced currents in multiply connected regions containing solid conductors are discussed. In particular, the formulation based on edge elements using the electric vector potential has been considered. The equations are explained using the language of circuit theory. It is observed that the edge values of $\boldsymbol{T}_0$ represent the loop currents in the loops surrounding the 'holes'. It is also shown that the iterative solution may be accelerated by over-specifying the number of loop currents in the loops around the 'holes'. The description of sources is derived for formulations expressed in terms of electric potentials $\boldsymbol{T}$ and $\boldsymbol{T}_0$. Selected results of induced current calculations are discussed and comparisons made.

1 Introduction

Many devices operate by utilising conduction currents created by electromotive forces, known as induced currents. Systems using such currents may be categorised as: (a) simply connected regions with solid conductors, for example the solid part of a magnetic core, (b) multiply connected regions with thin (filament) conductors, for example windings composed of stranded conductors and (c) multiply connected regions with solid conductors [1, 2], for example a solid core with holes, or windings composed of bars such as in a cage rotor of an induction motor. In (a), we are dealing with eddy currents. In (b), we have both induced and externally enforced currents. Currents imposed by external sources may also exist in the category (c) systems, whereas the induced currents are the eddy currents and/or currents circulating around the 'holes', for example currents i_{bi} in the system of Fig. 1. Thus, (a) and (b) may be treated as special cases of (c) and in this paper, we focus on the most general formulation by considering system (c). The application of a vector potential has been explored with the aim of deriving current distributions under the condition of a known distribution and time variation of the magnetic flux density.

2 Circuit representation of the T–T_0 method

The analysis of conduction current distributions in electrical devices may be undertaken using an electric scalar or electric vector potential. Moreover, it is convenient to express the finite element (FE) formulation involving these potentials using the circuit theory analogy [3]. In particular, the FE equations for the scalar potential and nodal elements represent the nodal equations of a conductance (electric edge) network with branches associated with element edges – Fig. 2*a*.

In contrast, the formulation involving the vector potential and edge elements is equivalent to a loop analysis of a resistance network (RN). The nodes of such a network are associated with the volume centres, whereas the branches pass through the facets (Fig. 2*b*); it may therefore be called an electric facet network. The equations for the edge elements of the classical $\boldsymbol{T}$ formulation are equivalent to loop equations for loop currents around element edges [3, 4], designated as i_{mi} in Fig. 1. In the case of a machine winding with multiply connected conductors, the fundamental circuits of such a network cannot be formed with these meshes only. In other words, the classical $\boldsymbol{T}$

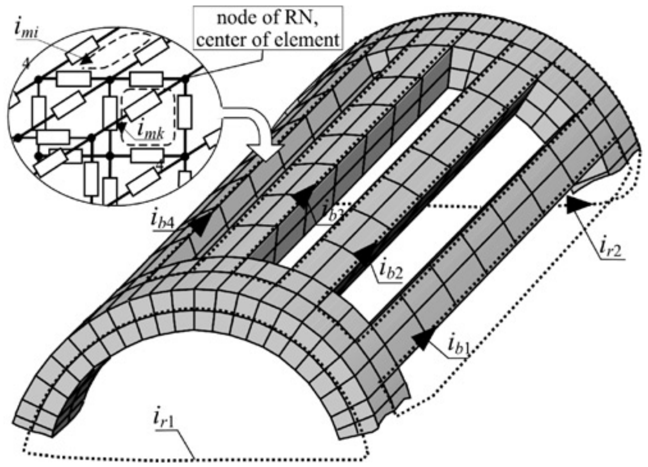


Figure 1 *Squirrel cage of an induction motor as a multiply connected conductor and its resistance model*

formulation is not capable of treating such machine windings, a typical example being the common squirrel cage of an induction motor. A common practice, when using the vector potential $\boldsymbol{T}$ to calculate currents in multiply connected regions, is to assume that the region is simply connected and the non-conducting 'holes' are filled with a conducting material with a very high value of resistivity. Such an approximation often leads to inaccurate results and slow convergence because of ill conditioning of the coefficient matrix. To rectify these shortcomings, a different approach is proposed in this paper where additional loops L_0, representing meshes with currents surrounding the 'holes', are introduced, for example loops with currents i_{bi} and i_{ri} in Fig. 1. These loop currents may be considered to be the edge values of a vector potential $\boldsymbol{T}_0$ [4, 5]. Thus, the loop equations for loop currents around edges and around 'holes' in the RN represent the edge element equations for the $\boldsymbol{T}-\boldsymbol{T}_0$ method. Finally, it is worth pointing out that the proposed approach may be applied to other methods, such as the finite cell method (FCM) [6]. The additional loops in the FCM are then formed from the branches associated with a particular 'cell'.

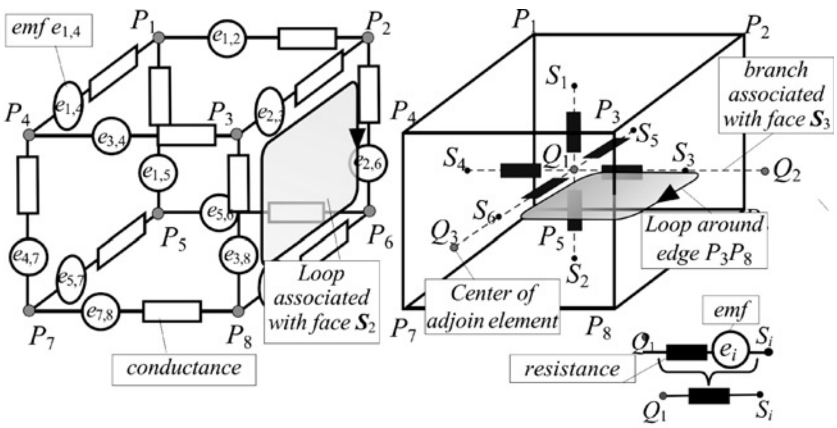
3 Loops for currents representing edge values of $\boldsymbol{T_0}$

The loops with currents representing the edge values of $\boldsymbol{T}_0$ must be chosen carefully to complement the missing loops in the fundamental set. To achieve this, a method put forward in [4] could be used, where a matrix $\boldsymbol{z}_e$ is formed of 'cuts' between the loop surface and the element edges (surface-edge or S-E method) – see, Fig. 3, or a method where a loop is defined by a matrix $\boldsymbol{z}_f$ of 'cuts' of the line L_0 with the element facets (line-facet or L-F method)–see, Fig. 4. Details of how matrices $\boldsymbol{z}_f$ and $\boldsymbol{z}_e$ are formed may be found in [7].

The S-E approach is more universal as it applies to both magnetic scalar Ω and magnetic vector potential $\boldsymbol{A}$ formulations, whereas the L-F method is restricted to cases where $\boldsymbol{A}$ is used as the solution potential. The analysis of this paper is focused on systems with known (enforced) flux distributions. For formulations employing the matrix $\boldsymbol{z}_e$, the equation describing induced currents may be written as

$$\begin{bmatrix} \boldsymbol{k}_e^{\mathrm{T}}\boldsymbol{R}\boldsymbol{k}_e & \boldsymbol{k}_e^{\mathrm{T}}\boldsymbol{R}\boldsymbol{k}_e\boldsymbol{z}_e \\ \boldsymbol{z}_e^{\mathrm{T}}\boldsymbol{k}_e^{\mathrm{T}}\boldsymbol{R}\boldsymbol{k}_e & \boldsymbol{z}_e^{\mathrm{T}}\boldsymbol{k}_e^{\mathrm{T}}\boldsymbol{R}\boldsymbol{k}_e\boldsymbol{z}_e \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{i}_m \\ \boldsymbol{i}_c \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_e \\ \boldsymbol{e}_{ce} \end{bmatrix} \tag{1}$$

where $\boldsymbol{R}$ is the matrix of branch resistances calculated using the interpolation functions of a facet element and $\boldsymbol{k}_e$ the transposed loop matrix for the RN, whereas $\boldsymbol{i}_m$ and $\boldsymbol{i}_c$ are the vectors of loop currents in the loops around edges and holes (edge values of $\boldsymbol{T}$, $\boldsymbol{T}_0$), respectively. The right-hand side of (1) represents sources for the RN, that is, electromotive forces $\boldsymbol{e}_e$ in meshes around edges with eddy currents, and electromotive forces $\boldsymbol{e}_{ce}$ in meshes around the 'holes'. The sources may be defined in terms of: (a) branch



a *b*

Figure 2 *Nodal and edge models of a hexahedron*

a Nodal model
b Edge model

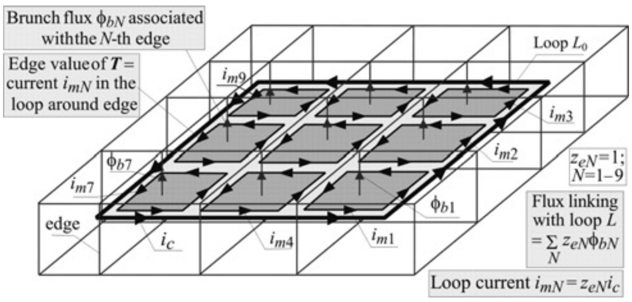


Figure 3 *Loop L_0 with current in space of an edge magnetic network [4]*

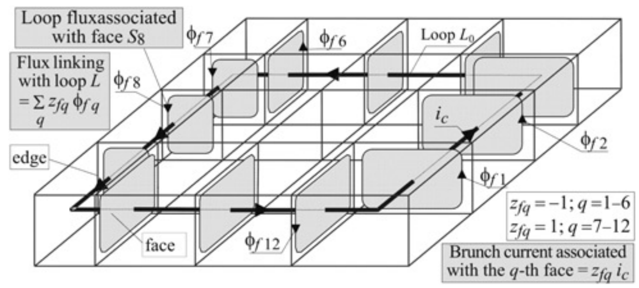


Figure 4 *Loop L_0 with current in space of a facet magnetic network [4]*

fluxes ϕ_s passing through facets, that is loops of the edge network (Fig. 5*a*), (b) loop fluxes ϕ_e around element edges, that is branches of the edge network (Fig. 5*b*) or (c) branch fluxes ϕ_b associated with the edges (Fig. 5*c*). For the edge P_2P_6, the flux $\phi_{b_{P_2P_6}}$ may be found from

$$\phi_{b_{P_2P_6}} = \iiint_{V_e} \boldsymbol{w}_{k_{P2P6}} \boldsymbol{B}\, \mathrm{d}V \tag{2}$$

where $\boldsymbol{w}_{k_{P_2P_6}}$ is the matrix of interpolating functions of the edge element, $\boldsymbol{B}$ the flux density and V_e the element volume.

In the case (a) of an enforced distribution of branch fluxes ϕ_s, the emfs $\boldsymbol{e}_e$ and $\boldsymbol{e}_{ce}$ are described as

$$\boldsymbol{e}_e = -\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{K}^{\mathrm{T}}\boldsymbol{\phi}_s \tag{3}$$

$$\boldsymbol{e}_{ce} = \boldsymbol{z}_e^{\mathrm{T}}\boldsymbol{e}_e = -\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{Z}_e^{\mathrm{T}}\boldsymbol{K}^{\mathrm{T}}\boldsymbol{\phi}_s \tag{4}$$

where $\boldsymbol{K}$ is a matrix transposing the branch values of the facet network into the values related to branches of the edge network [4].

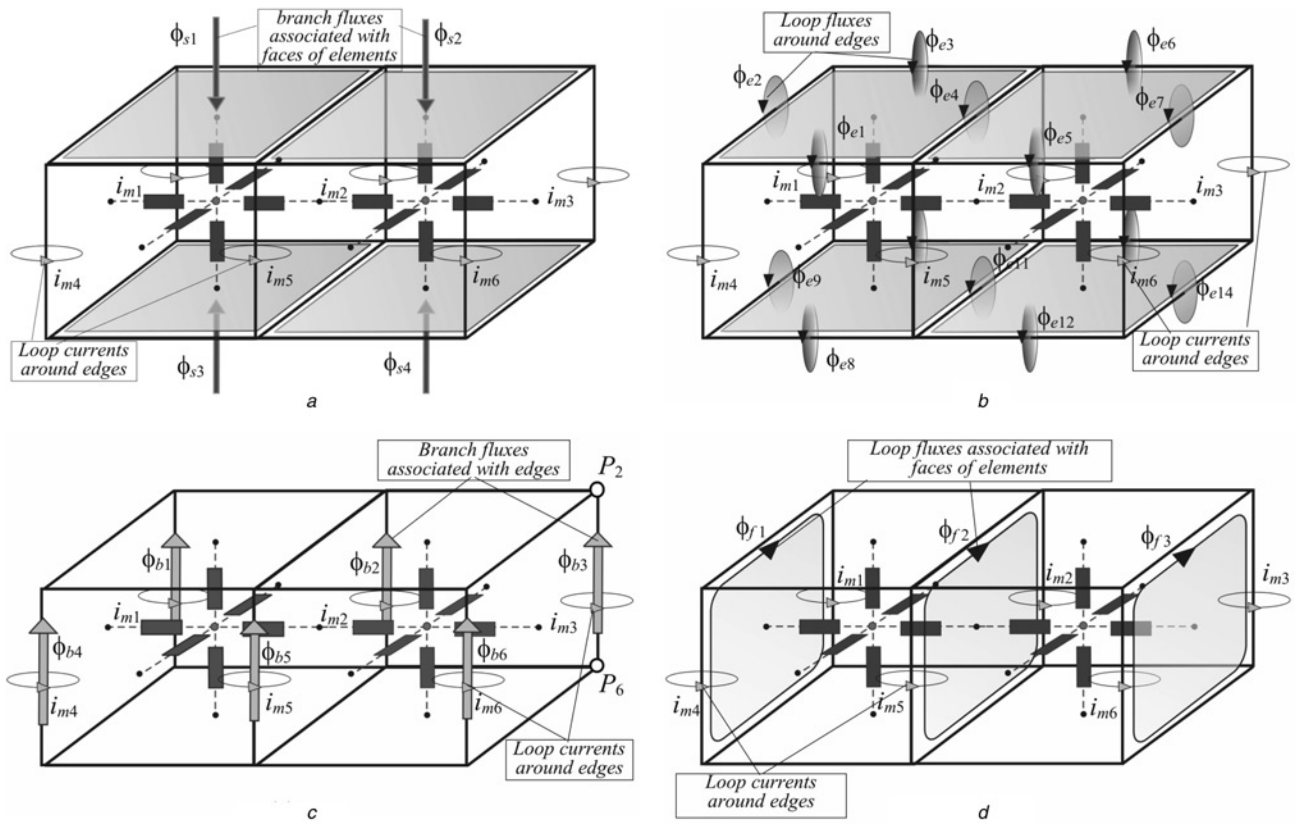


Figure 5 *Loop currents i_m around edges of elements*

a Branch fluxes ϕ_s associated with facets of elements
b Loop fluxes ϕ_e around edges of elements
c Branch fluxes ϕ_b associated with edges of elements
d Loop fluxes ϕ_f associated with facets of elements

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The emfs for the case (b) may be established by taking into account the relationship between the facet values of flux density $\boldsymbol{B}$ and the edge value of the vector potential $\boldsymbol{A}$, as given by $\boldsymbol{\phi}_s = \boldsymbol{k}_e\boldsymbol{\phi}_e$. This leads to

$$\boldsymbol{e}_e = -\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{K}^{\mathrm{T}}\boldsymbol{k}_e\boldsymbol{\phi}_e \tag{5}$$

$$\boldsymbol{e}_{ce} = \boldsymbol{z}_e^{\mathrm{T}}\boldsymbol{e}_e = -\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{z}_e^{\mathrm{T}}\boldsymbol{K}^{\mathrm{T}}\boldsymbol{k}_e\boldsymbol{\phi}_e \tag{6}$$

Finally, for the case (c), the flux associated with the edges passes through the loops of the RN and, hence, its derivative yields the loop emf. Thus, for the systems with an enforced $\boldsymbol{\phi}_b$, the electromotive forces are given by

$$\boldsymbol{e}_e = -\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\phi}_b \tag{7}$$

$$\boldsymbol{e}_{ce} = \boldsymbol{z}_e^{\mathrm{T}}\boldsymbol{e}_e = -\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{z}_e^{\mathrm{T}}\boldsymbol{\phi}_b \tag{8}$$

It was already noted that in order to describe the loop L_0, the L-F formulation could be used via the matrix $\boldsymbol{z}_f$. In this case, the equation describing the distribution of induced currents takes the form

$$\begin{bmatrix} \boldsymbol{k}_e^{\mathrm{T}}\boldsymbol{R}\boldsymbol{k}_e & \boldsymbol{k}_e^{\mathrm{T}}\boldsymbol{R}\boldsymbol{z}_f \\ \boldsymbol{z}_f^{\mathrm{T}}\boldsymbol{R}\boldsymbol{k}_e & \boldsymbol{z}_f^{\mathrm{T}}\boldsymbol{R}\boldsymbol{z}_f \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{i}_m \\ \boldsymbol{i}_c \end{bmatrix} = \begin{bmatrix} -\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{k}_e^{\mathrm{T}}\boldsymbol{\phi}_f \\ -\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{z}_e^{\mathrm{T}}\boldsymbol{\phi}_f \end{bmatrix} \tag{9}$$

In (9), the sources are formulated on the basis of the given distribution of the loop fluxes $\boldsymbol{\phi}_f$ associated with element facets, that is, loops of the edge network (Fig. 5*d*). The expression $-(\mathrm{d}/\mathrm{d}t)\boldsymbol{k}_e^{\mathrm{T}}\boldsymbol{\phi}_f$ describes the emfs in the loops around the edges with eddy currents, whereas the vector $-(\mathrm{d}/\mathrm{d}t)\boldsymbol{z}_e^{\mathrm{T}}\boldsymbol{\phi}_f$ represents electromotive forces in the loops around the 'holes'.

The magnetic flux distribution in a device or a system is normally not known. The external (impressed) flux will have an additional flux imposed on it resulting from unknown induced currents. As a consequence, the algorithms derived here for calculating induced current distributions must be supplemented by the procedures of solving the magnetic field equations.

4 Procedures for solving the $\boldsymbol{T}-\boldsymbol{T}_0$ equations

Consider iterative methods for solving (1) subject to sinusoidal time variation of a magnetic flux passing through a thin plate with two holes as depicted by Fig. 6. The position and the number of additional loops were varied to examine their influence on the convergence of the iterative scheme. A comparison was made of the solution error after a prescribed number of iterations for the SOR, ICCG and 'block ICCG' algorithms. In the case of the block ICCG algorithm, the loop equations for the $\boldsymbol{T}$ formulation were solved using a conventional ICCG algorithm. The results were substituted to the equations for the $\boldsymbol{T}_0$ potentials, which were then solved separately. The resulting edge values of $\boldsymbol{T}_0$ were used in the next step of the $\boldsymbol{T}$ calculations, and so on. The error was estimated as

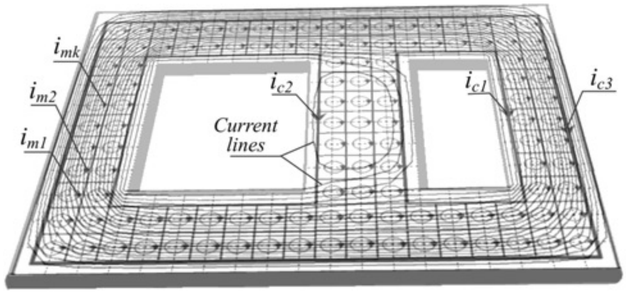

Figure 6 *Example of a conducting plate with two holes*

$$\delta = \frac{\sum_{i=1}^{N}\left|i_{mk,i} - i_{ma,i}\right|}{\sum_{i=1}^{N}\left|i_{ma,i}\right|} \tag{10}$$

where $i_{mk,i}$ is the calculated current in the ith branch after the kth iteration step, $i_{ma,i}$ the exact value of the current in the ith branch and N the total number of branches ($N = 292$ in the example of Fig. 6).

Figs. 7–9 summarise the results showing the errors as iterations progress for the three chosen iterative schemes and the four selected cases of additional loops. The fastest convergence was accomplished using the ICCG with three loops around holes (Fig. 6), where after only 16 iterations the solution was reached with accuracy to the level of rounding errors.

Finally, the results from the $\boldsymbol{T}-\boldsymbol{T}_0$ method were also compared against computations using the $\boldsymbol{T}$ formulation. In the $\boldsymbol{T}$ method, the plate was represented as a singly connected region with the non-conducting 'holes' filled with a conducting material of very low conductivity. The induced currents were calculated using both the SOR and

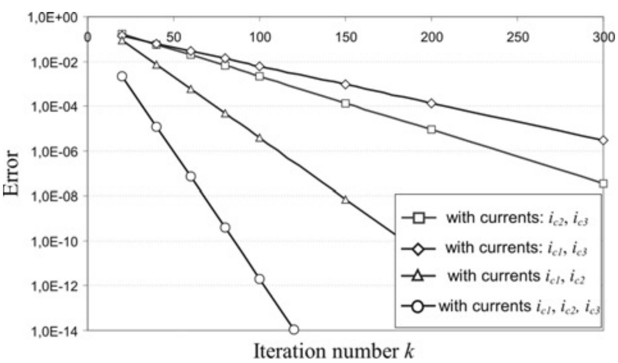

Figure 7 *Errors in current distribution after k iterations for SOR method*

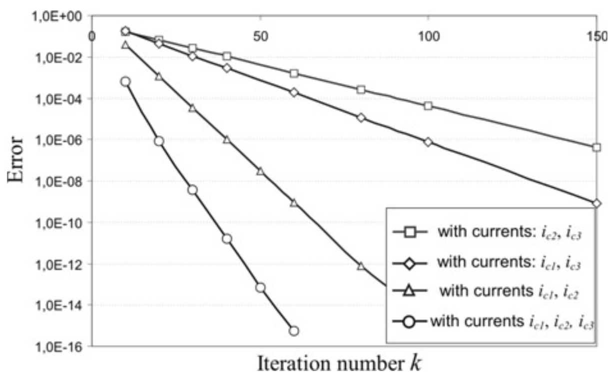

Figure 8 *Errors in current distribution after k iterations for 'block ICCG' method*

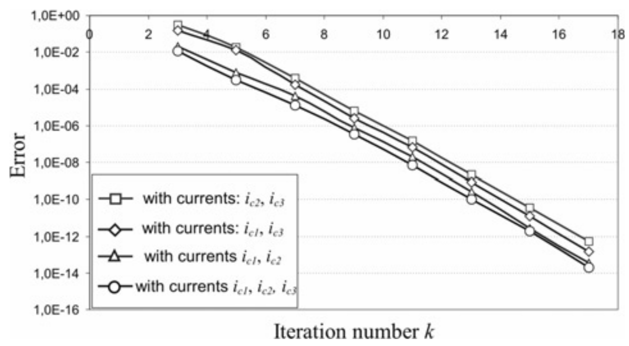

Figure 9 *Errors in current distribution after k iterations for ICCG method*

ICCG algorithms, whereas the error δ was estimated using expression (10). Calculations were done for various values of the ratio χ of plate conductivity σ_{con} to conductivity σ_{air} of the material filling the 'hole', that is, $\chi = \sigma_{\text{con}}/\sigma_{\text{air}}$. Figs. 10 and 11 provide a comparison of errors during iterations.

As may be seen, the convergence of both iterative schemes is significantly slower compared with the $\boldsymbol{T}$–$\boldsymbol{T}_0$ method (Fig. 9). It is also clear that convergence improves as χ becomes smaller, but unfortunately at the expense of worsened accuracy, as the smaller the value of χ, the further we depart with the conductivity of the 'artificial' filling material from the true conductivity of air, $\sigma_{\text{air}} = 0$. For example, for $\chi = 100$, the error in calculating the branch currents still exceeds 4%. The $\boldsymbol{T}$ method with the non-zero conductivity of the 'hole' is less accurate with slower convergence.

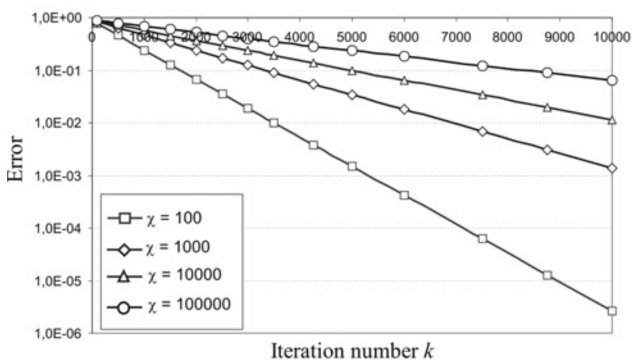

Figure 10 *Errors in current distribution after k iterations for SOR method*

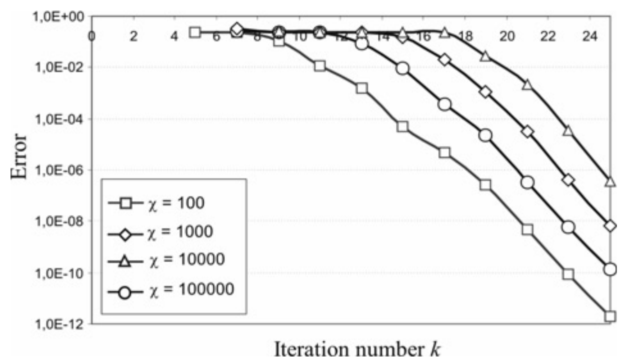

Figure 11 *Errors in current distribution after k iterations for ICCG method*

5 Conclusion

The use of the $\boldsymbol{T}$–$\boldsymbol{T}_0$ formulation has been explored in calculating induced currents in systems containing multiply connected conductors. It has been concluded that the use of the circuit representation of the FE method is particularly helpful when designing computational algorithms. It has also been shown that – when iteratively solving equations of the $\boldsymbol{T}$–$\boldsymbol{T}_0$ formulation, that is, equations describing current distributions in systems containing holes – it is beneficial to introduce superfluous loops. It emerges that by using iterative methods, it is possible to solve a system of equations described by a singular matrix of coefficients, in other words to obtain one solution of a multi-parameter set. After applying an appropriate iterative scheme (e.g. SOR and ICCG), one of the solutions is found. It has been established that the process of finding one of available solutions iteratively converges faster than an algorithm seeking one unique solution of a system without dependent loops.

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