Closed-loop data-driven simulation

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Abstract

Closed-loop data-driven simulation refers to the problem of finding the set of all responses of a closed-loop system to a given reference signal directly from an input/output trajectory of the plant and a representation of the controller. Conditions under which the problem has a solution are given and an algorithm for computing the solution is presented. The problem formulation and its solution are in the spirit of the deterministic subspace identification algorithms, *i.e.*, in the theoretical analysis of the method, the data is assumed exact (noise free). The results have applications in data-driven control, *e.g.*, testing controller's performance directly from closed-loop data of the plant in feedback with possibly different controller.

Keywords: System identification; Subspace methods; Persistency of excitation; Data-driven simulation and control.

1 Introduction

The data-driven simulation problem is defined as follows: given a trajectory of an unknown system, find the response of that system to a different input signal (under specified initial conditions). Of course, data-driven simulation can be reduced to classical model-based simulation by *identifying* the unknown system from the given trajectory. In the case of linear time-invariant (LTI) systems, however, the problem can be solved *without identifying* a representation of the system in an intermediate step [MWRM05a, MWRM05b]. The resulting algorithm involves solution of a linear system of equations in which the left hand side is a Hankel matrix constructed from the given trajectory and the right hand side is the desired trajectory. Despite the computational simplicity of the basic idea, the theoretical analysis and computational details are not trivial due to the need to ensure that the given data is sufficiently informative, the need to organize the computations recursively and to deal with the initial conditions. Indeed, a data-driven simulation problem may not be solvable. The main assumptions for solvability are controllability of the system and persistency of excitation of the input that has generated the given trajectory.

The concept of data-driven simulation and the resulting computational algorithms have applications in system identification, *e.g.*, computation of the impulse response from input/output data is a special data-driven simulation

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problem. Another application of data-driven simulation [MWVD06] is to give a system theoretic interpretation of the orthogonal and oblique projections, which appear in the subspace identification methods [VD96, VD92]. Finally, as shown in [MR07, MR08], data-driven simulation is the main building block of data-driven control methods, *i.e.*, methods that derive the control signal directly from a trajectory of the plant without identifying a model of the plant.

This paper further develops the concept of data-driven simulation to closed-loop data-driven simulation, defined as follows: given an input/output trajectory of an unknown system and a representation of a controller, find all responses of that closed-loop system to a given reference signal. Our motivation for studying the closed-loop data-driven simulation problem comes from unfalsified control [ST97]. Unfalsified control is an switching adaptive control method that selects in real-time a controller satisfying the performance specification from a set of candidate controllers. The main step in unfalsified control is testing the performance of a candidate controller without applying it on the plant. The performance of the candidate controller is evaluated *directly* from data collected of the plant (possibly operating in closed-loop with another controller). Data-driven simulation allows us to evaluate the controller performance by computing the closed-loop behavior of the plant with the given controller. The standard performance test in the unfalsified control setting makes no assumptions about the plant (therefore it is applicable for a general nonlinear time-varying system), however, it computes a single trajectory of the closed-loop system, so that the performance test can be conservative. In contrast, closed-loop data-driven control uses an LTI assumption about the plant but computes the full behaviors of the closed-loop system, so that it is non-conservative in the LTI case.

A standing assumption throughout the paper is that the data is generated by an LTI system of bounded complexity. Admittedly, this assumption is practically unrealistic, however, it is convenient for the theoretical study (*cf.*, deterministic subspace identification) and trivial modifications of the algorithm—replace solution of a linear system of equation by an approximate solution, rank test by a numerical rank test, *etc*.—leads to practically useful algorithms that can cope with noise on the data (*cf.*, stochastic subspace identification). We envisage that stochastic version of the results presented in this paper will appear in near future.

Notation

We use the following standard notation: $\mathbb{R}$ is the set of real numbers, $\mathbb{N}$ is the set of natural numbers, and $\mathbb{R}^{\mathtt{w}}$ is the $\mathtt{w}$-dimensional real vector space. $(\mathbb{R}^{\mathtt{w}})^{\mathbb{N}}$ denotes the set of functions from $\mathbb{N}$ to $\mathbb{R}^{\mathtt{w}}$, *i.e.*, $w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{N}}$ is the time series

$$w = \big(w(1), w(2), \dots, w(t), \dots\big), \qquad \text{where} \quad w(t) \in \mathbb{R}^{\mathtt{w}}.$$

$w \in (\mathbb{R}^{\mathtt{w}})^T$ is the finite sequence

$$w = \big(w(1), w(2), \dots, w(t), \dots, w(T)\big), \qquad \text{where} \quad w(t) \in \mathbb{R}^{\mathtt{w}},$$

however, with some abuse of notation, we will view $w \in (\mathbb{R}^{\mathtt{w}})^T$ also as a $\mathtt{w}T$-dimensional vector. The concatenation of the finite sequence w_{p} with the (possibly infinite) sequence w_{f} is denoted by $(w_{\mathrm{p}}, w_{\mathrm{f}})$. $A^{\dagger}$ is the pseudo-inverse of the matrix A and $\operatorname{col}\dim(A)$ is the number of columns of A.

The behavioural setting [Wil87] is especially suitable for solution of data-driven simulation and control problems because it treats a dynamical system as a set of trajectories (rather than equations) thus making explicit the relation between a trajectory and the system that generates the trajectory. In the behavioural setting, a discrete-time dynamical system $\mathscr{B}$ with $\mathtt{w}$ manifest variables (inputs and outputs) is a subset of the signal space $(\mathbb{R}^{\mathtt{w}})^{\mathbb{N}}$. In this paper, we assume that the manifest variables w have a *given* input/output partition

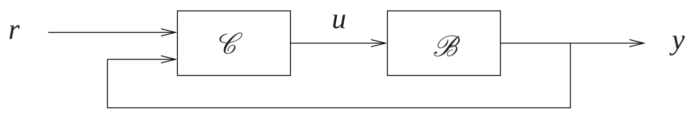
$$w = \begin{bmatrix} u \\ y \end{bmatrix},$$

where $u \in (\mathbb{R}^{\mathtt{m}})^{\mathbb{N}}$ is an input and $y \in (\mathbb{R}^{\mathtt{p}})^{\mathbb{N}}$ is an output. The *restriction* $\mathscr{B}|_T$ of the behavior $\mathscr{B}$ to the interval $[1,T]$ is defined as

$$\mathscr{B}|_T := \{\, w_{\mathrm{p}} \in (\mathbb{R}^{\mathtt{w}})^T \mid \text{there is } w_{\mathrm{f}} \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{N}} \text{ such that } (w_{\mathrm{p}}, w_{\mathrm{f}}) \in \mathscr{B} \,\},$$

i.e., there is an extension w_{f} of a finite trajectory $f_{\mathrm{p}} \in \mathscr{B}|_T$ of the system, to an infinite trajectory $w = (w_{\mathrm{p}}, w_{\mathrm{f}}) \in \mathscr{B}$.

The feedback interconnection of the plant $\mathscr{B} \subseteq (\mathbb{R}^{\mathtt{w}})^{\mathbb{N}}$ and a controller $\mathscr{C} \subseteq (\mathbb{R}^{\mathtt{r}+\mathtt{w}})^{\mathbb{N}}$



is given by

$$\mathscr{B}_{\mathscr{C}} = \mathscr{B}_{\mathrm{ext}} \cap \mathscr{C},$$

where

$$\mathscr{B}_{\mathrm{ext}} := \{ \begin{bmatrix} r \\ w \end{bmatrix} \in (\mathbb{R}^{\mathtt{r}+\mathtt{w}})^{\mathbb{N}} \mid w \in \mathscr{B} \}.$$

We consider linear, time-invariant, and finite dimensional plants and controllers. A kernel representation $R(\sigma)w = 0$, were σ is the backwards shift operator

$$\sigma w(t) := w(t+1),$$

is parameterized by the polynomial matrix R, and an image representation $w = M(\sigma)g$ is parameterized by the polynomial matrix M.

The Hankel matrix with t block rows, composed of the finite signal $w \in (\mathbb{R}^{\mathtt{w}})^T$ is denoted by

$$\mathscr{H}_t(w) := \begin{bmatrix} w(1) & w(2) & \cdots & w(T-t+1) \\ w(2) & w(3) & \cdots & w(T-t+2) \\ w(3) & w(4) & \cdots & w(T-t+3) \\ \vdots & \vdots & & \vdots \\ w(t) & w(t+1) & \cdots & w(T) \end{bmatrix}. \tag{1}$$

The signal $u = \big(u(1), \dots, u(T)\big)$ is called *persistently exciting* of order L if the Hankel matrix $\mathscr{H}_L(u)$ is of full row rank. The banded upper-triangular Toeplitz matrix with t block-columns, related to the polynomial

$$r(z) = r_0 + r_1 z + \cdots + r_n z^n$$

is denoted by

$$\mathscr{T}_t(r) := \begin{bmatrix} r_0 & r_1 & \cdots & r_n & 0 & \cdots & 0 \\ 0 & r_0 & r_1 & \cdots & r_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & r_0 & r_1 & \cdots & r_n \end{bmatrix}. \tag{2}$$

2 Closed-loop data-driven simulation

Problem 1 (Closed-loop data-driven simulation)**.** Given

- trajectory $w_{\rm d} = \big(w_{\rm d}(1),\dots,w_{\rm d}(T)\big) \in (\mathbb{R}^{\tt w})^T$ of a linear time-invariant system $\mathscr{B} \subset (\mathbb{R}^{\tt w})^{\mathbb{N}}$, with an input/output partition $w = \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right] \in \mathscr{B}$, $u \in (\mathbb{R}^{\tt m})^{\mathbb{N}}$ input, $y \in (\mathbb{R}^{\tt p})^{\mathbb{N}}$ output;
- linear time-invariant controller $\mathscr{C} \subset (\mathbb{R}^{{\tt r}+{\tt p}+{\tt m}})^{\mathbb{N}}$, with an input/output partition $\left[\begin{smallmatrix} r \\ w \end{smallmatrix}\right] \in \mathscr{C}$, $r \in (\mathbb{R}^{\tt r})^{\mathbb{N}}$, $y \in (\mathbb{R}^{\tt p})^{\mathbb{N}}$ inputs, $u \in (\mathbb{R}^{\tt m})^{\mathbb{N}}$ output; and
- reference signal $r_{\rm r} = \big(r_{\rm r}(1),\dots,r_{\rm r}(T_{\rm r})\big) \in (\mathbb{R}^{\tt r})^{T_{\rm r}}$

find the set of responses $w_{\rm r}$ of the closed-loop system $\mathscr{B}_{\mathscr{C}}$ to the reference signal $r_{\rm r}$.

Solution and computational algorithm

A closed-loop data-driven simulation algorithm aims to compute for given $w_{\rm d}$, $\mathscr{C}$, and $r_{\rm r}$, the signals $w_{\rm r}$, such that

$$\begin{bmatrix} r_{\rm r} \\ w_{\rm r} \end{bmatrix} \in \mathscr{B}_{\mathscr{C}}|_{T_{\rm r}} \quad \iff \quad \begin{cases} w_{\rm r} \in \mathscr{B}|_{T_{\rm r}} \\ \left[\begin{smallmatrix} r_{\rm r} \\ w_{\rm r} \end{smallmatrix}\right] \in \mathscr{C}|_{T_{\rm r}} \end{cases} \tag{3}$$

Assuming that the system $\mathscr{B}$ is controllable, it admits a minimal image representation

$$\mathscr{B} = \{\, w = M(\sigma)l \mid l \in (\mathbb{R}^{\tt m})^{\mathbb{N}} \,\}.$$

Consider a minimal kernel representation of the controller

$$\mathscr{C} = \{\, \begin{bmatrix} r \\ w \end{bmatrix} \mid R_r(\sigma)r + R_w(\sigma)w = 0 \,\}.$$

In terms of the image and kernel representations of the plant and controller, (3) becomes

$$\begin{cases} \text{there is } g, \text{ such that } w_{\rm r} = M(\sigma)g \\ R_r(\sigma)r_{\rm r} + R_w(\sigma)w_{\rm r} = 0. \end{cases} \tag{4}$$

We can and do assume that the controller $\mathscr{C}$ is specified by a kernel representation, however, the plant $\mathscr{B}$ is only implicitly specified by the trajectory $w_{\rm d}$ and we aim to avoid using a representation of $\mathscr{B}$. The crucial step for doing this is to replace the image representation in (4) by the equation

$$w_{\rm r} = \mathscr{H}_{T_{\rm r}}(w_{\rm d})g, \tag{5}$$

which depends only on the data $w_{\rm d}$. The equivalence of $w_{\rm r} = M(\sigma)g$ and (5) holds under the following assumptions

1. the system $\mathscr{B}$ is controllable,

2. the input component u_d of w_d is persistently exciting of order T_r plus the order of $\mathscr{B}$,

and is proved in [WRMM05, Theorem 1], see also [MWVD06, Section 8.4]. Therefore, under assumptions 1 and 2, the set of solutions w_r of the linear system of equations

$$\begin{aligned} w_\text{r} &= \mathscr{H}_{T_\text{r}}(w_\text{d})g \\ \mathscr{T}_{T_\text{r}}(R_w)w_\text{r} &= -\mathscr{T}_{T_\text{r}}(R_r)r_\text{r}. \end{aligned} \tag{6}$$

is equal to the set of trajectories w_r solving the closed-loop data-driven simulation problem.

Note 2 (Multi-output systems). In (6), we have replaced the difference operator $R(\sigma)$ by the structured matrix $\mathscr{T}_{T_\text{r}}(R)$. In the multi-output case, the structure of $\mathscr{T}$ is more complicated than the one shown in (2). In order to simplify the presentation and abstract from technical details, we assume that the system is single-output.

Substituting $w_\text{r} = \mathscr{H}_{T_\text{r}}(w_\text{d})g$ into the second equation of (6) gives the following system of equations

$$\underbrace{\mathscr{T}_{T_\text{r}}(R_w)\mathscr{H}_{T_\text{r}}(w_\text{d})}_{A}g = \underbrace{-\mathscr{T}_{T_\text{r}}(R_r)r_\text{r}}_{b}$$

The matrix A is of dimension $T_\text{r} \times (\mathtt{r}+\mathtt{w})T_\text{r}$, so that the system $Ag = b$ is underdetermined. Let g_0 be a particular solution, *e.g.*, the least-norm solution $g_0 = A^\dagger b$ and let N be a matrix whose columns span the null space of A. The set of solution of (6) for g is

$$\mathscr{G} := \{\, g_0 + Nz \mid z \in \mathbb{R}^{\operatorname{col\,dim}(N)} \,\}.$$

Then the set of responses w_r of the closed-loop system $\mathscr{B}_\mathscr{C}$ to the reference signal r_r is

$$\mathscr{W}_\text{r} = \mathscr{H}_{T_\text{r}}(w_\text{d})\mathscr{G} = \{\, \underbrace{\mathscr{H}_{T_\text{r}}(w_\text{d})g_0}_{w_{\text{r},0}} + \mathscr{H}_{T_\text{r}}(w_\text{d})Nz \mid z \in \mathbb{R}^{\operatorname{col\,dim}(N)} \,\}.$$

It is characterized by the particular response $w_{\text{r},0}$ and a subspace—the column span of the matrix $\mathscr{H}_{T_\text{r}}(w_\text{d})N$. Algorithm 1 summarizes the necessary steps for data-driven computation of $\mathscr{W}_\text{r}$ from w_d, R, and r_r.

We proved that under the assumptions on the data w_d and the plant $\mathscr{B}$, specified in the derivation of the algorithm, Algorithm 1 solves Problem 1.

Theorem 3. *Under the following assumptions:*

1. *the system $\mathscr{B}$ is controllable,*

2. *the input component u_d of w_d is persistently exciting of order T_r plus the order of $\mathscr{B}$,*

the set

$$\mathscr{W}_\text{r} := \{\, w_{\text{r},0} + N_w z \mid z \in \mathbb{R}^{\operatorname{col\,dim}(N_w)} \,\},$$

computed by Algorithm 1 is equal to the set of T_r samples long responses of the closed-loop system $\mathscr{B}_\mathscr{C}$ to the reference signal r_r, i.e.,

$$\mathscr{W}_\text{r} = \{\, w \in (\mathbb{R}^\mathtt{w})^{T_\text{r}} \mid \begin{bmatrix} r_\text{r} \\ w \end{bmatrix} \in \mathscr{B}_\mathscr{C}|_{T_\text{r}} \,\}.$$

Algorithm 1 Closed-loop data-driven simulation.

Input: trajectory $w_{\rm d} \in (\mathbb{R}^{\tt w})^T$ of an LTI system $\mathscr{B}$, parameter $\begin{bmatrix} R_r & R_w \end{bmatrix}$ of a minimal kernel representation of the controller $\mathscr{C}$, and reference signal $r_{\rm r} \in (\mathbb{R}^{\tt r})^{T_{\rm r}}$.

1: Compute the least-norm solution g_0 of the system of equations

$$\mathscr{T}_{T_{\rm r}}(R_w)\mathscr{H}_{T_{\rm r}}(w_{\rm d})g = -\mathscr{T}_{T_{\rm r}}(R_r)r_{\rm r}.$$

2: Let $w_{{\rm r},0} := \mathscr{H}_{T_{\rm r}}(w_{\rm d})g_0$.

3: Compute a matrix N which columns form a basis for the column span of $\mathscr{T}_{T_{\rm r}}(R_w)\mathscr{H}_{T_{\rm r}}(w_{\rm d})$.

4: Let N_w be a basis for the column span of $\mathscr{H}_{T_{\rm r}}(w_{\rm d})N$.

Output: $\mathscr{W}_{\rm r} := \{\, w_{{\rm r},0} + N_w z \mid z \in \mathbb{R}^{\operatorname{coldim}(N_w)} \,\}$ — the set of $T_{\rm r}$ samples long responses of the closed-loop system $\mathscr{B}_{\mathscr{C}}$ to the reference signal $r_{\rm r}$.

Simulation example

The data $w_{\rm d} = \left[\begin{smallmatrix} u_{\rm d} \\ y_{\rm d} \end{smallmatrix}\right]$, used for the closed-loop data driven simulation, is the first 10 samples from the step response of a randomly generated first order system $\mathscr{B}$ interconnected with the controller $\mathscr{C}_1 := \{ \left[\begin{smallmatrix} r \\ u \\ y \end{smallmatrix}\right] \mid u = r - y \}$. The aim is to compute the first $T_{\rm r} = 10$ samples of the step response of $\mathscr{B}_{\mathscr{C}_2}$, where $\mathscr{C}_2 := \{ \left[\begin{smallmatrix} r \\ u \\ y \end{smallmatrix}\right] \mid u = r + y \}$. For this purpose we use Algorithm 1, *i.e.*, we do not compute explicitly a representation of $\mathscr{B}$. Note that either of the systems $\mathscr{B}_{\mathscr{C}_1}$, $\mathscr{B}_{\mathscr{C}_2}$ can be unstable.

In order to ensure that assumption 2 is satisfied, we augment the given trajectory—the step response of $\mathscr{B}_{\mathscr{C}_1}$— with $T_{\rm r}$ zeros. This takes into account the zero initial conditions of the given trajectory and ensures that the initial conditions of the computed response $r_{\rm r}$ of $\mathscr{B}_{\mathscr{C}_1}$ are also zero (*i.e.*, $N_w = 0$). The results for a particular system $\mathscr{B}$ are shown in Figure 1. We verify that up to numerical errors $r_{\rm r}$ matches the step response s of $\mathscr{B}_{\mathscr{C}_2}$, obtained by model-based simulation. A Matlab file reproducing the simulation result is available from:

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http://users.ecs.soton.ac.uk/im/test_cdds.m
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3 Conclusions

We defined a new data-driven simulation problem, in which the closed-loop behavior of the unknown plant, in feedback with a given controller, is computed from a given trajectory of the plant and a representation of the controller. The proposed algorithm involves a solution of a linear system of equations and is, therefore, computationally fast and easy to implement. Future work will investigate modifications of the algorithm for recursive computation and noisy data, as well as application of closed-loop data-driven simulation in unfalsified and model predictive control.

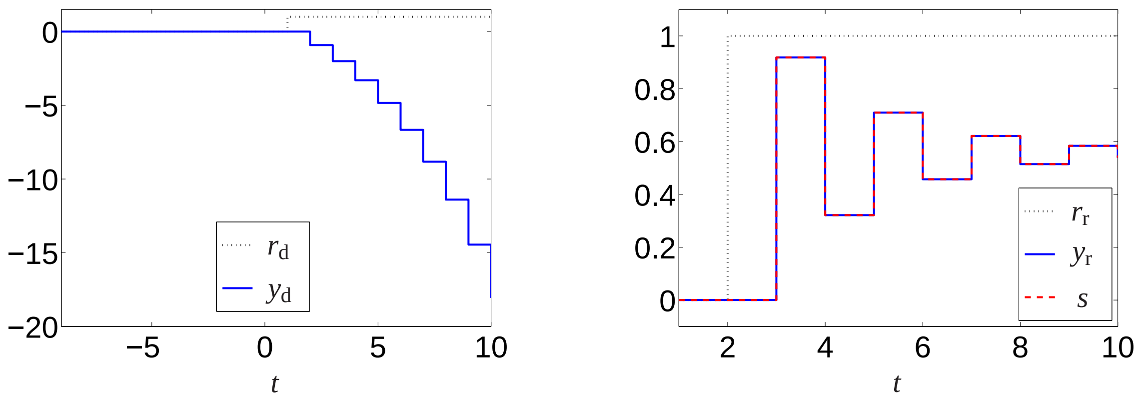

Figure 1: Step responses of $\mathscr{B}_{\mathscr{C}_1}$ (left) and $\mathscr{B}_{\mathscr{C}_2}$ (right).

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