Unfalsified control of linear time-invariant systems*

Ivan Markovsky†

School of Electronics and Computer Science

University of Southampton, SO17 1BJ, UK

Abstract

The unfalsified control concept is a data-driven assumptions-free control strategy. Its main tool is a controller falsification procedure, which tests the performance of a candidate controller directly from data of the plant in feedback with a possibly different controller. We pose and answer the questions: 1) how conservative is the falsification procedure in case the plant happens to be linear and time-invariant (LTI), 2) how can the LTI structure of the plant be taken into account in a controller falsification procedure. The first question is answered in the special case of first order plant and static controller. The example shows that the controller falsification test can be overly conservative when applied to an LTI plant. The answer to the second question is given for a general LTI plant and controller and leads to a new concept for testing controller's performance directly on data from the plant. The solution is based on a procedure for closed-loop data-driven simulation, i.e., construction of trajectories of a closed-loop system directly from data of the plant and a representation of the controller.

Keywords: system identification, persistency of excitation, data-driven simulation, data-driven control, unfalsified control.

1 Introduction

Data-driven control methods determine a control signal or a controller representation without using a model of the plant. The model is replaced by the observed data and prior hypothesis about the plant. Among the existing approaches for data-driven control, a particularly attractive one is the unfalsified control of Safonov and coworkers. As formulated in [ST97, SC01], unfalsified control uses *no prior hypothesis on the plant* apart from the observed data.

The unfalsified control is based on the observation that the ability of a candidate controller to meet desired performance specification can be tested using a trajectory of the plant without having a model of the plant or applying the controller on the plant. The controllers that (according to the test) fail to achieve the desired performance specification are discarded (falsified) and one of the remaining (unfalsified) controllers is used until it is falsified by the past measurements and replaced by a new unfalsified controller and so on. This leads to a switching adaptive control

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†Email: `im@ecs.soton.ac.uk`, Telephone: +44 (0) 23 8059 8715, Fax: +44 (0) 23 8059 4498

scheme. In this technical note, we do not consider a complete unfalsified switching adaptive control method but focus instead on its core ingredient—the test of a potential controller's performance directly from data of the closed-loop system, operating with a (possibly) different controller.

The standard test of the controller performance that uses only the data from the plant is based on what is called "fictitious reference" — a reference signal that together with the observed plant's output, could result (by proper choice of the controller's initial conditions) into an output of the controller that is equal to the observed plant's input. Therefore, according to the observed data of the plant, the tested controller driven by the fictitious reference (under suitable initial conditions) could have been in closed-loop with the plant. Consequently, the controller performance is verified against *this* closed-loop behavior. The controller is falsified on the basis of the observed data of the plant if the performance test (using the fictitious reference) fails. Otherwise, the controller is unfalsified by the data.

In Section 3, we review the controller falsification procedure, based on a fictitious reference signal. Since unfalsified control is a general control method it is natural to ask the question:

> Q1: How conservative is the fictitious reference signal when applied on a linear time-invariant (LTI) plant?

Although it is well known, see [MCMS07, DAL07], that the test based on a fictitious reference signal fails to detect instability of a destabilizing controller, its conservatism has not been quantified.

In Section 4 we quantify the conservatism of the fictitious reference test in the simplest possible case—first order plant and static controller. The analysis shows that in certain cases, the classical test, based on the fictitious reference test, can be arbitrary conservative in the sense that it can never falsify a destabilizing controller. Motivated by the conservatism of the fictitious reference test in the LTI case, we next address the question:

> Q2: How to test a controller performance, taking into account the prior knowledge (when available) that the plant is LTI?

Although tests for the controller performance are presented in the context of the direct unfalsified control [WHK99, Kos99], this work does not taking into account the LTI structure of the plant.

To the best of our knowledge questions Q1 and Q2 are novel. An answer to question Q2 is given in Section 5, where we propose a solution based on an extension of the data-driven simulation method of [MR08]. We gave the name "closed-loop data-driven simulation" to the resulting procedure that constructs the closed-loop behavior of the plant and the tested controller directly from a trajectory of the plant and controller representation. In the exact (noiseless) LTI case, the procedure gives the exact answer assuming that the data is persistently exciting and the plant is controllable. In the case of data generated by a nonlinear stochastic system, a minor modification of the procedure (use of pseudo-inverse instead of right-inverses on step 1 of Algorithm 1 and numerical rank instead of rank in steps 3 and 4) gives an approximate solution. (Note that the modification mentioned above is the basis for the transition from deterministic to approximate and stochastic subspace identification algorithms [VD96, MWVD06].) Once computed, the behavior of the plant–controller closed-loop system can be tested against any desired performance criterion by standard analysis methods. Thus “closed-loop data-driven simulation” gives a complete and non conservative answer to question Q2.

2 Preliminaries and notation

We use the behavioural language [Wil86, Wil91, PW98]. A dynamical system with $\mathtt{w}$ external variables (inputs and outputs) and time axis $\mathbb{T}$ is a subset of the signal space $(\mathbb{R}^{\mathtt{w}})^{\mathbb{T}}$ (i.e., the set of functions from $\mathbb{T}$ to $\mathbb{R}^{\mathtt{w}}$). In this paper, the time axis $\mathbb{T}$ is either the set of nonnegative real numbers $\mathbb{R}_+$ (continuous time) or the set of natural numbers $\mathbb{N}$ (discrete time). In the discrete-time case, a trajectory w of $\mathscr{B}$ is a vector time series $w = \big(w(1), w(2), \dots\big)$, where $w(t) \in \mathbb{R}^{\mathtt{w}}$, for all $t \in \mathbb{N}$. We assume that the manifest variables w have a given input/output partition. The number of inputs $\mathtt{m}$ and the number of outputs $\mathtt{p}$ of a system $\mathscr{B} \in (\mathbb{R}^{\mathtt{m}+\mathtt{p}})^{\mathbb{N}}$ are invariant of the representation. Modulo a permutation of the variables, any trajectory $w \in \mathscr{B}$ has an input/output partition

$$w = \operatorname{col}(u, y) := \begin{bmatrix} u \\ y \end{bmatrix},$$

where u is an input, i.e., it is free, and y is an output, i.e., it is determined by the input, the system, and the initial condition.

There are a number of representations of a linear, time-invariant, and finite dimensional system $\mathscr{B}$. Let σ denotes the backwards shift operator

$$\sigma w(t) := w(t+1).$$

In this paper, we use the

- kernel representation $R(\sigma)w = 0$, parameterized by the polynomial matrix $R \in \mathbb{R}^{\mathtt{p}\times(\mathtt{m}+\mathtt{p})}[z]$; and
- image representation $w = M(\sigma)g$, parameterized by the polynomial matrix $M \in \mathbb{R}^{(\mathtt{m}+\mathtt{p})\times\mathtt{m}}[z]$.

The lag $\mathbf{l}(\mathscr{B})$ of $\mathscr{B}$ is defined as the smallest degree of a kernel representation of $\mathscr{B}$ and is invariant of the system.

The Hankel matrix with t_1 block rows, composed of the signal $w \in (\mathbb{R}^{\mathtt{w}})^T$ is denoted by

$$\mathscr{H}_{t_1,t_2}(w) := \begin{bmatrix} w(1) & w(2) & \cdots & w(t_2) \\ w(2) & w(3) & \cdots & w(t_2+1) \\ w(3) & w(4) & \cdots & w(t_2+2) \\ \vdots & \vdots & & \vdots \\ w(t_1) & w(t_1+1) & \cdots & w(t_1+t_2-1) \end{bmatrix}. \tag{1}$$

If the index t_2 is skipped, the matrix $\mathscr{H}_{t_1}(w)$ is assumed to have the maximal possible number of columns $t_2 = T - t_1 + 1$. The signal $u = \big(u(1), \dots, u(T)\big)$ is called *persistently exciting* of order L if the Hankel matrix $\mathscr{H}_L(u)$ is of full row rank.

The banded upper-triangular Toeplitz matrix with t block-columns, related to the polynomial $r \in \mathbb{R}^{1\times \mathtt{r}}[z]$, $\deg(r) =: n$ is denoted by

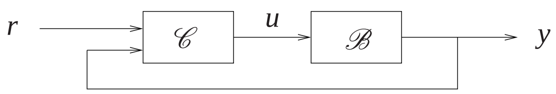
$$\mathscr{T}_t(r) := \begin{bmatrix} r_0 & r_1 & \cdots & r_n & 0 & \cdots & 0 \\ 0 & r_0 & r_1 & \cdots & r_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & r_0 & r_1 & \cdots & r_n \end{bmatrix}. \tag{2}$$

We use the same letter y for the signal $\big(y(1),\dots,y(t)\big)$ as well as for the vector $\operatorname{col}\big(y(1),\dots,y(t)\big)$, i.e., we identify the spaces $(\mathbb{R}^{\mathtt{p}})^t$ and $\mathbb{R}^{\mathtt{p}t}$. The outputs y that corresponding to a given input $u \in (\mathbb{R}^{\mathtt{m}})^t$, i.e., y such that $\operatorname{col}(u,y) \in \mathscr{B}_t$, form a subspace. This space is parameterized by the initial condition. For $t \geq \mathbf{l}(\mathscr{B})$, its dimension is $\mathbf{n}(\mathscr{B})$—the order of $\mathscr{B}$.

$A^\dagger$ denotes the Moore-Penrose pseudoinverse of a matrix A and $\operatorname{col\,span}(A)$ denotes the span of the columns of A.

3 Review of the controller falsification based on fictitious reference

In [ST97], the unfalsified control is defined for the feedback system[1]



where $\mathscr{B} \subseteq (\mathbb{R}^{\mathtt{m}+\mathtt{p}})^{\mathbb{T}}$ is the plant and $\mathscr{C} \subseteq (\mathbb{R}^{\mathtt{r}+\mathtt{p}+\mathtt{m}})^{\mathbb{T}}$ is the controller. We assume that in a trajectory $w \in \mathscr{B}$, $w =: (u,y)$, $u \in (\mathbb{R}^{\mathtt{m}})^{\mathbb{T}}$ is an input and $y \in (\mathbb{R}^{\mathtt{p}})^{\mathbb{T}}$ is an output. Similarity, in $(r,u,y) \in \mathscr{C}$, $r \in (\mathbb{R}^{\mathtt{r}})^{\mathbb{T}}$ and $y \in (\mathbb{R}^{\mathtt{p}})^{\mathbb{T}}$ are inputs and $u \in (\mathbb{R}^{\mathtt{m}})^{\mathbb{T}}$ is an output. The closed-loop system $\mathscr{B}_{\mathscr{C}}$, obtained by interconnecting $\mathscr{B}$ and $\mathscr{C}$, is given by

$$\mathscr{B}_{\mathscr{C}} = \mathscr{B}_{\text{ext}} \cap \mathscr{C},$$

where

$$\mathscr{B}_{\text{ext}} := \{\, (r,w) \in (\mathbb{R}^{\mathtt{r}+\mathtt{m}+\mathtt{p}})^{\mathbb{T}} \mid w \in \mathscr{B} \,\}.$$

The control specification can be formulated as a desired closed-loop behavior $\mathscr{B}_{\text{des}} \subseteq (\mathbb{R}^{\mathtt{r}+\mathtt{m}+\mathtt{p}})^{\mathbb{T}}$. A controller $\mathscr{C}$ is said to achieve the desired behavior $\mathscr{B}_{\text{des}}$ (on the plant $\mathscr{B}$) if $\mathscr{B}_{\mathscr{C}} \subseteq \mathscr{B}_{\text{des}}$, i.e.,

$$\mathscr{C} \text{ achieve } \mathscr{B}_{\text{des}} \text{ (on } \mathscr{B}) \quad :\Longleftrightarrow \quad \mathscr{B}_{\mathscr{C}} \subseteq \mathscr{B}_{\text{des}}. \tag{3}$$

Verification of (3) obviously requires knowledge of the plant's behavior $\mathscr{B}$. The aim of unfalsified control is to check if the controller fails to achieve the specification using only an observed trajectory w_{d} of $\mathscr{B}$.

Let $w_{\text{d}} = (u_{\text{d}}, y_{\text{d}}) \in \mathscr{B}$ be a given trajectory of $\mathscr{B}$ and let $\mathscr{W}_{\text{d,ext}}$ be the set of all possible extensions of w_{d} to a trajectory of $\mathscr{B}_{\text{ext}}$, i.e.,

$$\mathscr{W}_{\text{d,ext}} := \{\, (r, w_{\text{d}}) \in \mathscr{B}_{\text{ext}} \,\}. \tag{4}$$

[1]The presentation in [ST97] is restricted to the SISO case.

By construction $\mathscr{W}_{\mathrm{d,ext}} \subset \mathscr{B}_{\mathrm{ext}}$, so that $\mathscr{W}_{\mathrm{d,ext}} \cap \mathscr{C} \subset \mathscr{B}_{\mathrm{des}}$ is a necessary condition for $\mathscr{C}$ to achieve $\mathscr{B}_{\mathrm{des}}$ (see (3)). It follows that a sufficient condition for $\mathscr{C}$ to fail to achieve $\mathscr{B}_{\mathrm{des}}$ is

$$\mathscr{W}_{\mathrm{d,ext}} \cap \mathscr{C} \not\subset \mathscr{B}_{\mathrm{des}}. \tag{5}$$

The test (5) allows a controller to be falsified on the basis of the data alone without *any* assumption on the plant. Therefore, the test (5) is not only data-driven but also assumptions free.

The test (5) has a simple interpretation: $\mathscr{W}_{\mathrm{d,ext}} \cap \mathscr{C}$ is the problem of finding the set of reference signals $\mathscr{R}_{\mathscr{C}}(w_{\mathrm{d}})$ that are consistent with the data w_{d} and the controller $\mathscr{C}$, i.e.,

$$\mathscr{R}_{\mathscr{C}}(w_{\mathrm{d}}) = \{\, r \in (\mathbb{R}^{\mathtt{r}})^{\mathbb{T}} \mid (r, w_{\mathrm{d}}) \in \mathscr{C} \,\}.$$

The signals $r \in \mathscr{R}_{\mathscr{C}}(w_{\mathrm{d}})$ that satisfy (4) are called in [ST97] “fictitious reference signals”. Of course, for given $\mathscr{C}$ and w_{d}, $\mathscr{R}_{\mathscr{C}}(w_{\mathrm{d}})$ may be an empty set (in which case the data w_{d} is not sufficient to falsify the controller $\mathscr{C}$). Computing $\mathscr{R}_{\mathscr{C}}(w_{\mathrm{d}})$ for a general nonlinear controller $\mathscr{C}$ is a hard problem. For LTI controller, however, the problem is linear — $\mathscr{R}_{\mathscr{C}}(w_{\mathrm{d}})$ is either an empty set or an affine space. Concrete algorithms for computing $\mathscr{R}_{\mathscr{C}}(w_{\mathrm{d}})$ are given in [ST97]. Applied in real-time, these algorithms lead to an adaptive switching control strategy, see [WPSS05]. In this note, we do not consider the switching strategy and its real-time implementation but focus on its core ingredient—the fictitious reference test—in the case of an LTI plant.

4 Conservatism in the LTI case: an example

Consider the unfalsified control problem with a plant

$$\mathscr{B} := \{\, (u,y) \mid \tfrac{\mathrm{d}}{\mathrm{d}t} y = u \,\}$$

and a set of candidate controllers

$$\mathscr{C}_{\alpha} := \{\, (r,u,y) \mid u = -\alpha(r-y) \,\},$$

parameterized by $\alpha \in \mathbb{R}$, $\alpha \neq 0$. The considered performance specification is

$$\mathscr{B}_{\mathrm{des}} = \{\, (r,u,y) \mid \|r-y\|^2 + \|u\|^2 \leq \gamma \|r\|^2 \,\}, \tag{6}$$

where

$$\|r\|^2 := \int_0^{\infty} r^2(t)\,\mathrm{d}t,$$

and $\gamma \in \mathbb{R}_+$. The closed-loop system obtained by interconnecting $\mathscr{B}$ with $\mathscr{C}_{\alpha}$ is

$$\begin{aligned}\mathscr{B}_{\mathscr{C}_{\alpha}} = \{\, (r,u,y) \mid r \in \mathbb{R}^{\mathbb{R}_+},\ u = -\alpha(r-y),\ y(0) \in \mathbb{R},\\ \tfrac{\mathrm{d}}{\mathrm{d}t} y = -\alpha(r-y) \,\}\end{aligned}$$

or more explicitly

$$\mathscr{B}_{\mathscr{C}_\alpha} = \{\,(r,u,y) \mid r \in \mathbb{R}^{\mathbb{R}_+},\; u = -\alpha(r-y),\; y(0) \in \mathbb{R},$$
$$y(t) = \exp(\alpha t)y(0) - \alpha \int_0^t \exp\big(\alpha(t-\tau)\big) r(\tau)\mathrm{d}\tau,\; \text{for } t > 0\,\}.$$

The controller $\mathscr{C}_\alpha$ achieves $\mathscr{B}_{\text{des}}$ on $\mathscr{B}$ if

$$\gamma > 1 \quad \text{and} \quad \alpha \in [-\sqrt{\gamma - 1}, 0). \tag{7}$$

To see this, note that (6) is equivalent to $\|\mathscr{B}^{u,e}_{\mathscr{C}_\alpha}\|^2_\infty \leq \gamma$, where

$$\mathscr{B}^{u,e}_{\mathscr{C}_\alpha} := \{\,(r,u,e) \mid e = r - y,\; (r,u,y) \in \mathscr{B}_{\mathscr{C}_\alpha}\,\}.$$

We have

$$\|\mathscr{B}^{u,e}_{\mathscr{C}_\alpha}\|^2_\infty = \left\| \begin{bmatrix} -\alpha \\ 1 \end{bmatrix} \frac{s}{s-\alpha} \right\|^2_\infty = (\alpha^2 + 1) \left\| \frac{s}{s-\alpha} \right\|^2_\infty = \alpha^2 + 1$$

and for stability of $\mathscr{B}_{\mathscr{C}_\alpha}$, $\alpha < 0$.

For this example, there is a unique fictitious reference signal

$$\mathscr{R}_{\mathscr{C}_\alpha}(w_{\text{d}}) = \{\, r_{\text{d}} := y_{\text{d}} - u_{\text{d}}/\alpha \,\}.$$

for each controller $\mathscr{C}_\alpha$, $\alpha \neq 0$ and trajectory $w_{\text{d}} = (u_{\text{d}}, y_{\text{d}}) \in (\mathbb{R}^2)^{\mathbb{T}}$. Then the fictitious error signal is

$$e_{\text{d}} := r_{\text{d}} - y_{\text{d}} = -u_{\text{d}}/\alpha$$

and the controller $\mathscr{C}_\alpha$ is falsified if and only if

$$\|r_{\text{d}} - y_{\text{d}}\|^2 + \|u_{\text{d}}\|^2 > \gamma \|r_{\text{d}}\|^2 \iff (1/\alpha^2 + 1)\|u_{\text{d}}\|^2 > \gamma \|r_{\text{d}}\|^2$$
$$\iff \frac{\|u_{\text{d}}\|^2}{\|r_{\text{d}}\|^2} > \frac{\gamma \alpha^2}{\alpha^2 + 1}.$$

Note that $\|u_{\text{d}}\|^2 / \|r_{\text{d}}\|^2 \leq \|\mathscr{B}^u_{\mathscr{C}_\alpha}\|^2_\infty$, where

$$\mathscr{B}^u_{\mathscr{C}_\alpha} := \{\,(r,u) \mid (r,u,y) \in \mathscr{B}_{\mathscr{C}_\alpha}\,\}.$$

It turns out that the condition

$$\|\mathscr{B}^u_{\mathscr{C}_\alpha}\|^2_\infty > \frac{\gamma \alpha^2}{\alpha^2 + 1}$$

together with $\alpha < 0$ is equivalent to the condition that $\mathscr{C}_\alpha$ does not achieve the desired behavior $\mathscr{B}_{\text{des}}$. Indeed,

$$\|\mathscr{B}^u_{\mathscr{C}_\alpha}\|^2_\infty = \left\| \frac{-\alpha s}{s - \alpha} \right\|^2_\infty = \alpha^2$$

and

$$\alpha^2 > \frac{\gamma \alpha^2}{\alpha^2 + 1} \quad \implies \quad |\alpha| > \sqrt{\gamma - 1},$$

cf. (7). The above analysis shows that

$$c := \|\mathscr{B}^{u}_{\mathscr{C}_\alpha}\|^2_\infty - \|u_\mathrm{d}\|^2/\|r_\mathrm{d}\|^2 \tag{8}$$

is a measure for the conservatism of the fictitious reference test. Suppose that $\mathscr{C}_\alpha$ does not achieve $\mathscr{B}_\mathrm{des}$. The fictitious reference test will not falsify $\mathscr{C}_\alpha$ if and only if the following inequality holds

$$\|\mathscr{B}^{u}_{\mathscr{C}_\alpha}\|^2_\infty > \frac{\gamma\alpha^2}{\alpha^2+1} > \|u_\mathrm{d}\|^2/\|r_\mathrm{d}\|^2.$$

Let the data w_d be collected from the closed-loop system $\mathscr{B}_{\mathscr{C}_\beta}$ with reference signal r a step function and plant's initial condition $y(0) = 0$. Then

$$u_\mathrm{d}(t) = -\beta \exp(\beta t), \quad \text{and} \quad y_\mathrm{d}(t) = 1 - \exp(\beta t),$$

and the fictitious reference for a controller $\mathscr{C}_\alpha$ is

$$r_\mathrm{d} = 1 + \left(\frac{\beta}{\alpha} - 1\right)\exp(\beta t).$$

The conservatism measure c of the fictitious reference test in this case is

$$c = \alpha^2\left(1 - \frac{\beta^2}{\|\beta - \big(1-\exp(-\beta t)\big)\alpha\|^2}\right).$$

For $\alpha \neq \beta$, $c = \alpha^2$, so that the test can never falsify the controller $\mathscr{C}_\alpha$ although $\mathscr{C}_\alpha$ may fail to achieve the specification.

5 Closed-loop data-driven simulation

In analogy with the (open-loop) data-driven simulation problem [MWRM05, MR07, MR08], the closed-loop data-driven simulation problem is defined as follows.

Problem 1 (Closed-loop data-driven simulation)**.** Given

- trajectory $w_\mathrm{d} = \big(w_\mathrm{d}(1),\dots,w_\mathrm{d}(T)\big) \in (\mathbb{R}^\mathtt{w})^T$ of an LTI system $\mathscr{B} \subset (\mathbb{R}^\mathtt{w})^\mathbb{N}$, with an input/output partition $w = (u,y) \in \mathscr{B}$, $u \in (\mathbb{R}^\mathtt{m})^\mathbb{N}$ input, $y \in (\mathbb{R}^\mathtt{p})^\mathbb{N}$ output;
- LTI controller $\mathscr{C} \subset (\mathbb{R}^{\mathtt{r}+\mathtt{p}+\mathtt{m}})^\mathbb{N}$, with an input/output partition $(r,u,y) \in \mathscr{C}$, $r \in (\mathbb{R}^\mathtt{r})^\mathbb{N}$, $y \in (\mathbb{R}^\mathtt{p})^\mathbb{N}$ inputs, $u \in (\mathbb{R}^\mathtt{m})^\mathbb{N}$ output; and
- reference signal $r_\mathrm{r} = \big(r_\mathrm{r}(1),\dots,r_\mathrm{r}(T_\mathrm{r})\big) \in (\mathbb{R}^\mathtt{r})^{T_\mathrm{r}}$

find the set of responses w_r of the closed-loop system $\mathscr{B}_\mathscr{C}$ to the reference signal r_r.

As proven in the following proposition, under certain specified assumptions on the on the data w_r and the plant $\mathscr{B}$, Algorithm 1 solves Problem 1.

Algorithm 1 Closed-loop data-driven simulation.

Input: $w_\mathrm{d} \in (\mathbb{R}^{\mathtt{w}})^T$, $R \in \mathbb{R}^{1\times(\mathtt{r}+\mathtt{w})}[z]$, and $r_\mathrm{r} \in (\mathbb{R}^{\mathtt{r}})^{T_\mathrm{r}}$.

1: Compute the least-norm solution g_0 of the system of equations $\mathscr{T}_{T_\mathrm{r}}(R_w)\mathscr{H}_{T_\mathrm{r}}(w_\mathrm{d})g = -\mathscr{T}_{T_\mathrm{r}}(R_r)r_\mathrm{r}$.

2: Let $w_{\mathrm{r},0} := \mathscr{H}_{T_\mathrm{r}}(w_\mathrm{d})g_0$.

3: Compute a matrix N which columns form a basis for the column span of $\mathscr{T}_{T_\mathrm{r}}(R_w)\mathscr{H}_{T_\mathrm{r}}(w_\mathrm{d})$.

4: Let N_w be a basis for the column span of $\mathscr{H}_{T_\mathrm{r}}(w_\mathrm{d})N$.

Output: $w_{\mathrm{r},0}$ and N_w.

Proposition 2. *Under the following assumptions:*

1. *the system $\mathscr{B}$ is controllable,*

2. *the input component u_d of w_d is persistently exciting of order $T_\mathrm{r} + \mathbf{n}(\mathscr{B})$, where $\mathbf{n}(\mathscr{B})$ is the order of $\mathscr{B}$,*

the set

$$\mathscr{W}_\mathrm{r} := \{\, w_{\mathrm{r},0} + N_w z \mid z \in \mathbb{R}^{\operatorname{col}\dim(N_w)} \,\}.$$

computed by Algorithm 1 is equal to the set of T_r samples long responses of the closed-loop system $\mathscr{B}_\mathscr{C}$ to the reference signal r_r, i.e.,

$$\mathscr{W}_\mathrm{r} = \{\, w \in (\mathbb{R}^{\mathtt{w}})^{T_\mathrm{r}} \mid (r_\mathrm{r}, w) \in \mathscr{B}_\mathscr{C}|_{T_\mathrm{r}} \,\}.$$

Proof. A data-driven simulation algorithm aims to compute for given w_d, $\mathscr{C}$, and r_r, the signals w_r, such that

$$(r_\mathrm{r}, w_\mathrm{r}) \in \mathscr{B}_\mathscr{C} \quad \Longleftrightarrow \quad \begin{cases} w_\mathrm{r} \in \mathscr{B} \\ (r_\mathrm{r}, w_\mathrm{r}) \in \mathscr{C} \end{cases} \tag{9}$$

By assumption 2 the system $\mathscr{B}$ is controllable, so that it admits an image representation

$$\mathscr{B} = \operatorname{image}\big(M(\sigma)\big).$$

Consider a kernel representation of the controller

$$\mathscr{C} = \{\, (r, w) \mid R(\sigma)\operatorname{col}(r, w) = 0 \,\}.$$

In terms of the image and kernel representations of the plant and controller, (9) becomes

$$\begin{cases} \exists\, g, \text{ s.t. } w_\mathrm{r} = M(\sigma)g \\ R(\sigma)\operatorname{col}(r_\mathrm{r}, w_\mathrm{r}) = 0. \end{cases} \tag{10}$$

We can and do assume that the controller $\mathscr{C}$ is specified by a kernel representation, however, the plant $\mathscr{B}$ is only implicitly specified by the trajectory w_d. For a closed-loop data-driven simulation algorithm to qualify as "data-driven", we have to avoid using a representation of $\mathscr{B}$.

The closed-loop data-driven simulation algorithm solves (10) for $w_{\rm r}$, replacing the first equation in (10) by the equation

$$w_{\rm r} = \mathscr{H}_{T_{\rm r}}(w_{\rm d})g, \tag{11}$$

which does not involve a representation of $\mathscr{B}$. The equivalence of $w_{\rm r} = M(\sigma)g$ and (11) holds under assumptions 1 and 2, see [MWVD06, Section 8.4] and [WRMM05, Theorem 1]. Therefore, under assumptions 1 and 2, the set of solutions $w_{\rm r}$ of the linear system of equations

$$\begin{aligned} w_{\rm r} &= \mathscr{H}_{T_{\rm r}}(w_{\rm d})g \\ \mathscr{T}_{T_{\rm r}}(R)\,\mathrm{col}(r_{\rm r}, w_{\rm r}) &= 0 \end{aligned} \tag{12}$$

is equal to the set of trajectories $w_{\rm r}$ solving the closed-loop data-driven simulation problem.

Note 3 (Multi-output systems). In (12), we have replaced the difference operator $R(\sigma)$ by the structured matrix $\mathscr{T}_{T_{\rm r}}(R)$. In the multi-output case, the structure of $\mathscr{T}$ is more complicated than the one shown in (2). In order to simplify the presentation and abstract from technical details, here we assume that the system is single-output, so that $R \in \mathbb{R}^{1\times(\mathtt{r}+\mathtt{w})}[z]$.

Let $R =: \begin{bmatrix} R_r & R_w \end{bmatrix}$, where $R_r \in \mathbb{R}^{1\times \mathtt{r}}[z]$ and $R_w \in \mathbb{R}^{1\times \mathtt{w}}[z]$.

$$\mathscr{T}_{T_{\rm r}}(R)\,\mathrm{col}(r_{\rm r}, w_{\rm r}) = 0 \implies \mathscr{T}_{T_{\rm r}}(R_w)w_{\rm r} = -\mathscr{T}_{T_{\rm r}}(R_r)r_{\rm r}. \tag{13}$$

Substituting $w_{\rm r} = \mathscr{H}_{T_{\rm r}}(w_{\rm d})g$ into (13) gives the following system of equations

$$\underbrace{\mathscr{T}_{T_{\rm r}}(R_w)\mathscr{H}_{T_{\rm r}}(w_{\rm d})}_{A}\,g = \underbrace{-\mathscr{T}_{T_{\rm r}}(R_r)r_{\rm r}}_{b}$$

The matrix A is $T_{\rm r} \times (\mathtt{r}+\mathtt{w})T_{\rm r}$, so that the system $Ag = b$ is underdetermined. Let g_0 be a particular solution, e.g., the least-norm solution $g_0 = A^{\dagger}b$ and let N be a matrix whose columns span the null space of A. The set of solution of (12) for g is

$$\mathscr{G} := \{\, g_0 + Nz \mid z \in \mathbb{R}^{\mathrm{coldim}(N)} \,\}.$$

Then the set of responses $w_{\rm r}$ of the closed-loop system $\mathscr{B}_{\mathscr{C}}$ to the reference signal $r_{\rm r}$ is

$$\mathscr{W}_{\rm r} = \mathscr{H}_{T_{\rm r}}(w_{\rm d})\mathscr{G} = \{\, \underbrace{\mathscr{H}_{T_{\rm r}}(w_{\rm d})g_0}_{w_{\rm r,0}} + \mathscr{H}_{T_{\rm r}}(w_{\rm d})Nz \mid z \in \mathbb{R}^{\mathrm{coldim}(N)} \,\}.$$

It is characterized by the particular response $w_{\rm r,0}$ and a subspace—the column span of the matrix $\mathscr{H}_{T_{\rm r}}(w_{\rm d})N$. Algorithm 1 summarizes the necessary steps for data-driven computation of $\mathscr{W}_{\rm r}$ from $w_{\rm d}$, R, and $r_{\rm r}$. □

Example 4. We verify the correctness of Proposition 2 on the following example

$$\begin{aligned} &\text{plant:} && \mathscr{B} = \{\, (u,y) \in (\mathbb{R}^2)^{\mathbb{N}} \mid \sigma y - y = u \,\} \\ &\text{controller:} && \mathscr{C}_{\alpha} = \{\, (r,u,y) \mid u = -\alpha(r-y) \,\} \end{aligned}$$

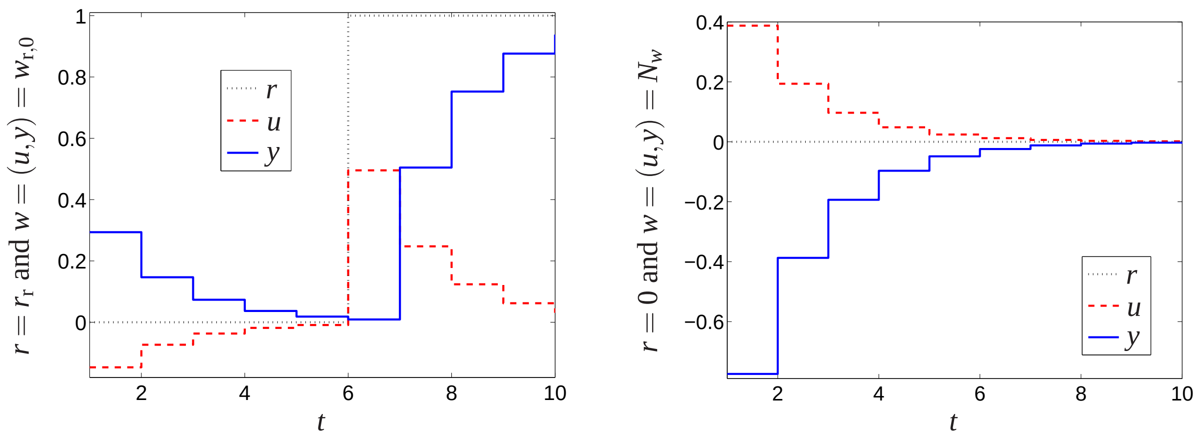


Figure 1: The trajectories $(r_{\mathrm{r}}, y_{\mathrm{r},0})$ and $(0, N_y)$ computed by the data-driven simulation algorithm. Any response of the closed-loop system $\mathscr{B}_{\mathscr{C}_\alpha}$ driven by the reference signal r_{r} is of the form $w = w_{\mathrm{r},0} + N_w g$, for some $g \in \mathbb{R}$.

(which is a discrete-time equivalent to the example considered in Section 4). The data w_{d} is a $T = 50$ samples long response of the closed-loop system $\mathscr{B}_{\mathscr{C}_\beta}$ with $\beta = 0.1$, to a zero mean random reference signal r under zero initial condition $y(0) = 0$. We are aiming to compute the $T_{\mathrm{r}} = 10$ samples long responses of the closed-loop $\mathscr{B}_{\mathscr{C}_\alpha}$ with $\alpha = 0.5$, to the reference signal

$$r_{\mathrm{r}}(t) = \begin{cases} 0, & \text{for } t = 1, \dots, 5 \\ 1, & \text{for } t = 6, \dots, 10. \end{cases}$$

A kernel representation of the controller $\mathscr{C}_\alpha$ is given by the matrix

$$R = \begin{bmatrix} \alpha & 1 & -\alpha \end{bmatrix}.$$

The system $\mathscr{B}$ is controllable and the input component u_{d} of the data w_{d} is persistently exciting of order 25, which is higher than $T_{\mathrm{r}} + \mathbf{n}(\mathscr{B}) = 11$, so that assumptions 1 and 2 hold. Therefore, denoting with $w_{\mathrm{r},0}$ and N_w the output of Algorithm 1 to the data w_{d}, R, r_{r} we should verify that

1. $w_{\mathrm{r},0}$ is a response of $\mathscr{B}_{\mathscr{C}_\alpha}$ driven by r_{r},

2. N_w is a zero input response of $\mathscr{B}_{\mathscr{C}_\alpha}$, and

3. $\operatorname{col\,dim}(N_w) = \mathbf{n}(\mathscr{B}) + \mathbf{n}(\mathscr{C}_\alpha) = 1$.

Items 1 and 2 ensure that

$$\mathscr{W}_{\mathrm{r}} \subseteq \{ y \in (\mathbb{R}^{\mathtt{w}})^{T_{\mathrm{r}}} \mid (r_{\mathrm{r}}, y) \in \mathscr{B}_{\mathscr{C}_\alpha}|_{T_{\mathrm{r}}} \}$$

Item 3 and the fact that the set in the right-hand side of (4) is an $\mathbf{n}(\mathscr{B}) + \mathbf{n}(\mathscr{C}_\alpha)$ dimensional space ensure that equality holds.

Under assumption 1 and 2, the data matrix $\mathscr{H}_{T_{\mathrm{r}}}(w_{\mathrm{d}}) \in \mathbb{R}^{\mathtt{w}T_{\mathrm{r}} \times (T - T_{\mathrm{r}} + 1)}$ has rank $\mathtt{m}T_{\mathrm{r}} + \mathbf{n}(\mathscr{B})$, see [WRMM05, Theorem 1]. In the example, $\mathscr{H}_{T_{\mathrm{r}}}(w_{\mathrm{d}})$ is 20×41 and indeed $\operatorname{rank}(H) = 11$. In the SISO case (see Note 3), the matrix

$\mathscr{T}_{T_r}(R_w) \in \mathbb{R}^{T_r \times (\mathtt{r}+\mathtt{w})}$ is obviously full rank. In the example, the matrix $A = \mathscr{T}_{T_r}(R_w)\mathscr{H}_{T_r}(w_d)$ is 10×41 and is full rank.

Let N be a matrix whose columns span the null space of A. The matrix $\mathscr{H}_{T_r}(w_d)N \in \mathbb{R}^{20\times 31}$ has rank equal to 1, which demonstrates that item 3 in the list above holds. The problem of verifying that $(r_r, w_{r,0})$ and $(0, N_w)$ are trajectories of $\mathscr{B}_{\mathscr{C}_\alpha}$ is a state estimation problem: verify that there are initial conditions under which $(r_r, w_{r,0})$ and $(0, N_w)$ are trajectories of $\mathscr{B}_{\mathscr{C}_\alpha}$. It turns out that the initial condition of $\mathscr{B}_{\mathscr{C}_\alpha}$ corresponding to $(r_r, w_{r,0})$ is $y(0) = 0.2949$ and the the initial condition corresponding to $(0, N_w)$ is $y(0) = -0.7746$. The trajectories are shown in Figure 1.

6 Discussion and conclusions

The reason for the conservatism of the fictitious reference test is intuitively clear: in the case of a static controller, the test derives a *single* trajectory of the closed-loop system. Consequently, the performance of the controller is verified against this trajectory only. Using the LTI hypothesis, we can augment the computed closed-loop trajectory with all time shifts and subsequently with all linear combinations of the resulting set of trajectory. This construction is done systematically by the closed-loop data-driven simulation procedure. Moreover, according to Proposition 2, under assumptions 1 and 2, *all* trajectories of the closed-loop system are constructed in this way. Therefore, under these assumptions a test of the controller performance, based on the proposed closed-loop data-driven simulation procedure is *non conservative.*

A question for further research is find conditions under which the data collected in closed-loop is persistently exciting of sufficiently high order, i.e., ensure that assumption 2 holds. One possible way of ensuring persistency of excitation is to artificially inject a noise signal at the input of the plant. Of course, such a solution will lead to deterioration of the controller performance. The dual role of the input in adaptive control: a) achieve performance specification, b) persistently excite the plant—is a well known issue (sometimes called *dual control.*). Another direction for further research is related to the computational issue of checking efficiently the performance of a (large) number of controllers on the same data w_d. The final goal is applying the data-driven simulation algorithm in a (switching) adaptive control scheme.

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