An algorithm for closed-loop data-driven simulation

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Abstract: Closed-loop data-driven simulation refers to the problem of constructing trajectories of a closed-loop system directly from data of the plant and a representation of the controller. Conditions under which the problem has a solution are given and an algorithm for computing the solution is presented. The problem formulation and its solution are in the spirit of the deterministic identification algorithms, *i.e.*, in the theoretical analysis of the method, the data is assumed exact (noise free).

Keywords: Simulation, system identification, behaviors.

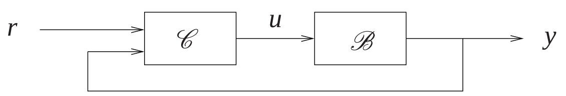
PRELIMINARIES AND NOTATION

We use the behavioural language, see Willems (1986). A discrete-time dynamical system with $\mathtt{w}$ manifest variables is a subset of the signal space $(\mathbb{R}^{\mathtt{w}})^{\mathbb{N}}$, *i.e.*, the set of functions from the set of natural numbers $\mathbb{N}$ to $\mathbb{R}^{\mathtt{w}}$. We assume that the manifest variables w have a *given* input/output partition

$$w = \operatorname{col}(u,y) := \left[\begin{smallmatrix} u \\ y \end{smallmatrix}\right],$$

where $u \in (\mathbb{R}^{\mathtt{m}})^{\mathbb{N}}$ is an input and $y \in (\mathbb{R}^{\mathtt{p}})^{\mathbb{N}}$ is an output.

The feedback interconnection of the plant $\mathscr{B} \subseteq (\mathbb{R}^{\mathtt{m}+\mathtt{p}})^{\mathbb{N}}$ and a controller $\mathscr{C} \subseteq (\mathbb{R}^{\mathtt{r}+\mathtt{p}+\mathtt{m}})^{\mathbb{N}}$



is given by

$$\mathscr{B}_{\mathscr{C}} = \mathscr{B}_{\text{ext}} \cap \mathscr{C},$$

where

$$\mathscr{B}_{\text{ext}} := \{\, (r,w) \in (\mathbb{R}^{\mathtt{r}+\mathtt{m}+\mathtt{p}})^{\mathbb{N}} \mid w \in \mathscr{B} \,\}.$$

We consider linear, time-invariant, and finite dimensional plants and controllers. A kernel representation $R(\sigma)w = 0$, were σ is the backwards shift operator

$$\sigma w(t) := w(t+1),$$

is parameterized by the polynomial matrix $R \in \mathbb{R}^{\mathtt{p}\times(\mathtt{m}+\mathtt{p})}[z]$, and an image representation $w = M(\sigma)g$ is parameterized by the polynomial matrix $M \in \mathbb{R}^{(\mathtt{m}+\mathtt{p})\times\mathtt{m}}[z]$.

The Hankel matrix with t block rows, composed of the signal $w \in (\mathbb{R}^{\mathtt{w}})^T$ is denoted by

$$\mathscr{H}_t(w) := \begin{bmatrix} w(1) & w(2) & \cdots & w(T-t+1) \\ w(2) & w(3) & \cdots & w(T-t+2) \\ w(3) & w(4) & \cdots & w(T-t+3) \\ \vdots & \vdots & & \vdots \\ w(t) & w(t+1) & \cdots & w(T) \end{bmatrix}.$$

The signal $u = \big(u(1),\dots,u(T)\big)$ is called *persistently exciting* of order L if the Hankel matrix $\mathscr{H}_L(u)$ is of full row rank.

The banded upper-triangular Toeplitz matrix with t block-columns, related to the polynomial $r \in \mathbb{R}^{1\times\mathtt{r}}[z]$, $\deg(r) =: n$ is denoted by

$$\mathscr{T}_t(r) := \begin{bmatrix} r_0 & r_1 & \cdots & r_n & 0 & \cdots & 0 \\ 0 & r_0 & r_1 & \cdots & r_n & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & 0 \\ 0 & \cdots & 0 & r_0 & r_1 & \cdots & r_n \end{bmatrix}.$$

CLOSED-LOOP DATA-DRIVEN SIMULATION

The data-driven simulation problem is defined in Markovsky et al. (2005), where its applications in subspace system identification Van Overschee and De Moor (1996); Markovsky et al. (2006) are presented. Its fundamental role in data-driven control is shown in Markovsky and Rapisarda (2007, 2008). This paper further develops the concept of data-driven simulation to closed-loop data-driven simulation, defined as follows.

Problem 1. (Closed-loop data-driven simulation). Given

- trajectory $w_{\text{d}} = \big(w_{\text{d}}(1),\dots,w_{\text{d}}(T)\big) \in (\mathbb{R}^{\mathtt{w}})^T$ of an LTI system $\mathscr{B} \subset (\mathbb{R}^{\mathtt{w}})^{\mathbb{N}}$, with an input/output partition $w = (u,y) \in \mathscr{B}$, $u \in (\mathbb{R}^{\mathtt{m}})^{\mathbb{N}}$ input, $y \in (\mathbb{R}^{\mathtt{p}})^{\mathbb{N}}$ output;
- LTI controller $\mathscr{C} \subset (\mathbb{R}^{\mathtt{r}+\mathtt{p}+\mathtt{m}})^{\mathbb{N}}$, with an input/output partition $(r,u,y) \in \mathscr{C}$, $r \in (\mathbb{R}^{\mathtt{r}})^{\mathbb{N}}$, $y \in (\mathbb{R}^{\mathtt{p}})^{\mathbb{N}}$ inputs, $u \in (\mathbb{R}^{\mathtt{m}})^{\mathbb{N}}$ output; and
- reference signal $r_{\text{r}} = \big(r_{\text{r}}(1),\dots,r_{\text{r}}(T_{\text{r}})\big) \in (\mathbb{R}^{\mathtt{r}})^{T_{\text{r}}}$

find the set of responses w_{r} of the closed-loop system $\mathscr{B}_{\mathscr{C}}$ to the reference signal r_{r}.

Solution and computational algorithm

For given w_{d}, $\mathscr{C}$, r_{r}, we aim to compute the signals w_{r}, such that

$$(r_{\text{r}}, w_{\text{r}}) \in \mathscr{B}_{\mathscr{C}} \quad \Longleftrightarrow \quad \begin{cases} w_{\text{r}} \in \mathscr{B} \\ (r_{\text{r}}, w_{\text{r}}) \in \mathscr{C} \end{cases} \tag{1}$$

Assumption that $\mathscr{B}$ is controllable, there is $M \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}[z]$, such that

$$\mathscr{B} = \operatorname{image}\big(M(\sigma)\big).$$

Consider a kernel representation of the controller

$$\mathscr{C} = \{\, (r,w) \mid R(\sigma)\operatorname{col}(r,w) = 0 \,\}.$$

In terms of the image and kernel representations of the plant and controller, (1) becomes

$$\begin{cases} \exists\, g, \text{ s.t. } w_{\mathrm{r}} = M(\sigma)g \\ R(\sigma)\operatorname{col}(r_{\mathrm{r}}, w_{\mathrm{r}}) = 0. \end{cases} \tag{2}$$

We assume that the controller $\mathscr{C}$ is specified by a kernel representation, however, the plant $\mathscr{B}$ is only implicitly specified by the trajectory w_{d}. In order to avoid using a representation of $\mathscr{B}$, we solves (2) for w_{r}, replacing the first equation by

$$w_{\mathrm{r}} = \mathscr{H}_{T_{\mathrm{r}}}(w_{\mathrm{d}})g. \tag{3}$$

The equivalence of $w_{\mathrm{r}} = M(\sigma)g$ and (3) holds under the following assumptions:

(1) the system $\mathscr{B}$ is controllable,
(2) the input component u_{d} of w_{d} is persistently exciting of order $T_{\mathrm{r}} + \mathbf{n}(\mathscr{B})$, where $\mathbf{n}(\mathscr{B})$ is the order of $\mathscr{B}$,

see (Markovsky et al., 2006, Section 8.4) and (Willems et al., 2005, Theorem 1). Therefore, under assumptions 1 and 2, the set of solutions w_{r} of the linear system of equations

$$\begin{aligned} w_{\mathrm{r}} &= \mathscr{H}_{T_{\mathrm{r}}}(w_{\mathrm{d}})g \\ \mathscr{T}_{T_{\mathrm{r}}}(R)\operatorname{col}(r_{\mathrm{r}}, w_{\mathrm{r}}) &= 0 \end{aligned} \tag{4}$$

is equal to the set of trajectories w_{r} solving the closed-loop data-driven simulation problem.

Let $R =: [R_r \;\; R_w]$, where $R_r \in \mathbb{R}^{1\times \mathtt{r}}[z]$ and $R_w \in \mathbb{R}^{1\times \mathtt{w}}[z]$.

$$\mathscr{T}_{T_{\mathrm{r}}}(R)\operatorname{col}(r_{\mathrm{r}}, w_{\mathrm{r}}) = 0 \implies \mathscr{T}_{T_{\mathrm{r}}}(R_w)w_{\mathrm{r}} = -\mathscr{T}_{T_{\mathrm{r}}}(R_r)r_{\mathrm{r}}. \tag{5}$$

Substituting $w_{\mathrm{r}} = \mathscr{H}_{T_{\mathrm{r}}}(w_{\mathrm{d}})g$ into (5) gives the following system of equations

$$\underbrace{\mathscr{T}_{T_{\mathrm{r}}}(R_w)\mathscr{H}_{T_{\mathrm{r}}}(w_{\mathrm{d}})}_{A}\, g = \underbrace{-\mathscr{T}_{T_{\mathrm{r}}}(R_r)r_{\mathrm{r}}}_{b}$$

The matrix A is $T_{\mathrm{r}} \times (\mathtt{r}+\mathtt{w})T_{\mathrm{r}}$, so that the system $Ag = b$ is underdetermined. Let g_0 be a particular solution, *e.g.*, the least-norm solution $g_0 = A^{\dagger}b$ and let N be a matrix whose columns span the null space of A. The set of solution of (4) for g is

$$\mathscr{G} := \{\, g_0 + Nz \mid z \in \mathbb{R}^{\operatorname{col}\dim(N)} \,\}.$$

Then the set of responses w_{r} of the closed-loop system $\mathscr{B}_{\mathscr{C}}$ to the reference signal r_{r} is

$$\mathscr{W}_{\mathrm{r}} = \mathscr{H}_{T_{\mathrm{r}}}(w_{\mathrm{d}})\mathscr{G} = \{\, \underbrace{\mathscr{H}_{T_{\mathrm{r}}}(w_{\mathrm{d}})g_0}_{w_{\mathrm{r},0}} + \mathscr{H}_{T_{\mathrm{r}}}(w_{\mathrm{d}})Nz \mid z \in \mathbb{R}^{\operatorname{col}\dim(N)} \,\}.$$

It is characterized by the particular response $w_{\mathrm{r},0}$ and a subspace—the column span of the matrix $\mathscr{H}_{T_{\mathrm{r}}}(w_{\mathrm{d}})N$. Algorithm 1 summarizes the necessary steps for data-driven computation of $\mathscr{W}_{\mathrm{r}}$ from w_{d}, R, and r_{r}.

Algorithm 1 Closed-loop data-driven simulation.

Input: $w_{\mathrm{d}} \in (\mathbb{R}^{\mathtt{w}})^{T}$, $R \in \mathbb{R}^{1\times(\mathtt{r}+\mathtt{w})}[z]$, and $r_{\mathrm{r}} \in (\mathbb{R}^{\mathtt{r}})^{T_{\mathrm{r}}}$.

1: Compute the least-norm solution g_0 of the system of equations $\mathscr{T}_{T_{\mathrm{r}}}(R_w)\mathscr{H}_{T_{\mathrm{r}}}(w_{\mathrm{d}})g = -\mathscr{T}_{T_{\mathrm{r}}}(R_r)r_{\mathrm{r}}$.
2: Let $w_{\mathrm{r},0} := \mathscr{H}_{T_{\mathrm{r}}}(w_{\mathrm{d}})g_0$.
3: Compute a matrix N which columns form a basis for the column span of $\mathscr{T}_{T_{\mathrm{r}}}(R_w)\mathscr{H}_{T_{\mathrm{r}}}(w_{\mathrm{d}})$.
4: Let N_w be a basis for the column span of $\mathscr{H}_{T_{\mathrm{r}}}(w_{\mathrm{d}})N$.

Output: $w_{\mathrm{r},0}$ and N_w.

We have shown that under assumptions 1 and 2, the set

$$\mathscr{W}_{\mathrm{r}} := \{\, w_{\mathrm{r},0} + N_w z \mid z \in \mathbb{R}^{\operatorname{col}\dim(N_w)} \,\}.$$

computed by Algorithm 1 is equal to the set of T_{r} samples long responses of the closed-loop system $\mathscr{B}_{\mathscr{C}}$ to the reference signal r_{r}, *i.e.*,

$$\mathscr{W}_{\mathrm{r}} = \{\, w \in (\mathbb{R}^{\mathtt{w}})^{T_{\mathrm{r}}} \mid (r_{\mathrm{r}}, w) \in \mathscr{B}_{\mathscr{C}}|_{T_{\mathrm{r}}} \,\}.$$

Simulation example

The data $w_{\mathrm{d}} = (u_{\mathrm{d}}, y_{\mathrm{d}})$, used for the closed-loop data driven simulation, is the first 10 samples from the step response of a randomly generated first order system $\mathscr{B}$ interconnected with the controller $\mathscr{C}_1 := \{\, (r,u,y) \mid u = -y \,\}$. The aim is to compute the first $T_{\mathrm{r}} = 10$ samples of the step response of $\mathscr{B}_{\mathscr{C}_2}$, where $\mathscr{C}_2 := \{\, (r,u,y) \mid u = y \,\}$. For this purpose we use Algorithm 1, *i.e.*, we do not compute explicitly a representation of $\mathscr{B}$. Note that either of the systems $\mathscr{B}_{\mathscr{C}_1}$, $\mathscr{B}_{\mathscr{C}_2}$ can be unstable.

In order to ensure that assumption 2 is satisfied, we augment the given trajectory—the step response of $\mathscr{B}_{\mathscr{C}_1}$—with T_{r} zeros. This takes into account the zero initial conditions of the given trajectory and ensures that the initial conditions of the computed response r_{r} of $\mathscr{B}_{\mathscr{C}_1}$ are also zero (*i.e.*, $N_w = 0$). The results for a particular system $\mathscr{B}$ are shown in Figure 1. We verify that up to numerical errors r_{r} matches the step response s of $\mathscr{B}_{\mathscr{C}_2}$, obtained by model-based simulation. A MATLAB file reproducing the simulation result is available from:

`http://users.ecs.soton.ac.uk/im/test_cdds.m`

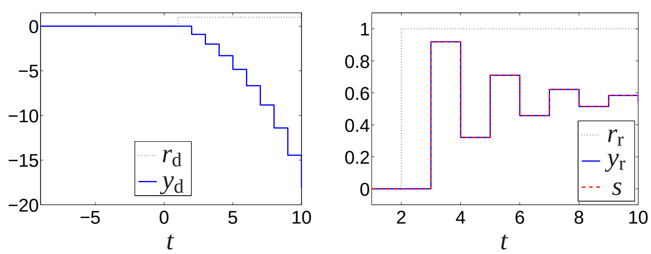


Fig. 1. Step responses of $\mathscr{B}_{\mathscr{C}_1}$ (left) and $\mathscr{B}_{\mathscr{C}_2}$ (right).

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