Nonlinear High Gain Separation Principles and Fast Sampling Results ensuring Robust Stability

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Abstract

A semi-global nonlinear separation principle is described which presents conditions under which a stabilizing controller based on the output and its derivatives (typically the state or partial state) can be replaced by a controller based on measurement of the output only. The results include the case of high gain observer reconstruction of the output derivatives and the case of approximating the output derivatives via numerical derivatives for which various realizable schemes based on discrete sampling are given. The same sampling constructions and results are also applied directly to the output measurement itself, giving rise to fast sampling theorems. The proofs are based on estimating gap distances between the original controller and the reconstructed approximation and conditions based on the robust stability margin.

1 Introduction

The long established linear separation principle states that for every stabilizing state feedback controller, a corresponding output feedback controller can be constructed using a suitable observer and realizing the original state feedback with the corresponding observer states. In the linear context, it is similarly well known and long established that digital controllers can be designed via a process of fast sample and hold emulation of a continuous time design. The purpose of this paper is to provide a wide-ranging extension of both these results in a general nonlinear context.

There is a substantive literature which addresses the generalisation of the linear separation principle to nonlinear systems. Typical results utilize high gain linear observers to reconstruct the state of the system which can then be used under appropriate conditions to construct a suitable stabilizing feedback. Such results have been obtained by a state space analysis based on singular perturbation theory and require a time scale separation of the observer dynamics from the system dynamics. The separation principles we present here are related to the above, but give rise to different conditions and include both the case of controllers based on reconstructions of the state via observers and those based on numerical differentiation. The approach is technically very different; here we utilize gap and graph perturbation techniques from the theory of nonlinear robust control, as opposed

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to the state-space driven techniques of [3], see also [22, 23, 21]. One immediate benefit of the operator approach taken is that the results apply to any plant which can be stabilized by memoryless feedback of the output and its derivatives; it is not necessary for an underlying finite dimensional state-space model to be known, or even to exist. Furthermore, again in contrast to [3], the analysis inherently includes the effects of disturbances in both the input and output channels and guarantees robust stability [15].

The formal treatment of the stability of controllers based on numerical differentiation appears to have been first undertaken in [18] (limited to linear plants which are minimum phase and either relative degree one or two), although note that such schemes for implementing state feedbacks for nonlinear systems are common in applications. The analysis in [18] involves a detailed state space construction of a Lyapunov-Krasovskii function and would appear to be hard to generalize. Subsequently an alternative approach based on the gap metric was developed in [11] which established global results in a variety of signal space settings. The analysis in the particular case of linear minimum phase systems was rather complete. The results of [11] are limited in the general nonlinear context since they require global closed loop gain stability, and are thus typically restricted to plants and controllers which have linear growth conditions.

This paper provides a general semi-global analysis, thus removing the growth restrictions from the analysis in [11]. The passage from global to semi-global is not elementary. The underlying robust stability theorem utilizes Schauder fixed point theory, and gives rise to a requirement of compactness of a certain operator arising in the analysis, and detailed analysis is required to establish this compactness in a variety of signal space settings. The resulting output feedback controller can be constructed from a variety of different operations to reconstruct the derivatives. We demonstrate that appropriate reconstructions include the basic Euler approximation considered in [11, 18]; two sampled versions of the Euler approximation scheme and methods based on high gain observers.

We remark that it has been well argued e.g. in [17] that such differentiation schemes (or other approximate reconstructions such as high gain observers) may be essential for many nonlinear systems, simply because it seems that it is only possible to construct (exact) nonlinear observers for certain specialized classes of plants. We further remark that there is a wide literature in both the control and signal processing concerning nonlinear estimation and reconstruction using differentiation, see e.g. [9] and the references therein, however, the results in [11, 18] and in this paper are distinguished by the consideration of closed loop robust stability.

The fast sampling results established in this paper are obtained by applying the same sampling constructions utilized in the sampled versions of the Euler approximations, but now applied directly to the output channel. In particular, we give conditions under which an output feedback controller can be replaced by a sampled data controller via a process of zero or first order hold sampling of the original controller. This substantially widens the signal setting of the previous input-output approach to sampled data controllers [4] and contrasts to the wider literature on state-space methods for this problem [2, 8, 24, 25, 26, 27, 28, 32] in a similar manner to the separation principle results: the results apply equally in finite and infinite dimensional contexts and the analysis inherently includes the effects of disturbances in both the input and output channels and automatically guarantees positive robust stability margins in the sense of the gap metric [15].

From the vantage point of nonlinear robust stability theory, this paper provides another substantive illustration of the power of the gap metric and robust stability theory [5, 15, 20]. This follows on from other major applications of the framework to the analysis of nonlinear oscillators [16], the analysis of robustness properties of adaptive controllers [6, 7, 10, 12, 13] and iterative learning controllers [10], and a previous analysis of sampled data controllers [4].

The paper is structured as follows. In section 2 we introduce the notation and system theoretic properties. In section 3 we investigate the properties of Euler operators, and show that both observer based and numerical derivative based constructions have the required properties. In section 4 we state and prove the nonlinear separation principles and discuss a number of corollaries and examples. Section 5 relates the required notion of gain stability in the Sobolev space setting to the standard notion of L^p gain stability under a relative degree assumption. Section 6 considers the sampling process as applied also to the output measurement itself and establishes conditions for robust stability under fast sampling. Section 7 shows the previously developed results for the initial condition free case also imply results for the case of non-zero initial conditions under reachability assumptions. Conclusions are given in section 8, and an appendix contains the statement and proof of the underlying robust stability result.

2 Background and notation

2.1 Function spaces

Let $\mathrm{map}(E, F)$ be the set of all maps from the set E to the set F. We define the domain of a mapping $x \in \mathrm{map}(E, F)$ as $\mathrm{dom}(x) = E$. Let $N \geq 1$ and let $I \subset \mathbb{R}_+$ be an interval. Then we let $BUC(I, \mathbb{R}^N)$ denote the space of uniformly continuous functions $x \colon I \to \mathbb{R}^N$ with the uniform norm $\|x\|_{BUC(I,\mathbb{R}^N)} := \sup_{t \in I} |x(t)|$ and $L^\infty(I, \mathbb{R}^N)$ denote the space of all bounded functions $x \colon I \to \mathbb{R}^N$ with the norm $\|x\|_{L^\infty(I,\mathbb{R}^N)} := \operatorname{ess\,sup}_{t \in I} |x(t)|$. When I is compact, we let $C(I, \mathbb{R}^N)$ denote the space $BUC(I, \mathbb{R}^N)$ since all continuous functions on I are bounded and uniformly continuous. For $1 \leq p < \infty$ we let $L^p(I, \mathbb{R}^N)$ denote the space of all measurable functions $x \colon I \to \mathbb{R}^N$ with $\int_I |x(t)|^p \, \mathrm{d}t < \infty$ and with norm $x \mapsto \|x\|_{L^p(I,\mathbb{R}^N)} := \left(\int_I |x(t)|^p \, \mathrm{d}t\right)^{\frac{1}{p}}$, We let $L^p_{\mathrm{loc}}(I, \mathbb{R}^N)$ denote the set of all functions $x \colon I \to \mathbb{R}^N$ with $\int_K |x(t)|^p \mathrm{d}t < \infty$ for all compact $K \subset I$. For $0 \leq r \leq \infty$ let $C^r(I, \mathbb{R}^N)$ denote the set of all uniformly continuous, r-times differential functions from I to $\mathbb{R}^N$. Suppose $0 \in I \subset \mathbb{R}_+$, define

$$C^r_0(I, \mathbb{R}^N) := \{y \in C^r(I, \mathbb{R}^N) \mid y^{(j)}(0) = 0, \ 0 \leq j \leq r - 1\},$$

and then let

$$CW^{r,p}(I, \mathbb{R}^N) := \left\{ y \in C^r(I, \mathbb{R}^N) \;\middle|\; y^{(i)} \in L^p(I, \mathbb{R}^N), \ 0 \leq i \leq r, \ \sum_{i=0}^{r} \|y^{(i)}\|_{L^p(I,\mathbb{R})} < \infty \right\},$$

$$CW^{r,p}_0(I, \mathbb{R}^N) := CW^{r,p}(I, \mathbb{R}^N) \cap C^r_0(I, \mathbb{R}^N).$$

with norm $\|\cdot\|_{W^{r,p}(I,\mathbb{R}^N)} = \|\cdot\|_{CW^{r,p}(I,\mathbb{R}^N)} = \|\cdot\|_{CW^{r,p}_0(I,\mathbb{R}^N)}$, defined by the mapping

$$x \mapsto \|x\|_{W^{r,p}(I,\mathbb{R}^N)} := \sum_{i=0}^{r} \|x^{(i)}\|_{L^p(I,\mathbb{R}^N)}.$$

We let $W^{r,p}(I,\mathbb{R}^N)$ denote the Sobolev spaces of r-times weakly differential functions which is equal to the completion of $C^r(I,\mathbb{R}^N)$ w.r.t. to the norm $\|\cdot\|_{W^{r,p}(I,\mathbb{R}^N)}$, see [1], noting that the weak derivative $y^{(i)}$ coincides with the classical derivative when $y \in C^i$. We let $W_0^{r,p}(I,\mathbb{R}^N)$ denote the closure of $C_0^r(I,\mathbb{R}^N)$ in $W^{r,p}(I,\mathbb{R}^N)$. Note that for intervals $I \subset \mathbb{R}_+$, the spaces $L^p(I,\mathbb{R}^N)$, $W^{r,p}(I,\mathbb{R}^N)$ and $W_0^{r,p}(I,\mathbb{R}^N)$, $1 \le p \le \infty$ are Banach spaces. If $I \subset \mathbb{R}_+$ is compact then $C(I,\mathbb{R}^N)$, $CW^{r,\infty}(I,\mathbb{R}^N)$, $CW_0^{r,\infty}(I,\mathbb{R}^N)$ are complete.

The Sobolev embedding theorem [1, Ch. 5] includes the statement that if $rp > 1$, then $W^{r,p}(I,\mathbb{R}^N)$ is embedded into $CW^{r,\infty}(I,\mathbb{R}^N)$, that is there exists $M > 0$ such that for every element $[y] \in W^{r,p}(I,\mathbb{R}^N)$ (noting that elements of $W^{r,p}(I,\mathbb{R}^N)$ are equivalence classes of functions equal a.e.) there exists $x \in CW^{r,\infty}(I,\mathbb{R}^N)$ such that $x \in [y]$, that is $y = x$ a.e., and $\|x\|_{CW^{r,\infty}(I,\mathbb{R}^N)} \le M\|y\|_{W^{r,p}(I,\mathbb{R}^N)}$. Consequently $W_0^{r,p}(I,\mathbb{R}^N)$ is embedded into $CW_0^{r,\infty}(I,\mathbb{R}^N)$, this is established as follows. For $[y] \in W_0^{r,p}(I,\mathbb{R}^N)$, there exists $y_n \in CW_0^{r,p}(I,\mathbb{R}^N)$, $n \ge 1$, such that $\|y_n - y\|_{W_0^{r,p}(I,\mathbb{R}^N)} \to 0$ as $n \to \infty$. Since $CW^{r,\infty}(I,\mathbb{R}^N)$ is embedded in $W^{r,p}(I,\mathbb{R}^N)$, it follows that there exists $x \in CW^{r,\infty}(I,\mathbb{R}^N)$, $x \in [y]$ such that $\|y_n - x\|_{CW^{r,\infty}(I,\mathbb{R}^N)} \le M\|y_n - x\|_{W^{r,p}(I,\mathbb{R}^N)} \to 0$ as $n \to \infty$. In particular this shows that $x^{(i)}(0) = 0$, $0 \le i \le r$ since $y_n^{(i)}(0) = 0$ for $0 \le i \le r$, thus $x \in CW_0^{r,\infty}(I,\mathbb{R}^N)$ as required.

2.2 Systems

The material in this section is based on [15, Sec. II], [10, Sec. 2], [11, Sec. 2].

Let $\mathcal{X}$ be a nonempty set. For $0 < \omega \le \infty$ let $\mathcal{S}_\omega$ denote the set of all locally integrable maps in $\mathrm{map}([0,\omega) \to \mathcal{X})$. For ease of notation define $\mathcal{S} := \mathcal{S}_\infty$. For $0 < \tau < \omega \le \infty$ define a truncation operator T_τ and a restriction operator R_τ as follows:

$$T_\tau : \mathcal{S}_\omega \to \mathcal{S}\,, \quad v \mapsto T_\tau v := \left(t \mapsto \begin{cases} v(t), & t \in [0,\tau) \\ 0, & \text{otherwise} \end{cases} \right),$$
$$R_\tau : \mathcal{S}_\omega \to \mathcal{S}_\tau\,, \quad v \mapsto R_\tau v := (t \mapsto v(t), \quad t \in [0,\tau))\,.$$

We define $\mathcal{V} \subset \mathcal{S}$ to be a *signal space* if, and only if, it is a vector space. Suppose additionally that $\mathcal{V}$ is a normed vector space and that the norm $\|\cdot\| = \|\cdot\|_\mathcal{V}$ is (also) defined for signals of the form $T_\tau v$, $v \in \mathcal{V}$, $\tau > 0$. We can define a norm $\|\cdot\|_\tau$ on $\mathcal{S}_\tau$ by $\|v\|_\tau = \|T_\tau v\|$, for $v \in \mathcal{S}_\tau$. We associate spaces as follows:

- $\mathcal{V}[0,\tau) = \{v \in \mathcal{S}_\tau \mid \exists\, w \in \mathcal{V} \text{ with } \|T_\tau w\|_\mathcal{V} < \infty \;:\; v = R_\tau w\}$, for $\tau > 0$,
- $\mathcal{V}_e = \{v \in \mathcal{S} \mid \forall\, \tau > 0 \;:\; R_\tau v \in \mathcal{V}[0,\tau)\}$, *the extended space*;
- $\mathcal{V}_\omega = \{v \in \mathcal{S}_\omega \mid \forall\, \tau \in (0,\omega) \;:\; R_\tau v \in \mathcal{V}[0,\tau)\}$, for $0 < \omega \le \infty$; and
- $\mathcal{V}_a = \bigcup_{\omega \in (0,\infty]} \mathcal{V}_\omega$, *the ambient space.*

A mapping $Q\colon \mathcal{U}_a \to \mathcal{Y}_a$ is said to be *causal* if, and only if,

$$\forall\, x, y \in \mathcal{U}_a \;\forall\, \tau \in \mathrm{dom}(x) \cap \mathrm{dom}(Qx) \;:\; [R_\tau x = R_\tau y \;\Rightarrow\; R_\tau Q x = R_\tau Q y]\,.$$

Let $P : \mathcal{U}_a \to \mathcal{Y}_a$ and $C : \mathcal{Y}_a \to \mathcal{U}_a$ be causal mappings representing the plant and the controller, respectively. Consider the system of equations

$$[P, C] \ : \quad y_1 = Pu_1, \quad u_2 = Cy_2, \quad u_0 = u_1 + u_2, \quad y_0 = y_1 + y_2 \tag{2.1}$$

corresponding to the closed-loop feedback configuration in Figure 1.

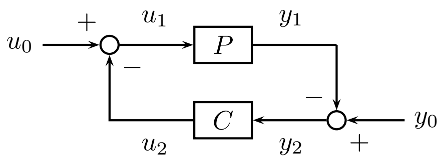

Figure 1: The closed-loop system $[P, C]$

For $w_0 = (u_0, y_0)^T \in \mathcal{W} := \mathcal{U} \times \mathcal{Y}$ a pair $(w_1, w_2) = ((u_1, y_1)^T, (u_2, y_2)^T) \in \mathcal{W}_a \times \mathcal{W}_a$, $\mathcal{W}_a := \mathcal{U}_a \times \mathcal{Y}_a$, is a solution for $[P, C]$ if, and only if, (2.1) holds on $\mathrm{dom}(w_1, w_2)$.
Let $\mathcal{X}_{w_0} := \{(w_1, w_2) \in \mathcal{W}_a \times \mathcal{W}_a \mid (w_1, w_2) \text{ solves } (2.1)\}$ be the set of all solutions, which may be empty. The closed loop $[P, C]$ is said to have the *existence property* if $\mathcal{X}_{w_0} \neq \emptyset$, and the *uniqueness property* if

$$\forall \ w_0 \in \mathcal{W} \ : \ (\hat{w}_1, \hat{w}_2), (\tilde{w}_1, \tilde{w}_2) \in \mathcal{X}_{w_0} \implies (\hat{w}_1, \hat{w}_2) = (\tilde{w}_1, \tilde{w}_2) \quad \text{on} \quad \mathrm{dom}(\hat{w}_1, \hat{w}_2) \cap \mathrm{dom}(\tilde{w}_1, \tilde{w}_2).$$

For each $w_0 \in \mathcal{W}$, define $0 < \omega_{w_0} \leq \infty$, by the property $(0, \omega_{w_0}) := \bigcup_{(\hat{w}_1, \hat{w}_2) \in \mathcal{X}_{w_0}} \mathrm{dom}(\hat{w}_1, \hat{w}_2)$ and define $(w_1, w_2) \in \mathcal{W}_a \times \mathcal{W}_a$, with $\mathrm{dom}(w_1, w_2) = (0, \omega_{w_0})$, by the property $(w_1, w_2)|_{(0,t)} \in \mathcal{X}_{w_0}$ for all $t \in (0, \omega_{w_0})$. This induces the operator

$$H_{P,C} : \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a, \quad w_0 \mapsto (w_1, w_2).$$

For $\Omega \subset \mathcal{W}$ the closed loop system $[P, C]$ given by (2.1), is said to be:

- *locally well posed on* Ω if, and only if, it has the existence and uniqueness properties and the operator $H_{P,C}\big|_{\Omega} : \Omega \to \mathcal{W}_a \times \mathcal{W}_a$, $w_0 \mapsto (w_1, w_2)$, is causal;
- *regularly well posed on* Ω if, and only if, it is locally well posed and

$$\forall \ w_0 \in \Omega \quad \Big[\ \omega_{w_0} < \infty \quad \Rightarrow \quad \big\| R_\tau H_{P,C} w_0 \big\|_{\mathcal{W}_\tau \times \mathcal{W}_\tau} \to \infty \text{ as } \tau \to \omega_{w_0} \ \Big]. \tag{2.2}$$

- *globally well posed on* Ω if, and only if, it is locally well posed on Ω and $H_{P,C}(\Omega) \subset \mathcal{W}_e \times \mathcal{W}_e$;

For the plant operator P and the controller operator C define the *graph* $\mathcal{G}_P$ of the plant and the *graph* $\mathcal{G}_C$ of the controller, respectively, as follows:

$$\mathcal{G}_P := \left\{ \begin{pmatrix} u \\ Pu \end{pmatrix} \in \mathcal{W}_a \ \middle| \ u \in \mathcal{U}, \, Pu \in \mathcal{Y} \right\} \subset \mathcal{W}, \quad \mathcal{G}_C := \left\{ \begin{pmatrix} Cy \\ y \end{pmatrix} \in \mathcal{W}_a \ \middle| \ Cy \in \mathcal{U}, \, y \in \mathcal{Y} \right\} \subset \mathcal{W}.$$

P is said to be *stabilizable* [10] (or causally extendible in [11, 14]) if, and only if for all $w_1 = (u_1, y_1) \in \mathcal{W}_a$ satisfying $Pu_1 = y_1$ and for all $\tau \in \mathrm{dom}(w_1)$, there exists $w_1' \in \mathcal{G}_P$ such that $R_\tau w_1 = R_\tau w_1'$.

Next define the operators

$$\Pi_{P//C}\colon \mathcal{W} \to \mathcal{W}_a, \quad w_0 \mapsto w_1, \qquad \text{and} \qquad \Pi_{C//P}\colon \mathcal{W} \to \mathcal{W}_a, \quad w_0 \mapsto w_2.$$

Clearly, $H_{P,C} = \left(\Pi_{P//C}, \Pi_{C//P}\right)$ and $\Pi_{P//C} + \Pi_{C//P} = I$.

For normed signal spaces $\mathcal{X}$, $\mathcal{V}$, let $\mathcal{B}_R = \mathcal{B}_{R,\mathcal{X}}(0) \subset \mathcal{X}$ denote the ball centred at 0 and of radius $r > 0$ in $\mathcal{X}$, and define the following:

- A causal operator $Q\colon \mathcal{X} \to \mathcal{V}_a$ is called *gain stable on* $\mathcal{B}_R$ if, and only if, $Q(\mathcal{X}) \subset \mathcal{V}$, $Q(0) = 0$, and

$$\left\| Q|_{\mathcal{B}_R} \right\|_{\mathcal{X},\mathcal{V}} := \sup \left\{ \frac{\|R_\tau Q x\|_\tau}{\|R_\tau x\|_\tau} \;\middle|\; x \in \mathcal{X}\,,\ \|R_\tau x\|_\tau \le R \ \ \tau > 0\,,\ R_\tau x \neq 0 \right\} < \infty\,.$$

- A causal operator $Q\colon \mathcal{X} \to \mathcal{V}_a$ is called *globally gain-function stable* if, and only if, $Q(\mathcal{X}) \subset \mathcal{V}$ and the nonlinear so-called *gain-function*

$$g\,[Q] : [0,\infty) \to [0,\infty), \qquad r \mapsto g\,[Q]\,(r) := \sup \left\{ \|R_\tau Q x\|_\tau \;\middle|\; x \in \mathcal{X}\,,\ \|R_\tau x\|_\tau \le r\,,\ \tau > 0 \right\}\,,$$

  is defined.

For normed signal spaces $\mathcal{U}$, $\mathcal{Y}$ and $\mathcal{W} := \mathcal{U} \times \mathcal{Y}$ and the causal operator $P\colon \mathcal{U}_a \to \mathcal{Y}_a$ and $C\colon \mathcal{Y}_a \to \mathcal{U}_a$ define the following:

- The closed-loop system $[P, C]$ given by (2.1) with the associated operator $H_{P,C}\colon \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a$ is said to be $\mathcal{W}$-*stable* if, and only if, it is globally well posed on $\mathcal{W}$ and $H_{P,C}(\mathcal{W}) \subset \mathcal{W} \times \mathcal{W}$.

- The closed-loop system $[P, C]$ given by (2.1) with the associated operator $H_{P,C}\colon \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a$ is said to be *gain stable on* $\mathcal{B}_{R,\mathcal{W}}(0)$ if, and only if, it is globally well posed on $\mathcal{B}_{R,\mathcal{W}}(0)$ and $H_{P,C}$ is gain stable on $\mathcal{B}_{R,\mathcal{W}}(0)$.

- The closed-loop system $[P, C]$ given by (2.1) with the associated operator $H_{P,C}\colon \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a$ is said to be *globally gain-function stable* if, and only if, it is globally well posed on $\mathcal{W}$ and $H_{P,C}$ is globally gain-function stable.

- Consider the causal operator $P\colon \mathcal{U}_a \to \mathcal{Y}_a$ and a one parameter family of operators $\{C[\lambda]\colon \mathcal{Y}_a \to \mathcal{U}_a\}_{\lambda>0}$. The parameterized closed-loop system $[P, C[\cdot]]$ given by (2.1) is said to be *semi-globally gain stable* if, and only if, for all $R > 0$ there exists $\lambda > 0$ such that the closed loop system $[P, C[\lambda]]$ is gain stable on $\mathcal{B}_{R,\mathcal{W}}(0)$.

3 Euler approximations and their properties

For $1 \le p \le \infty$, $N \ge 1$ and $1 \le k \le r \le \infty$, let $\mathcal{U} := L^p(\mathbb{R}_+, \mathbb{R})$ and $\mathcal{Y} := CW_0^{r,p}(\mathbb{R}_+, \mathbb{R}^N)$ or $\mathcal{Y} := W_0^{r,p}(\mathbb{R}_+, \mathbb{R}^N)$. We define the differentiation operator ∂^i, $i \le r$:

$$\partial^i\colon \mathcal{Y}_a \to \partial^i \mathcal{Y}_a \subset L_a^p(\mathbb{R}_+, \mathbb{R}^N), \quad : \quad \partial^i y = y^{(i)},$$

Let $\partial_k \colon \mathcal{Y}_a \to \mathcal{Y}_a^k$ be defined by $\partial_k = (I, \partial^1, \dots \partial^k)$, where $\mathcal{Y}^k = \mathcal{V} \times \cdots \times \mathcal{V}$ is the Cartesian product of k copies of $\mathcal{V} = L^p(\mathbb{R}_+, \mathbb{R}^N)$. In this paper we consider causal, $\mathcal{U} \times \mathcal{Y}$-stable controllers of the form

$$C_F \colon \mathcal{Y}_a \to \mathcal{U}_a \quad : \quad C_F = F \circ \partial_k \tag{3.1}$$

where $F \colon \mathcal{Y}_a^k \to \mathcal{U}_a$ and $F(\mathcal{Y}^k) \subset \mathcal{U}$. The Euler controller is defined to be:

$$C_F^{\mathrm{Euler}}[h] \colon \mathcal{Y}_a \to \mathcal{U}_a \quad : \quad C_F^{\mathrm{Euler}}[h] = F \circ \Delta_{k,h}. \tag{3.2}$$

where the Euler operator $\Delta_{k,h} \colon \mathcal{Y}_a \to \mathcal{Y}_a^k$ belongs to a suitable class of approximations to the differentiation operator which will be described below. We will consider the output feedback controller (3.2) to form an approximation of the output derivative feedback controller (3.1), (in a manner which we will make precise in section 4) and we will study the question of when the stability of a closed loop system $[P, C_F]$ also guarantees the stability of $[P, C_F^{\mathrm{Euler}}[h]]$.

As a concrete example, given a (locally Lipschitz continuous) function $f \colon \mathbb{R}^2 \to \mathbb{R}$, we define the nonlinear proportional-derivative (PD) feedback

$$\begin{aligned} C_f : \mathcal{Y}_a &\to \mathcal{U}_a , \\ y &\mapsto u := f\left(y, y^{(1)}\right) , \end{aligned} \tag{3.3}$$

and the Euler controller:

$$\begin{aligned} C_f^{\mathrm{Euler}}[h] : \mathcal{Y}_a &\to \mathcal{U}_a , \\ y &\mapsto u := f\left(y, \tfrac{1}{h}(y(\cdot) - y(\cdot - h))\right) , \end{aligned} \tag{3.4}$$

where the signal $y^{(1)}$ (which is potentially unavailable for measurement) is replaced by an approximate reconstruction, thus requiring an output measurement of y only. We will later consider higher order and sampled versions of the above Euler controller, together with constructions based on high gain observers.

We now make precise the notion of $\Delta_{k,h}$ forming an approximation to ∂_k.

Definition 3.1 *Let $1 \le p \le \infty$, $N \ge 1$, $1 \le k \le r \le \infty$ and $h > 0$. Let $\mathcal{Y} = W^{r,p}(\mathbb{R}_+, \mathbb{R}^N)$, $CW^{r,p}(\mathbb{R}_+, \mathbb{R}^N)$, $W_0^{r,p}(\mathbb{R}_+, \mathbb{R}^N)$ or $CW_0^{r,p}(\mathbb{R}_+, \mathbb{R}^N)$. The map*

$$\Delta_{k,h} = (\Delta_{k,h}^0, \Delta_{k,h}^1, \dots, \Delta_{k,h}^k) : \mathcal{Y}_a \to \mathcal{Y}_a^k \tag{3.5}$$

is said to be an $\mathcal{Y}$ Euler operator *if it is linear, causal and if the inequality*

$$\left\| R_T(\Delta_{k,h}^i y - \partial^i y) \right\|_{L^p([0,T),\mathbb{R}^N)} \le \gamma_i(h) \left\| R_T y \right\|_{W^{r,p}([0,T),\mathbb{R}^N)} \tag{3.6}$$

holds for all $y \in \mathcal{Y}$, $T > 0$ and $0 \le i \le k$. The constant $\gamma_i(h)$ is called an Euler approximation constant*, and we define $\gamma(h) = \sum_{i=0}^k \gamma_i(h)$.*

Examples of Euler-operators and their Euler approximation constants now follow.

3.1 Euler operators from numerical approximations

In the first three examples, we specify two operators $\Delta^0_{k,h}, \Delta^1_{k,h} \; : \; \mathcal{Y}_a \to L^p_a(\mathbb{R}_+, \mathbb{R}^N)$, $k \geq 0$, $h > 0$ and define the operators $\Delta^\mu_{k,h}$, $\mu \geq 0$, by:

$$\begin{aligned} \Delta^\mu_{k,h} : \mathrm{map}(\mathbb{R}_+, \mathbb{R}^N) &\to \mathrm{map}(\mathbb{R}_+, \mathbb{R}^N)\,, \\ y &\mapsto \Delta^\mu_{k,h}(y) := \begin{cases} \Delta^{\mu-1}_{k,h}(\Delta_{k,h}(y)) & \text{if } \mu \geq 2 \\ \Delta^1_{k,h}(y) & \text{if } \mu = 1 \\ \Delta^0_{k,h}(y) & \text{if } \mu = 0\,. \end{cases} \end{aligned} \tag{3.7}$$

where

$$y \mapsto \Delta^0_{k,h}(y) \; := \; y \tag{3.8}$$

$$y \mapsto \Delta^1_{k,h}(y) \; := \; \left[\tfrac{1}{h}(\delta_h \circ D_{-h} - \delta_h)\right](y) = \tfrac{1}{h}(\delta_h(D_{-h}(y)) - \delta_h y), \tag{3.9}$$

for some $\delta_h \; : \; \mathrm{map}(\mathbb{R}_+, \mathbb{R}^N) \to \mathrm{map}(\mathbb{R}_+, \mathbb{R}^N)$ to be specified and where D_τ, $\tau \in \mathbb{R}$, denotes the delay operator:[1]

$$\begin{aligned} D_\tau \; : \; \mathrm{map}(\mathbb{R}_+, \mathbb{R}^N) &\to \mathrm{map}(\mathbb{R}_+, \mathbb{R}^N) \\ y &\mapsto D_\tau y := \begin{cases} 0 & \text{on } [0, \tau), \text{ if } \tau > 0 \\ y(\,\cdot - \tau) & \text{on } [\tau, \infty), \text{ if } \tau > 0 \\ y(\,\cdot - \tau) & \text{on } [0, \infty), \text{ if } \tau \leq 0\,. \end{cases} \end{aligned} \tag{3.10}$$

For $i \geq 1$, it is useful to observe that provided δ_h commutes with both D_h and D_{-h} then:

$$\Delta^i_{k,h}(y) = \tfrac{1}{h^i} \left(\sum_{j=0}^{i} (-1)^j \tbinom{i}{j} \delta^j_h(y) \right), \tag{3.11}$$

where

$$\delta^j_h(y)(t) = \delta_h D_{(j-1)h}(y)(t), \qquad j \geq 0. \tag{3.12}$$

It is trivial to see that if $\Delta^0_{k,h}$ is given by (3.8), then the Euler constant $\gamma_0(h)$ is zero for all $h > 0$.

1. The standard Euler operator is specified by taking δ_h to be the pure delay of length h:

$$y \;\mapsto\; \delta_h(y) := D_h(y). \tag{3.13}$$

   Thus the standard Euler operator is simply the Euler formula for the numerical derivative: $\Delta^0_{k,h}(y)(t) = \frac{1}{h}(y(t) - y(t-h))$. It is trivial to see that δ_h commutes with both D_h and D_{-h}, so (3.12) holds. We show in Theorem 3.4 below (see also [11] for the case of $N = 1$), that for either $1 \leq p < \infty$ and $1 \leq k \leq r - 2$, or $p = \infty$ and $1 \leq k \leq r - 1$, $\Delta_{k,h}$ is an $\mathcal{Y}$ Euler operator and that the Euler approximation constants for $\mathcal{Y} = CW^{r,p}_0(\mathbb{R}_+, \mathbb{R}^N)$ and $\mathcal{Y} = W^{r,p}_0(\mathbb{R}_+, \mathbb{R}^N)$ are given in the first row of table 1.

[1] For the function spaces $L^p(\mathbb{R}_+, \mathbb{R}^N)$, $W^{r,p}(\mathbb{R}_+, \mathbb{R}^N)$ where elements are equivalence classes of functions equal a.e., D_τ is naturally extended via: $D_\tau[x] = [\tilde{D}_\tau x]$ where $\tilde{D}_\tau \colon \mathrm{map}(\mathbb{R}_+, \mathbb{R}^N) \to \mathrm{map}(\mathbb{R}_+, \mathbb{R}^N)$ denotes the delay operator defined on $\mathrm{map}(\mathbb{R}_+, \mathbb{R}^N)$.

$y \mapsto \delta_h(y)$	Euler approximation constants for $CW_0^{r,p}(\mathbb{R}_+,\mathbb{R}^N)$ and $W_0^{r,p}(\mathbb{R}_+,\mathbb{R}^N)$ $1 \le i \le r-2$ if $p<\infty$, $1 \le i \le r-1$ if $p=\infty$	
$\delta_h = D_h$	$\gamma_i(h) = ihN(2(1+ihp))^{\frac{1}{p}}$	(3.14)
$\delta_h = H_h^0 \circ S_h^0, \quad rp>1$	$\gamma_i(h) = (i+1)hN\,(2(1+(i+1)hp))^{\frac{1}{p}}$	(3.15)
$\delta_h = H_h^1 \circ S_h^0, \quad rp>1$	$\gamma_i(h) = (2i+1)hN(2(1+(2i+1)hp))^{\frac{1}{p}}$	(3.16)
$\delta_h = H_h^0 \circ S_h^1$	$\gamma_i(h) = (i+2)hN\,(2(1+(i+2)hp))^{\frac{1}{p}}$	(3.17)
$\delta_h = H_h^1 \circ S_h^1$	$\gamma_i(h) = (3i+1)hN\,(2(1+(3i+1)hp))^{\frac{1}{p}}$	(3.18)

Table 1: Sampled Euler operators and their Euler approximation constants for $i \ge 1$.

Whilst the standard Euler operator is convenient for analysis, for implementation it suffers the serious drawback it can only be realised by storing the signal y on the interval $[t-h,t)$. This motivates the formalisation of the notion of sample and hold.

Let $h>0$ be the sample period. The perfect sampling operator corresponds to the mapping from the signal to the sequence of signal values at the sampling times. Since $W_0^{r,p}(\mathbb{R}_+,\mathbb{R}^N) \subset L^p(\mathbb{R}_+,\mathbb{R}^N)$, the formal definition requires the Sobolev embedding theorem. Hence, we define the perfect sampling operator S_h^0 as follows:

$$S_h^0\colon \mathcal{Y}_a \to \mathrm{map}(\mathbb{Z},\mathbb{R}^N) \;:\quad (S_h^0 y)(j) = \begin{cases} E_{jh}y, & j>0,\ jh \in \mathrm{dom}(y) \\ 0, & \text{otherwise,} \end{cases} \qquad j \in \mathbb{Z} \tag{3.19}$$

where in the case of $\mathcal{Y} = W_0^{r,p}(\mathbb{R}_+,\mathbb{R}^N)$, $rp>1$, the pointwise evaluation operator $E_t\colon \mathcal{Y}_a \to \mathbb{R}^N$ is interpreted via the Sobolev embedding theorem, that is for any $t \in \mathbb{R}$ and $\tau>0$ satisfying $0 \le t < \tau \in \mathrm{dom}(y)$, $E_t([y])$ is defined to be pointwise evaluation of the element $x \in C([0,\tau),\mathbb{R}^N) \cap [R_\tau y]$, that is $E_t([y]) = x(t)$.

The integrally sampled operator, $S_h^1\colon \mathcal{Y} \to \mathbb{R}^N$ is defined by

$$S_h^1\colon \mathcal{Y}_a \to \mathrm{map}(\mathbb{Z},\mathbb{R}^N) \;:\quad (S_h^1 y)(j) = \begin{cases} \frac{2}{h}\int_0^{\frac{h}{2}} y((j-\frac{1}{2})h+s)\,ds, & \text{if } j>1,\ jh \in \mathrm{dom}(y), \\ 0, & \text{otherwise,} \end{cases} \qquad j \in \mathbb{Z} \tag{3.20}$$

and reflects the action of the typical implementation of a sampling procedure; that is to take the average value of the signal over a short period.

We let $H_h^0\colon \mathrm{map}(\mathbb{Z},\mathbb{R}^N) \to \mathcal{U}_a$ denote the zero-order hold, and $H_h^1\colon \mathrm{map}(\mathbb{Z},\mathbb{R}^N) \to \mathcal{U}_a$ denote the first order hold, defined for $x \in \mathrm{map}(\mathbb{Z},\mathbb{R}^N)$ as follows:

$$H_h^0(x)(t) = x\left(\frac{\lfloor t-h\rfloor_h}{h}\right) \tag{3.21}$$

$$H_h^1(x)(t) = \left(\frac{t-\lfloor t\rfloor_h}{h}\right) x\left(\frac{\lfloor t-h\rfloor_h}{h}\right) + \left(1-\frac{t-\lfloor t\rfloor_h}{h}\right) x\left(\frac{\lfloor t-2h\rfloor_h}{h}\right) \tag{3.22}$$

where $\lfloor t\rfloor_h = \max\{s \in \mathbb{R} \mid s=jh,\ j \in \mathbb{Z},\ s \le t\}$.

2. The sampled Euler operators are specified by taking δ_h as in the last four entries of table 1 and correspond respectively to the cases of zero order hold perfect sampling, first order hold perfect sampling, zero order hold with integral sampling and first order hold with integral sampling. In all four cases, δ_h commutes with both D_h and D_{-h}, so (3.12) holds and hence the sampled Euler operators can be realised as a causal operator acting on the sampled signal y, that is at the time instants $t = jh$, $j \in \mathbb{N}$, and the current time $t \geq 0$. Furthermore, it only requires memory of at most the last r samples, that is, it only requires storage of $(Sy)(\lfloor t-h \rfloor_h), \ldots, (Sy)(\lfloor t-rh \rfloor_h)$, where $S = S_h^0$ or $S = S_h^1$. The Euler approximation constants given in the second column of table 1 are established in Theorem 3.4 below.

3.1.1 Proof of the Euler operator properties for the numerical approximations

To establish the relevant norm bounds in inequality (3.6), we establish pointwise bounds on $\Delta^i_{k,h}(y) - y^{(i)}$ in terms of $y^{(i+1)}$ using the Mean Value Theorem. Lemma 3.3 then establishes the relevant norm bound from the pointwise bounds. The following result, Proposition 3.2 is quoted directly from [11, Prop. 3.2]: note that this result comes from a detailed analysis, and is not an elementary estimate.

Proposition 3.2 For $y \in C(\mathbb{R}_{\geq 0}, \mathbb{R})$ and $\varrho > 0$, define the function

$$M_\varrho[y] \;:\; \mathbb{R}_+ \to \mathbb{R}\,, \qquad t \mapsto \max_{\tau \in [t-\varrho, t]} |y(\tau)|\;, \quad \text{where } y(s) = 0 \text{ if } s < 0\,. \tag{3.23}$$

Then, for every $y \in CW_0^{1,p}(\mathbb{R}_+, \mathbb{R})$ and $1 \leq p < \infty$,

$$\forall\, T > 0 \;:\; \|M_\varrho[y]\|^p_{L^p([0,T),\mathbb{R})} \;\leq\; 2\|y\|^{p-1}_{L^p([0,T),\mathbb{R})} \left(\|y\|_{L^p([0,T),\mathbb{R})} + \varrho p \|\dot{y}\|_{L^p([0,T),\mathbb{R})}\right)\,. \tag{3.24}$$

The following lemma uses Proposition 3.2 to give bounds on the Euler approximation constants.

Lemma 3.3 *Let $1 \leq p \leq \infty$, $0 \leq r \leq \infty$ and $\mathcal{Y} = CW_0^{r,p}(\mathbb{R}_+, \mathbb{R}^N)$ or $\mathcal{Y} = W_0^{r,p}(\mathbb{R}_+, \mathbb{R}^N)$. If there exists $c(h) > 0$ and $\varrho(h) > 0$ such that for all $t \geq 0$, $y \in C_0^r(\mathbb{R}_+, \mathbb{R}^N)$ and $\nu \in \{1, \ldots, N\}$,*

$$|\Delta^i_{k,h}(y_\nu)(t) - y_\nu^{(i)}(t)| \leq c(h) M_{\varrho(h)}[y_\nu^{(i+1)}](t) \quad \text{for all} \quad \begin{array}{ll} 0 \leq i \leq k < r-1 & \text{if } 1 \leq p < \infty \\ 0 \leq i \leq k < r & \text{if } p = \infty, \end{array}$$

then

$$\gamma_i(h) \leq \begin{cases} c(h) N (2(1 + \varrho(h) p))^{\frac{1}{p}} & \text{if } 1 \leq p < \infty,\; 0 \leq i \leq k < r-1 \\ c(h) N & \text{if } p = \infty,\; 0 \leq i \leq k < r. \end{cases} \tag{3.25}$$

Proof. Let $\mathcal{Y} = CW_0^{r,p}(\mathbb{R}_+, \mathbb{R}^N)$ and let $T > 0$. Then, for $1 \leq p < \infty$, Proposition 3.2 yields, for all $\nu \in \{1, \ldots, N\}$,

$$\begin{aligned} \|R_T(\Delta^i_{k,h}(y_\nu) - y_\nu^{(i)})\|_{L^p[0,T)} &\leq c(h) \|R_T M_{\varrho(h)}[y_\nu^{(i+1)}]\|_{L^p[0,T)} \\ &\leq c(h) \left(2\|R_T y_\nu^{(i+1)}\|^{p-1}_{L^p[0,T)} (\|R_T y_\nu^{(i+1)}\|_{L^p[0,T)} + \varrho(h) p \|R_T y_\nu^{(i+2)}\|_{L^p[0,T)})\right)^{\frac{1}{p}} \\ &\leq c(h)(2(1+\varrho(h)p))^{\frac{1}{p}} \|R_T y_\nu\|_{CW^{r,p}[0,T)} \qquad \forall\, 0 \leq i \leq k \leq r-1, \end{aligned}$$

and hence

$$\|R_T(\Delta^i_{k,h}(y) - y^{(i)})\|_{L^p} \leq c(h)N(2(1+\varrho(h)p))^{\frac{1}{p}}\|R_T y\|_{CW^{r,p}[0,T)} \qquad \forall\, 0 \leq i \leq k \leq r-1.$$

For $p = \infty$, we have, for all $\nu \in \{1,\dots,N\}$,

$$\|R_T M_{\varrho(h)}[y^{(i+1)}]\|_{L^\infty[0,T)} = \|R_T y^{(i+1)}\|_{L^\infty[0,T)}$$

hence

$$\|R_T(\Delta^i_{k,h}(y) - y^{(i)})\|_{L^\infty[0,T)} \leq c(h)N\|R_T y\|_{CW^{r,\infty}[0,T)}\,.$$

Let $\mathcal{Y} = W^{r,p}_0(\mathbb{R}_+,\mathbb{R}^N)$ and let $\gamma_i(h)$ be given by equation (3.25). By the density of $CW^{r,p}_0([0,T),\mathbb{R}^N)$ in $W^{r,p}_0([0,T),\mathbb{R}^N)$, and since $\Delta^i_{k,h}$, ∂^i are bounded linear operators from $W^{r,p}_0([0,T),\mathbb{R}^N)$ to $L^p([0,T),\mathbb{R}^N)$, it follows that there exist $y_n \in CW^{r,p}_0([0,T),\mathbb{R}^N)$, $n \geq 1$, such that $y_n \to y$ as $n \to \infty$ and:

$$\begin{aligned}
\|R_T(\Delta^i_{k,h}(y) \;&-\; \partial^i y)\|_{L^p[0,T)} \\
&\leq \|R_T(\Delta^i_{k,h}(y) - \Delta^i_{k,h}(y_n))\|_{L^p[0,T)} + \|R_T(\Delta^i_{k,h}(y_n) - \partial^i y_n)\|_{L^p[0,T)} + \|R_T(\partial^i y_n - \partial^i y)\|_{L^p[0,T)} \\
&\leq \|\Delta^i_{k,h}\|_{W^{r,p}_0,L^p}\|R_T(y-y_n)\|_{L^p[0,T)} + \gamma_i(h)\|R_T y_n\|_{W^{r,p}_0[0,T)} + \|\partial^i\|_{W^{r,p}_0,L^p}\|R_T(y-y_n)\|_{L^p[0,T)} \\
&\to \gamma_i(h)\|R_T y\|_{W^{r,p}_0[0,T)} \quad \text{as } n \to \infty,
\end{aligned}$$

hence the Euler approximation constants for $\mathcal{Y} = W^{r,p}_0$ are also given by equation (3.25). □

Theorem 3.4 *Let $1 \leq p \leq \infty$, $0 \leq k < r \leq \infty$. Let $\mathcal{U} = L^p(\mathbb{R}_+,\mathbb{R}^N)$ and $\mathcal{Y} = CW^{r,p}_0(\mathbb{R}_+,\mathbb{R}^N)$ or $\mathcal{Y} = W^{r,p}_0(\mathbb{R}_+,\mathbb{R}^N)$. If $\Delta_{k,h}$ is defined by* (3.5), (3.7), (3.8), (3.9) *and table 1, then $\Delta_{k,h}$ is an $\mathcal{Y}$ Euler operator and the Euler approximation constants are given in table 1.*

Proof. *Step 1*: Let $1 \leq i \leq k < r$. We first consider $\Delta^i_{k,h}$ defined by (3.7), (3.8), (3.9) and $\delta_h = D_h$. Let $t \geq 0$ and let $y \in C^i_0(\mathbb{R}_+,\mathbb{R}^N)$. By i applications of the Mean Value Theorem, there exist $\xi^i_j : \mathbb{R}_{\geq 0} \to (0, jh]^N$, for $j \in \{1,\dots,i\}$, such that, for all $\nu \in \{1,\dots,N\}$,

$$\begin{aligned}
\left|\Delta^i_{k,h}(y_\nu)(t) - y^{(i)}_\nu(t)\right| &= \left|\Delta^{i-1}_{k,h}\left(\frac{1}{h}\big(y_\nu(\cdot) - y_\nu(\cdot - h)\big)\right)(t) - y^{(i)}_\nu(t)\right| \\
&= \left|\Delta^{i-1}_{k,h}\left(y^{(1)}_\nu\right)(t - (\xi^i_1)_\nu(t)) - y^{(i)}_\nu(t)\right| \\
&\;\;\vdots \\
&= \left|\left(\frac{1}{h}\Big(\big(y^{(i-1)}_\nu\big)(t - (\xi^i_{i-1})_\nu(t)) - \big(y^{(i-1)}_\nu\big)(t - (\xi^i_{i-1})_\nu(t) - h)\Big)\right) - y^{(i)}_\nu(t)\right| \\
&= \left|y^{(i)}_\nu\left(t - (\xi^i_i)_\nu(t)\right) - y^{(i)}_\nu(t)\right|
\end{aligned}$$

and by a further application of the Mean Value Theorem, there exist $\xi^{i+1}_i : \mathbb{R}_{\geq 0} \to (0, ih]^N$ such that, for all $\nu \in \{1,\dots,N\}$, $\left|y^{(i)}_\nu\left(t - (\xi^i_i)_\nu(t)\right) - y^{(i)}_\nu(t)\right| = \left|(\xi^i_i)_\nu(t) \cdot y^{(i+1)}_\nu(t - (\xi^i_{i+1})_\nu(t))\right| \leq ih\left|y^{(i+1)}_\nu(t - (\xi^i_{i+1})_\nu(t))\right| \leq ihM_{ih}[y^{(i+1)}](t)$. Hence with $c(h) = ih$ and $\varrho(h) = ih$, Lemma 3.3 gives the Euler approximation constant bound (3.14).

Step 2: We now consider $\Delta^i_{k,h}$ defined by (3.7), (3.8), (3.9) and $\delta_h = H^0_h \circ S^0_h$. By applications of the Mean Value Theorem, there exists $\xi^i_1 : R_{\geq 0} \to (0, \lfloor t \rfloor_h - \lfloor t-h \rfloor_h)]^N = (0,h]^N$ and $\xi^i_2 : R_{\geq 0} \to (0, \lfloor t \rfloor_h - \lfloor t-2h \rfloor_h)]^N = (0,2h]^N$ such that, for all $\nu \in \{1,\dots,N\}$,

$$\Delta^1_{k,h}(y_\nu)(t) = \frac{1}{h}\left(y_\nu(\lfloor t \rfloor_h) - y_\nu(\lfloor t-h \rfloor_h)\right) = y^{(1)}_\nu(\lfloor t \rfloor_h - (\xi^i_1)_\nu(t)) \,. \tag{3.26}$$

and

$$\Delta^2_{k,h}(y_\nu)(t) = \frac{1}{h}\left(y^{(1)}_\nu(\underbrace{\lfloor \lfloor t \rfloor_h - (\xi^i_1)_\nu(t) \rfloor_h}_{= \lfloor t \rfloor_h - \lfloor (\xi^i_1)_\nu(t) \rfloor_h}) - y^{(1)}_\nu(\lfloor t \rfloor_h - \lfloor (\xi^i_1)_\nu(t) \rfloor_h - h) \right) = y^{(2)}_\nu(\lfloor t \rfloor_h - (\xi^i_2)_\nu(t)) \,.$$

Then, by the same analysis as in step 1, there exists $\xi^i_{i+1} : R_{\geq 0} \to (0, t - \lfloor t \rfloor_h + ih]^N \subset (0,(i+1)h]^N$ such that, for all $\nu \in \{1,\dots,N\}$,

$$\begin{aligned} \left|\Delta^i_{k,h}(y_\nu)(t) - y^{(i)}_\nu(t)\right| &\leq (i+1)h \left| y^{(i+1)}_\nu(t - (\xi^i_{i+1})_\nu(t))\right| \\ &\leq (i+1)hM_{(i+1)h}[y^{(i+1)}_\nu] \end{aligned}$$

and thus with, $c(h) = (i+1)h$ and $\varrho(h) = (i+1)h$, Lemma 3.3 gives the Euler approximation constants (3.15) as required.

Step 3: We now consider $\Delta^i_{k,h}$ defined by (3.7), (3.8), (3.9) and $\delta_h = H^1_h \circ S^0_h$. First observe that

$$\Delta^1_{k,h}(y)(t) = \frac{1}{h}\left(\frac{t - \lfloor t \rfloor_h}{h}\left(y(\lfloor t \rfloor_h) - y(\lfloor t-h \rfloor_h)\right) + \left(1 - \frac{t - \lfloor t \rfloor_h}{h}\right)\left(y(\lfloor t-h \rfloor_h) - y(\lfloor t-2h \rfloor_h)\right)\right) \,.$$

By an application of the Mean Value Theorem there exists $\xi^i_1 : \mathbb{R} \to (0, \lfloor t \rfloor_h - \lfloor t-h \rfloor_h]^N = (0,h]^N$ such that, for all $\nu \in \{1,\dots,N\}$,

$$\Delta^1_{k,h}(y_\nu)(t) = \frac{t - \lfloor t \rfloor_h}{h} y^{(1)}_\nu(\lfloor t \rfloor_h - (\xi^i_1)_\nu(t)) + \left(1 - \frac{t - \lfloor t \rfloor_h}{h}\right) y^{(1)}_\nu(\lfloor t-h \rfloor_h - (\xi^i_1)_\nu(t-h)) \,.$$

Then it follows that

$$\begin{aligned} \Delta^2_{k,h}(y_\nu)(t) = &\frac{1}{h}\left(\frac{t - \lfloor t \rfloor_h}{h} y^{(1)}_\nu(\lfloor t-h \rfloor_h - (\xi^i_1)_\nu(t-h)) \right. \\ &+ \left(1 - \frac{t - \lfloor t \rfloor_h}{h}\right) y^{(1)}_\nu(\lfloor t-h \rfloor_h - (\xi^i_1)_\nu(t-h)) \\ &- \frac{t - \lfloor t \rfloor_h}{h} y^{(1)}_\nu(\lfloor t-h \rfloor_h - (\xi^i_1)_\nu(t-h) - h) \\ &\left. + \left(1 - \frac{t - \lfloor t \rfloor_h}{h}\right) y^{(1)}_\nu(\lfloor t-h \rfloor_h - (\xi^i_1)_\nu(t-h) - 2h) \right) \end{aligned}$$

and hence an application of the Mean Value Theorem yields the existence of $\xi^i_2 : \mathbb{R} \to (h,2h]^N$ such that, for all $\nu \in \{1,\dots,N\}$,

$$\Delta^2_{k,h}(y_\nu)(t) = \frac{t - \lfloor t \rfloor_h}{h} y^{(2)}_\nu(\lfloor t-h \rfloor_h - (\xi^i_2)_\nu(t-h)) + \left(1 - \frac{t - \lfloor t \rfloor_h}{h}\right) y^{(2)}_\nu(\lfloor t-2h \rfloor_h - (\xi^i_1)_\nu(t-2h)) \,.$$

Following the same analysis as in step 1 it follows that there exists $\xi_i^i : \mathbb{R} \to ((i-1)h, ih]^N$ such that, for all $\nu \in \{1, \ldots, N\}$,

$$\Delta_{k,h}^i(y_\nu)(t) = \frac{t - \lfloor t \rfloor_h}{h} y_\nu^{(i)}(\lfloor t-(i-1)h \rfloor_h - (\xi_i^i)_\nu(t-(i-1)h)) + \left(1 - \frac{t - \lfloor t \rfloor_h}{h}\right) y_\nu^{(i)}(\lfloor t - ih \rfloor_h - (\xi_i^i)_\nu(t-ih)) .$$

Observe that by the Intermediate Value Theorem there exists $\zeta^i : \mathbb{R}_{\geq 0} \to [2(i-1)h, (2i+1)h]^N$ such that, for all $\nu \in \{1, \ldots, N\}$,

$$\Delta_{k,h}^i(y_\nu)(t) = y^{(i)}(t - (\zeta^i)_\nu(t)) . \tag{3.27}$$

Finally, by another application of the Mean Value Theorem there exists $\xi_{i+1}^i : \mathbb{R}_{\geq 0} \to (0, (2i+1)h]^N$ such that, for all $\nu \in \{1, \ldots, N\}$,

$$\left|\Delta_{k,h}^i(y_\nu)(t) - y^{(i)}(t)\right| \leq (2i+1)h \left|y_\nu^{(i+1)}(t - (\xi_{i+1}^i)_\nu(t))\right| \leq (2i+1)hM_{(2i+1)h}[y_\nu^{(i+1)}]$$

and thus, with $c(h) = (2i+1)h$ and $\varrho(h) = (2i+1)h$, Lemma 3.3 gives the Euler approximation constants (3.16) as required.

Step 4: We now consider $\Delta_{k,h}^i$ defined by (3.7), (3.8), (3.9) and $\delta_h = H_h^0 \circ S_h^1$. Let $t \geq 0$. Observe that, for all $\nu \in \{1, \ldots, N\}$,

$$\begin{aligned}\Delta_{k,h}^1(y_\nu)(t) &= \frac{1}{h}\left(\frac{2}{h}\int_0^{\frac{h}{2}} D_{\frac{h}{2}} y_\nu(\lfloor t-h \rfloor_h + h + s)\mathrm{d}s - \frac{2}{h}\int_0^{\frac{h}{2}} D_{\frac{h}{2}} y_\nu(\lfloor t-h \rfloor_h + s)\mathrm{d}s\right) \\ &= \frac{2}{h}\int_0^{\frac{h}{2}} \frac{1}{h}\left(y_\nu(\lfloor t \rfloor_h - \tfrac{h}{2} + s) - y_\nu(\lfloor t-h \rfloor_h - \tfrac{h}{2} + s)\right)\mathrm{d}s .\end{aligned}$$

By an application of the Mean Value Theorem there exists $\xi_1^i : \mathbb{R}_{\geq 0} \to (\frac{h}{2}, \lfloor t \rfloor_h - \lfloor t-h \rfloor_h + \frac{h}{2}]^N = (\frac{h}{2}, \frac{3h}{2}]^N$ such that, for all $\nu \in \{1, \ldots, N\}$,

$$\Delta_{k,h}^1(y_\nu)(t) = \frac{2}{h}\int_0^{\frac{h}{2}} y_\nu^{(1)}\left(\lfloor t \rfloor_h + s - (\xi_1^i)_\nu(t)\right)\mathrm{d}s .$$

Then by the Integral Mean Value Theorem, there exists $\eta_1^i : \mathbb{R}_{\geq 0} \in [0, \frac{h}{2}]^N$ such that, for all $\nu \in \{1, \ldots, N\}$,

$$\Delta_{k,h}^1(y_\nu)(t) = y_\nu^{(1)}\left(\lfloor t \rfloor_h - (\xi_1^i)_\nu(t) - (\eta_1^i)_\nu(t)\right) . \tag{3.28}$$

By another application of the Mean Value Theorem and the Integral Mean Value Theorem there exist $\xi_2^i : \mathbb{R}_{\geq 0} \to (\frac{3h}{2}, \frac{5h}{2}]^N$ and $\eta_2^i : \mathbb{R}_{\geq 0} \in [0, \frac{h}{2}]^N$ such that, for all $\nu \in \{1, \ldots, N\}$,

$$\begin{aligned}\Delta_{k,h}^2(y_\nu)(t) &= \frac{1}{h}\Bigg(\frac{2}{h}\int_0^{\frac{h}{2}} y_\nu^{(1)}\left(\lfloor t-h \rfloor_h + h - \frac{h}{2} - \lfloor (\xi_1^i)_\nu(t) + (\eta_1^i)_\nu(t) \rfloor_h + s\right)\mathrm{d}s \\ &\qquad - \frac{2}{h}\int_0^{\frac{h}{2}} y_\nu^{(1)}\left(\lfloor t-h \rfloor_h - \frac{h}{2} - \lfloor (\xi_1^i)_\nu(t) + (\eta_1^i)_\nu(t) \rfloor_h + s\right)\mathrm{d}s\Bigg) \\ &= \frac{2}{h}\int_0^{\frac{h}{2}} y_\nu^{(2)}\left(\lfloor t \rfloor_h + s - (\xi_2^i)_\nu(t)\right)\mathrm{d}s \\ &= y_\nu^{(2)}\left(\lfloor t \rfloor_h - (\xi_2^i)_\nu(t) - (\eta_2^i)_\nu(t)\right) .\end{aligned}$$

Applying the Mean Value Theorem and the Integral Mean Value Theorem another $i-2$ times gives the existence of $\xi_i^i : \mathbb{R}_{\geq 0} \to (\frac{(2i-1)h}{2}, \frac{(2i+1)h}{2}]^N$ and $\eta_i^i : \mathbb{R}_{\geq 0} \in [0, \frac{h}{2}]^N$ such that, for all $\nu \in \{1, \dots, N\}$,

$$\Delta_{k,h}^i(y_\nu)(t) = y_\nu^{(i)} \left(\lfloor t \rfloor_h - (\xi_i^i)_\nu(t) - (\eta_i^i)_\nu(t) \right)$$

and thus applying the Mean Value Theorem once again there exists $\xi_{i+1}^i : \mathbb{R}_{\geq 0} \to (0, (i+2)h]^N$ such that, for all $\nu \in \{1, \dots, N\}$,

$$\left| \Delta_{k,h}^i(y_\nu)(t) - y_\nu^{(i)}(t) \right| \leq (i+2)h \left| y_\nu^{(i)} \left(t - (\xi_{i+1}^i)_\nu(t) \right) \right| \leq (i+2)h M_{(i+2)h}[y_\nu^{(i+1)}]$$

and thus, with $c(h) = (i+2)h$ and $\varrho(h) = (i+2)h$, Lemma 3.3 gives the Euler approximation constants (3.17) as required.

Step 5: Finally consider the case of the integrally sampled operator, i.e. $\Delta_{k,h}^i$ defined by (3.7), (3.8), (3.9) and $\delta_h = H_h^1 \circ S_h^1$. Observe that, for all $\nu \in \{1, \dots, N\}$,

$$\begin{aligned} \Delta_{k,h}^1(y_\nu)(t) = \frac{1}{h} \Bigg(\frac{2}{h} \int_0^{\frac{h}{2}} D_{\frac{h}{2}} \Bigg[& \frac{t - \lfloor t \rfloor_h}{h} \left(y_\nu(\lfloor t \rfloor_h + s) - y_\nu(\lfloor t - h \rfloor_h + s) \right) \\ & + \left(1 - \frac{t - \lfloor t \rfloor_h}{h} \right) \left(y_\nu(\lfloor t - h \rfloor_h + s) - y_\nu(\lfloor t - 2h \rfloor_h + s) \right) \Bigg] \mathrm{d}s \Bigg) . \end{aligned}$$

By the Mean Value Theorem there exists $\xi_1^i : \mathbb{R} \to \left(\frac{h}{2}, \frac{3h}{2} \right]^N$ such that, for all $\nu \in \{1, \dots, N\}$,

$$\begin{aligned} \Delta_{k,h}^1(y_\nu)(t) = & \frac{t - \lfloor t \rfloor_h}{h} \frac{2}{h} \int_0^{\frac{h}{2}} y_\nu^{(1)} \left(\lfloor t \rfloor_h + s - (\xi_1^i)_\nu(t) \right) \mathrm{d}s \\ & + \left(1 - \frac{t - \lfloor t \rfloor_h}{h} \right) \frac{2}{h} \int_0^{\frac{h}{2}} y_\nu^{(1)} \left(\lfloor t \rfloor_h + s - h - (\xi_1^i)_\nu(t-h) \right) \mathrm{d}s . \end{aligned}$$

Then by the Integral Mean Value Theorem there exists $\eta_1^i : \mathbb{R} \to \left(0, \frac{h}{2} \right]^N$ such that, for all $\nu \in \{1, \dots, N\}$,

$$\begin{aligned} \Delta_{k,h}^1(y_\nu)(t) = & \frac{t - \lfloor t \rfloor_h}{h} y_\nu^{(1)} \left(\lfloor t \rfloor_h - (\xi_1^i)_\nu(t) - (\eta_1^i)_\nu(t) \right) \\ & + \left(1 - \frac{t - \lfloor t \rfloor_h}{h} \right) y_\nu^{(1)} \left(\lfloor t \rfloor_h - h - (\xi_1^i)_\nu(t-h) - (\eta_1^i)_\nu(t) \right) \end{aligned}$$

and thus the Intermediate Value Theorem yields existence of $\zeta_1^i : \mathbb{R}_{\geq 0} \to (0, 3h]^N$ such that, for all $\nu \in \{1, \dots, N\}$,

$$\Delta_{k,h}^1(y_\nu)(t) = y_\nu^{(1)} \left(\lfloor t \rfloor_h - (\zeta_1^i)_\nu(t) \right) .$$

Applying this analysis $i-1$ times more it follows that there exists $\zeta_i^i : \mathbb{R}_{\geq 0} \to (0, (3i+1)h]^N$ such that, for all $\nu \in \{1, \dots, N\}$,

$$\Delta_{k,h}^i(y_\nu)(t) = y_\nu^{(i)} \left(t - (\zeta_i^i)_\nu(t) \right)$$

and hence by the Mean Value Theorem there exists $\xi_{i+1}^i : \mathbb{R}_{\geq 0} \to (0, (3i+1)h]^N$ such that, for all $\nu \in \{1, \dots, N\}$,

$$\left| \Delta_{k,h}^i(y_\nu)(t) - y_\nu^{(i)}(t) \right| \leq (3i+1)h \left| y_\nu^{(i+1)}(t - (\xi_{i+1}^i)_\nu(t)) \right| \leq (3i+1)h M_{(3i+1)h}[y_\nu^{(i+1)}]$$

and thus, with $c(h) = (3i+1)h$ and $\varrho(h) = (3i+1)h$, Lemma 3.3 gives the Euler approximation constants (3.18) as required. □

3.2 High gain observers are Euler operators

An Euler operator of a very different nature is that of a high gain observer, which we consider in the SISO setting ($N = 1$).

3. For the high gain observer construction, we define $\Delta_{k,h} \colon \mathcal{Y}_a \to L^p_a(\mathbb{R}_+, \mathbb{R})$, $1 \le p \le \infty$ as follows:

$$
\begin{aligned}
y \mapsto \Delta_{k,h}(y) \quad : \quad \dot{x}_1 &= x_2 + \frac{\alpha_1}{h}(x_1 - y) \\
&\vdots \\
\dot{x}_k &= x_{k+1} + \frac{\alpha_k}{h^k}(x_1 - y) \\
\dot{x}_{k+1} &= \frac{\alpha_{k+1}}{h^{k+1}}(x_1 - y) \\
x(0) &= 0 \\
\Delta^0_{k,h}(y) &= y \\
\Delta^i_{k,h}(y)(t) &= x_i(t), \qquad 1 \le i \le k \le r-1.
\end{aligned}
\tag{3.29}
$$

where the polynomial $s^{k+1} + \alpha_1 s^k + \ldots \alpha_k s + \alpha_{k+1}$ is Hurwitz. Letting $e_i = x_i - y^{(i-1)}$, $1 \le i \le k+1$, $e = (e_1, \ldots e_{k+1})^T$, we have

$$\dot{e} = A_h e + B y^{(k+1)}, \tag{3.30}$$

where

$$
A_h = \begin{pmatrix} \frac{\alpha_1}{h} & 1 & 0 & \cdots & 0 \\ \frac{\alpha_2}{h^2} & 0 & 1 & \ddots & \vdots \\ \vdots & \vdots & & \ddots & 0 \\ \frac{\alpha_k}{h^k} & 0 & 0 & \cdots & 1 \\ \frac{\alpha_{k+1}}{h^{k+1}} & 0 & 0 & \cdots & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix}.
$$

To obtain a bound on the Euler approximation constant, observe that for $1 \le i \le k \le r-1$,

$$\|\Delta^i_{k,h} - \partial^i\|_{W^{r,2},L^2} = \|C_{i+1}(sI - A_h)^{-1}B\|_{H^\infty} = h^{k+1-i}\|C_{i+1}(sI - A_1)^{-1}B\|_{H^\infty},$$

where C_i is the row vector with 1 in the ith column, and 0's elsewhere. Hence the Euler approximation constant is bounded by:

$$
\begin{aligned}
\gamma(h) &= \sum_{i=1}^{k+1} h^{k+i-1}\|C_{i+1}(sI - A_1)^{-1}B\|_{H^\infty} && \text{if } p = 2,\ k \le r-1, \\
\gamma(h) &= (2k+3)\sum_{i=1}^{k+1} h^{k+i-1}\|C_{i+1}(sI - A_1)^{-1}B\|_{H^\infty} && \text{if } p = \infty,\ k \le r-1,
\end{aligned}
$$

where the result for $p = \infty$ follows from the fact that $\|P\|_{L^\infty(\mathbb{R}_+,\mathbb{R}),L^\infty(\mathbb{R}_+,\mathbb{R})} \le (2n+1)\|P\|_{L^2(\mathbb{R}_+,\mathbb{R}),L^2(\mathbb{R}_+,\mathbb{R})}$ where n is the minimal state dimension of P, [30].

In all the above examples of Euler operators the Euler approximation constants are defined and approach zero as $h \to 0$. The results that we will establish place bounds on the required size of $h > 0$; furthermore, the role of h as an important parameter in determining trade-offs between the region of stability and sensitivity to disturbances will be explicit.

3.3 Regular Euler operators

In order to establish semi-global results, we will require an additional compactness property. This property is not required for the global results of [11] or for the global result presented later in section 4.2, and arises here from application of the Schauder fixed point theorem in the underlying robust stability theorem (Theorem 9.1).

Definition 3.5 *Let $1 \le p \le \infty$ and $k \ge 0$, $h > 0$, $\tau > 0$. A $\mathcal{Y}$ Euler operator $\Delta_{k,h}$, (3.5), is said to be* regular *if the operators $Q_i^\tau : \mathcal{Y} \to L^p([0,\tau),\mathbb{R}^N)$, $0 \le i \le k$ defined by*

$$Q_i^\tau y = R_\tau(\Delta_{k,h}^i - \partial^i)y, \qquad y \in \mathcal{Y}$$

are compact for all $\tau > 0$.

Here, recall that an operator is said to be compact if it is continuous and maps bounded sets into relatively compact sets, and where a set is said to be relatively compact if it has compact closure.

We first consider the high gain observer construction from 3. above, where for $1 \le p \le \infty$ the required continuity and compactness follows from the properties of linear systems.

Proposition 3.6 *Let $p = 2$ or $p = \infty$, $1 \le k < r < \infty$, $h > 0$ and let $\mathcal{Y} = W_0^{r,p}(\mathbb{R}_+,\mathbb{R}^N)$ or $\mathcal{Y} = CW_0^{r,p}(\mathbb{R}_+,\mathbb{R}^N)$. Let $\Delta_{k,h}$ be defined by (3.5), (3.29). Then $\Delta_{k,h}$ is a regular $\mathcal{Y}$ Euler operator.*

Proof. Consider equation (3.30). As the map $y \mapsto e_i$ is bounded and linear on finite intervals for $1 \le i \le r-1$ it follows that Q_i^τ is continuous. Since equation (3.30) forms a strictly proper LTI system, the map $y \mapsto e_i$ is compact on finite intervals, and hence the map Q_i^τ is also compact. Since $Q_0^\tau \equiv 0$ is compact, the result follows. □

We now consider the numerical derivative based constructions from 1., 2. above. In the setting of continuous function spaces, the required compactness for $1 \le p < \infty$ is established via the Arzela-Ascoli theorem.

Proposition 3.7 *Let $1 \le r < \infty$ and suppose either $1 \le p < \infty$ and $1 \le k \le r-2$, or $p = \infty$ and $1 \le k \le r-1$. Let $h > 0$ and suppose $rp > 1$. Let $\mathcal{Y} = CW_0^{r,p}(\mathbb{R}_+,\mathbb{R}^N)$. Let $\Delta_{k,h}$ be defined by (3.5), (3.7), (3.8), (3.9) where δ_h is either given by $\delta_h = D_h$, $\delta_h = H_h^1 \circ S_h^0$ or $\delta_h = H_h^1 \circ S_h^1$. Then $\Delta_{k,h}$ is a regular $\mathcal{Y}$ Euler operator.*

Proof. Let $\tau > 0$ and $0 \le i \le r-1$. In all three cases, where δ_h is either given by $\delta_h = D_h$, $\delta_h = H_h^1 \circ S_h^0$ or $\delta_h = H_h^1 \circ S_h^1$, Q_i^τ is a bounded linear operator, so Q_i^τ is continuous. It remains to show that Q_i^τ maps bounded sets onto relatively compact sets. This is trivial if $i = 0$, so suppose $i \ge 1$.

Let $R > 0$, let $\mathcal{B}_{R,\mathcal{Y}}(0) = \{y \in \mathcal{Y} \mid \|y\| \le R\}$ and let $\mho = Q_i^\tau(\mathcal{B}_{R,\mathcal{Y}}(0))$. By definition of $\mathcal{Y}$ it follows that every element of $x \in \mho$ is bounded and uniformly continuous, hence there exists a unique $x^* \in C([0,\tau],\mathbb{R}^N)$ such that $R_\tau x^* = x$. Let $\Omega = \{x^* \in C([0,\tau],\mathbb{R}^N) \mid R_\tau x^* \in \mho\} \subset C([0,\tau],\mathbb{R}^N)$. By [29, Theorem A4], $\overline{\Omega}$ is compact in $C([0,\tau],\mathbb{R}^N)$ if, and only if, Ω is totally bounded as a subset of $C([0,\tau],\mathbb{R}^N)$. The Arzela-Ascoli theorem (see for example [29, Theorem A5]) gives that Ω is totally bounded in $C([0,\tau],\mathbb{R}^N)$ if and only if, Ω satisfies two conditions: (i) Ω is pointwise bounded, i.e. for all $t \in [0,\tau]$, $\sup\{|z(t)| \in \mathbb{R}_+ \mid z \in \Omega\} < \infty$, and (ii) Ω is equicontinuous on $[0,\tau]$.

To establish (i), let $y \in \mathcal{B}_{R,\mathcal{Y}}(0)$. Then $z = Q_i^\tau y \in \Omega \subset C([0,\tau],\mathbb{R}^N)$ and for all $t \in [0,\tau]$,

$$|Q_i^\tau(y)(t)| = |\Delta_{k,h}^i(y)(t) - (\partial^i y)(t)| \le \gamma_i(h)\|y\|_{W^{i+1,\infty}([0,\tau],\mathbb{R}^N)} = \gamma_i(h)\|y\|_{CW_0^{i+1,\infty}([0,\tau],\mathbb{R}^N)} \tag{3.31}$$

where $\gamma_i(h)$ is the appropriate Euler approximation constant, see table 1. By the Sobolev embedding theorem, there exists $M > 0$ such that $\|y\|_{CW_0^{i+1,\infty}([0,\tau],\mathbb{R}^N)} \le M\|y\|_{W_0^{i+1,p}([0,\tau],\mathbb{R}^N)}$, hence $|(Q_i^\tau y)(t)| \le M\gamma_i(h)\|y\|_{\mathcal{Y}}$. Since this holds for all $y \in \mathcal{B}_{R,\mathcal{W}}(0)$ this establishes (i). (ii) follows from Proposition 3.8 below.

This establishes that $\overline{\Omega}$ is compact as a subset of $C([0,\tau],\mathbb{R}^N)$. Let $\{x_j\}_{j\ge1}$ be a sequence with values in $\overline{\mho}$. Since $R_\tau\Omega = \mho$, $\overline{\mho} = \overline{R_\tau\Omega} \subset R_\tau\overline{\Omega}$ it follows that $\{x_j\}_{j\ge1}$ takes values in $R_\tau\overline{\Omega}$. Hence $\{x_j^*\}_{j\ge1}$ is a sequence with values in $\overline{\Omega}$. Since $\overline{\Omega}$ is compact in $C([0,\tau],\mathbb{R}^N)$, it follows that $\{x_j^*\}_{j\ge1}$ has a convergent subsequence in $C([0,\tau],\mathbb{R}^N)$. Then since $x_j = R_\tau x_j^*$, it follows that $\{x_j\}_{j\ge1}$ has a convergent subsequence in $BUC([0,\tau),\mathbb{R}^N)$, and so $\overline{\mho}$ is compact in $BUC([0,\tau),\mathbb{R}^N)$.

We conclude the proof by observing that this also implies the compactness of $\overline{\mho}$ as a subset of $L^p([0,\tau),\mathbb{R}^N)$: by compactness any sequence in $\overline{\mho}$ has a convergent subsequence in $BUC([0,\tau),\mathbb{R}^N)$, which is also convergent in $\mathcal{U}_\tau = L^p([0,\tau),\mathbb{R}^N)$ since convergent sequences in $BUC([0,\tau),\mathbb{R}^N)$ are also convergent in $L^p([0,\tau),\mathbb{R}^N)$. This completes the proof. □

Proposition 3.8 *Let $1 \le p \le \infty$, $1 \le k < r < \infty$, $h > 0$ and let $\mathcal{Y} = CW_0^{r,p}(\mathbb{R}_+,\mathbb{R}^N)$. Let $\Delta_{k,h}$ be defined by* (3.5), (3.7), (3.8), (3.9) *and where δ_h is given by either $\delta_h = D_h$, $\delta_h = H_h^1 \circ S_h^0$ (if $rp > 1$), or $\delta_h = H_h^1 \circ S_h^1$. Let $\tau > 0$, $R > 0$, and let $\Omega = Q_i^\tau(\mathcal{B}_{R,\mathcal{Y}}(0))$. Then Ω is equicontinuous on* $[0,\tau]$.

Proof. To establish equicontinuity we have to show that

$$\forall\, \varepsilon > 0 \quad \forall\, t_0 \in [0,\tau] \quad \exists\, T = T_{t_0}(\varepsilon) > 0 \quad \forall\, t \in [t_0 - T, t_0 + T] \cap [0,\tau] \quad \forall\, z \in \Omega : \; |z(t_0) - z(t)| < \varepsilon . \tag{3.32}$$

So, let $\tau > 0$, $\varepsilon > 0$, $t_0 \in [0,\tau]$ and $t \in [t_0 - T, t_0 + T] \cap [0,\tau]$. Let $y \in \mathcal{B}_{R,\mathcal{Y}}(0)$. Then by equation (3.11),

$$\sum_{i=0}^k |Q_i^\tau y(t_0) - Q_i^\tau y(t)| \;\le\; \sum_{i=0}^k \left(|y^{(i)}(t_0) - y^{(i)}(t)| + \tfrac{1}{h^i} \sum_{j=0}^i \binom{i}{j} \left| (\delta_h^j y)(t_0) - (\delta_h^j y)(t) \right| \right).$$

Let $\frac{1}{p} + \frac{1}{q} = 1$. Let $0 \le i \le k$. It follows from the fundamental theorem of calculus and the Hölder inequality that:

$$\left| y^{(i)}(t_0) - y^{(i)}(t) \right| \le \int_{t_0}^t |y^{(i+1)}(s)|\,\mathrm{d}s \le \|1\|_{L^q([t_0,t],\mathbb{R}^N)} \left\| y^{(i+1)} \right\|_{L^p([t_0,t],\mathbb{R}^N)} \le \|y\|_{W^{r,p}([t_0,t],\mathbb{R}^N)} N^{\frac{1}{q}} |t - t_0|^{\frac{1}{q}}. \tag{3.33}$$

Let $0 \le j \le i$. In the case of $\delta_h = D_h$, as given by (3.13), we have $\delta_h^j y(t) = D_{jh} y(t)$, so similarly we have:

$$\left| (\delta_h^j y)(t_0) - (\delta_h^j y)(t) \right| \;=\; |y(t_0 - jh) - y(t - jh)| \le \|y\|_{W^{r,p}([t_0 - jh, t - jh],\mathbb{R}^N)} N^{\frac{1}{q}} |t - t_0|^{\frac{1}{q}}.$$

In the case of $\delta_h = H_h^1 \circ S_h^0$ by equation (3.12), we have:

$$\delta_h^j y(t) = \left(\left(\frac{t - \lfloor t \rfloor_h}{h} \right) S^0_{\lfloor t - jh \rfloor_h} + \left(1 - \frac{t - \lfloor t \rfloor_h}{h} \right) S^0_{\lfloor t - jh - h \rfloor_h} \right)(y)(t).$$

We consider two cases: a) there exists $\alpha_{t_0} > 0$ such that $\lfloor t \rfloor_h = \lfloor t_0 \rfloor_h$ for all $t_0 - \alpha_{t_0} < t < t_0 + \alpha_{t_0}$, or b) $t_0 = hi$ for some $i \in \mathbb{N}$. We first consider case a). Since $y(\lfloor t_0 - jh \rfloor_h) = y(\lfloor t - jh \rfloor_h)$ and $y(\lfloor t_0 - jh - h \rfloor_h) = y(\lfloor t - jh - h \rfloor_h)$, by the fundamental theorem of calculus and the Hölder inequality we have:

$$\begin{aligned}
\left|(\delta_h^j y)(t_0) - (\delta_h^j y)(t)\right| &= \left|\left(\frac{t_0 - t}{h}\right)(y(\lfloor t_0 - jh \rfloor_h) - y(\lfloor t_0 - jh - h \rfloor_h))\right| \\
&\le \frac{1}{h}|t_0 - t| \int_{\lfloor t_0 - jh - h \rfloor_h}^{\lfloor t_0 - jh \rfloor_h} |y^{(1)}(s)| \, \mathrm{d}s \\
&\le h^{\frac{1-q}{q}} N^{\frac{1}{q}} |t_0 - t| \, \|y\|_{W^{r,p}([\lfloor t_0 - jh - h \rfloor_h, \lfloor t_0 - jh \rfloor_h], \mathbb{R}^N)} .
\end{aligned}$$

We now consider case b). Let $\alpha_{t_0} = h$. If $t_0 - \alpha_{t_0} < t < t_0$ then since $y(\lfloor t_0 - jh - h \rfloor_h) = y(\lfloor t - jh \rfloor_h)$ and $t - \lfloor t \rfloor_h = h - (t_0 - t)$, similarly to the above we obtain:

$$\begin{aligned}
\left|(\delta_h^j y)(t_0) - (\delta_h^j y)(t)\right| &= \left|\left(1 - \frac{t - \lfloor t \rfloor_h}{h}\right)(y(\lfloor t - jh \rfloor_h) - y(\lfloor t - jh - h \rfloor_h))\right| \\
&= \left|\left(\frac{t_0 - t}{h}\right)(y(\lfloor t - jh \rfloor_h) - y(\lfloor t - jh - h \rfloor_h))\right| \\
&\le h^{\frac{1-q}{q}} N^{\frac{1}{q}} |t_0 - t| \, \|y\|_{W^{r,p}([\lfloor t - jh - h \rfloor_h, \lfloor t - jh \rfloor_h], \mathbb{R}^N)} .
\end{aligned}$$

If $t_0 + \alpha_{t_0} > t \ge t_0$, we obtain

$$\left|(\delta_h^j y)(t_0) - (\delta_h^j y)(t)\right| \le h^{\frac{1-q}{q}} N^{\frac{1}{q}} |t_0 - t| \, \|y\|_{W^{r,p}([\lfloor t_0 - jh - h \rfloor_h, \lfloor t_0 - jh \rfloor_h], \mathbb{R}^N)}$$

as in case a).

In the case of $\delta_h = H_h^1 \circ S_h^1$, by the interval Mean Value Theorem, we have for $\nu \in \{1, \dots, N\}$:

$$\begin{aligned}
\delta_h^j y_\nu(t) &= \left(\left(\frac{t - \lfloor t \rfloor_h}{h}\right) S^1_{\lfloor t - jh \rfloor_h} + \left(1 - \frac{t - \lfloor t \rfloor_h}{h}\right) S^1_{\lfloor t - jh - h \rfloor_h}\right)(y_\nu)(t) \\
&= \left(\left(\frac{t - \lfloor t \rfloor_h}{h}\right) S^0_{\lfloor t - jh \rfloor_h - \eta_1^\nu(t)} + \left(1 - \frac{t - \lfloor t \rfloor_h}{h}\right) S^0_{\lfloor t - jh - h \rfloor_h - \eta_2^\nu(t)}\right)(y_\nu)(t)
\end{aligned}$$

where $\eta_1^\nu(t), \eta_2^\nu(t) \in [0, h/2]$. Analogously to the above, we can then show that there exists $\alpha_{t_0} > 0$ such that for all $t_0 - \alpha_{t_0} < t < t_0 + \alpha_{t_0}$:

$$\left|(\delta_h^j y)(t_0) - (\delta_h^j y)(t)\right| \le \begin{cases} N^{\frac{1}{q}} h^{\frac{1-q}{q}} |t_0 - t| \, \|y\|_{W^{r,p}([\lfloor t_0 - jh - h \rfloor_h - h/2, \lfloor t_0 - jh \rfloor_h], \mathbb{R}^N)} & \text{in case a)} \\ N^{\frac{1}{q}} h^{\frac{1-q}{q}} |t_0 - t| \, \|y\|_{W^{r,p}([\lfloor t - jh - h \rfloor_h - h/2, \lfloor t - jh \rfloor_h], \mathbb{R}^N)} & \text{in case b).} \end{cases}$$

Hence in all cases we have established a bound of the form:

$$\sum_{i=0}^{k} |Q_i^\tau y(t_0) - Q_i^\tau y(t)| \le \|y\|_{W^{r,p}([0,\tau], \mathbb{R})} \left(C_1 |t - t_0|^{\frac{1}{q}} + C_2 |t - t_0|\right) , \tag{3.34}$$

for some $C_1, C_2 > 0$. Thus inequality (3.32) holds for $T < \min\left\{\alpha_{t_0}, \left(\frac{\varepsilon}{2C_1 R}\right)^q, \frac{\varepsilon}{2C_2 R}\right\}$ thus establishing the required equicontinuity. □

In the setting of the function spaces $\mathcal{Y} = W_0^{r,p}(\mathbb{R}_+, \mathbb{R}^N)$, the required compactness for $1 \le p < \infty$ can be established from [1, Theorem 2.21].

Proposition 3.9 *Let $1 \le r < \infty$ and suppose either $1 \le p < \infty$ and $1 \le k \le r-2$, or $p = \infty$ and $1 \le k \le r-1$. Let $h > 0$ and suppose $rp > 1$. Let $\mathcal{Y} = W_0^{r,p}(\mathbb{R}_+, \mathbb{R}^N)$. Let $\Delta_{k,h}$ be defined by (3.5), (3.7), (3.8), (3.9) where δ_h is given by either $\delta_h = D_h$, $\delta_h = H_h^1 \circ S_h^0$ or $\delta_h = H_h^1 \circ S_h^1$. Then $\Delta_{k,h}$ is a regular $\mathcal{Y}$ Euler operator.*

Proof. Let $\tau > 0$ and $0 \le i \le r-1$. As in Proposition 3.7, the continuity of Q_i^τ in all three cases follows from the fact that Q_i^τ is a bounded linear operator. It remains to show that Q_i^τ maps bounded sets onto relatively compact sets.

Let $\varepsilon > 0$. By [1, Theorem 2.21] it suffices to show that there exists $\delta > 0$ and $0 < t_1 < t_2 < \tau$ such that for all $z = Q_i^\tau y$, $y \in \mathcal{B}_{R,\mathcal{W}}(0)$ and for all $0 < t_0 \le \delta$,
i) $\int_0^{t_1} |z(t)|^p \, dt + \int_{t_2}^\tau |z(t)|^p \, dt < \varepsilon^p$, and
ii) $\int_0^\tau |z(t+t_0) - z(t)|^p \, dt < \varepsilon^p$.

We first establish i). Similarly to the proof of Proposition 3.7, we know there exists $M > 0$ such that for all $0 \le t \le \tau$, and for all $y \in \mathcal{B}_{R,\mathcal{W}}(0)$, $|(Q_i^\tau y)(t)| \le M\gamma_i(h)\|y\|_{\mathcal{Y}}$, hence

$$\int_0^{t_1} |z(t)|^p \, dt + \int_{t_2}^\tau |z(t)|^p \, dt \le (t_1 + \tau - t_2)\left(M\gamma_i(h)\|y\|_{\mathcal{Y}}\right)^p,$$

and i) follows. We now establish ii). By the proof of Proposition 3.8, we know for all $\varepsilon > 0$ there exists $\delta > 0$ such that $|z(t+t_0) - z(t)| < \varepsilon$ for all $z \in Q_i^\tau \mathcal{B}_{R,\mathcal{W}}(0) \cap CW_0^{r,p}(\mathbb{R}_+, \mathbb{R}^N)$ and $0 < t_0 \le \delta$. Hence,

$$\int_0^\tau |z(t+t_0) - z(t)|^p \, dt \le \int_0^\tau \varepsilon^p \, dt \le \tau\varepsilon^p$$

and this suffices to complete the proof since $CW_0^{r,p}(\mathbb{R}_+, \mathbb{R}^N)$ is dense in $W_0^{r,p}(\mathbb{R}_+, \mathbb{R}^N)$. □

4 A high gain nonlinear separation principle

For suitable signal spaces $\mathcal{U}$ and $\mathcal{Y}$ we consider causal controllers of the form

$$\begin{aligned} C_F \colon \mathcal{Y}_a \to \mathcal{U}_a \quad &: \quad C_F = F \circ \partial \\ C_F^{\text{Euler}}[h] \colon \mathcal{Y}_a \to \mathcal{U}_a \quad &: \quad C_F^{\text{Euler}} = F \circ \Delta_{k,h}, \end{aligned} \tag{4.1}$$

where $\partial \colon \mathcal{Y}_a \to \mathcal{Y}_a^k$ denotes the differentiation operator and $\Delta_{k,h} \colon \mathcal{Y}_a \to \mathcal{Y}_a^k$ is a regular $\mathcal{Y}$ Euler operator, where recall that $\mathcal{Y}^k$ is defined to the Cartesian product of k copies of $L^p(\mathbb{R}_+, \mathbb{R}^N)$. We will additionally assume that $F \colon \mathcal{Y}_a^k \to \mathcal{U}_a$ is causal and satisfies a local Lipschitz condition, namely that there exists a function $\Lambda_F \colon \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, monotonically increasing in both arguments, such that:

$$\|R_\tau F(x) - R_\tau F(y)\|_{\mathcal{U}[0,\tau)} \le \Lambda_F(\|R_\tau x\|_{\mathcal{Y}^k[0,\tau)}, \|R_\tau x - R_\tau y\|_{\mathcal{Y}^k[0,\tau)})\|R_\tau x - R_\tau y\|_{\mathcal{Y}^k[0,\tau)}, \quad x, y \in \mathcal{Y}^k, \tau \ge 0. \tag{4.2}$$

We now investigate the inference of stability of the closed loop $[P, C_F^{\text{Euler}}[h]]$ from the stability of the closed loop $[P, C_F]$.

4.1 Regional stability

We now give the main result of the paper, which establishes a regional version of the nonlinear separation principle when the closed loop system starts at rest.

Theorem 4.1 *Let $1 \leq p \leq \infty$, $1 \leq k < r \leq \infty$, $M, N \geq 1$ and let $\mathcal{W} := \mathcal{U} \times \mathcal{Y}$ where $\mathcal{W} = L^p(\mathbb{R}_+, \mathbb{R}^M) \times W_0^{r,p}(\mathbb{R}_+, \mathbb{R}^N)$ or $\mathcal{W} = L^\infty(\mathbb{R}_+, \mathbb{R}^M) \times CW_0^{r,\infty}(\mathbb{R}_+, \mathbb{R}^N)$. Let $R > \varrho > 0$. Suppose $F\colon \mathcal{Y}_a^k \to \mathcal{U}_a$ is causal and locally Lipschitz with $F(0) = 0$ and is such that the controller $C_F : \mathcal{Y}_a \to \mathcal{U}_a$ given by (3.1) applied to some causal operator $P : \mathcal{U}_a \to \mathcal{Y}_a$, with $P(0) = 0$, yields a closed-loop system $[P, C_F]$ which is gain stable on $\mathcal{B}_{R,\mathcal{W}}(0)$. Let $0 < \alpha := \left\|\Pi_{C_F//P}|_{\mathcal{B}_{R,\mathcal{W}}(0)}\right\|_{\mathcal{W},\mathcal{W}} < \infty$. Suppose $h > 0$ is such that*

$$\lambda := \gamma(h)\Lambda_F\left(\alpha R, \gamma(h)\alpha R\right) \leq \frac{R-\varrho}{R\alpha}. \tag{4.3}$$

Suppose that $[P, C_F^{\mathrm{Euler}}[h]]$ has the uniqueness property, where $C_F^{\mathrm{Euler}}[h]\colon \mathcal{Y}_a \to \mathcal{U}_a$ is given by (3.2) and $\Delta_{k,h}$ is a regular $\mathcal{Y}$ Euler operator. Then the closed-loop system $[P, C_F^{\mathrm{Euler}}[h]]$ is gain stable on $\mathcal{B}_{\varrho,\mathcal{W}}(0)$ and

$$\left\|\Pi_{C_F^{\mathrm{Euler}}[h]//P}\Big|_{\mathcal{B}_{\varrho,\mathcal{W}}(0)}\right\|_{\mathcal{W},\mathcal{W}} \leq \frac{\alpha R(1+\lambda)}{\varrho}. \tag{4.4}$$

Proof. Since F is locally Lipschitz, $F\left(\partial y\right), F\left(\Delta_{k,h} y\right) \in \mathcal{U}$ for all $y \in \mathcal{Y}$. Hence the graphs of C_F and $C_F^{\mathrm{Euler}}[h]$ are of the form

$$\begin{aligned} \mathcal{G}_{C_F} &= \left\{ w \in \mathcal{W}_a \;\middle|\; w = \begin{pmatrix} F\left(\partial y\right) \\ y \end{pmatrix},\ y \in \mathcal{Y} \right\}, \\ \mathcal{G}_{C_F^{\mathrm{Euler}}[h]} &= \left\{ w \in \mathcal{W}_a \;\middle|\; w = \begin{pmatrix} F\left(\Delta_{k,h} y\right) \\ y \end{pmatrix},\ y \in \mathcal{Y} \right\}. \end{aligned}$$

Consider the causal, surjective mapping

$$\Phi_h : \mathcal{G}_{C_F} \cap \{x \in \mathcal{W} \mid \|R_\tau x\|_{\mathcal{W}[0,\tau)} \leq \alpha R,\ \tau > 0\} \to \mathcal{G}_{C_F^{\mathrm{Euler}}[h]}, \quad \begin{pmatrix} F\left(\partial y\right) \\ y \end{pmatrix} \mapsto \begin{pmatrix} F\left(\Delta_{k,h}(y)\right) \\ y \end{pmatrix}. \tag{4.5}$$

Suppose $\tau > 0$ and $\alpha R \geq \left\| R_\tau \begin{pmatrix} F\left(\partial y\right) \\ y \end{pmatrix} \right\|_{\mathcal{W}[0,\tau)}$. Then $\left\| R_\tau \begin{pmatrix} F\left(\partial y\right) \\ y \end{pmatrix} \right\|_{\mathcal{W}[0,\tau)} \geq \|R_\tau y\|_{\mathcal{Y}[0,\tau)}$ and by (3.6),

$$\begin{aligned} \left\| R_\tau (\Phi_h - I)|_{\mathcal{G}_{C_F} \cap \mathcal{B}_{\alpha R,\mathcal{W}}(0)} \begin{pmatrix} F\left(\partial y\right) \\ y \end{pmatrix} \right\|_{\mathcal{W}[0,\tau)} &= \left\| R_\tau \begin{pmatrix} F\left(\Delta_{k,h}(y)\right) - F\left(\partial y\right) \\ 0 \end{pmatrix} \right\|_{\mathcal{W}[0,\tau)} \\ &= \|R_\tau(F(\Delta_{k,h} y) - F(\partial y))\|_{\mathcal{U}[0,\tau)} \\ &\leq \Lambda_F\left(\|R_\tau \partial y\|_{\mathcal{Y}^k[0,\tau)}, \|R_\tau(\Delta_{k,h} y - \partial y)\|_{\mathcal{Y}^k[0,\tau)}\right) \\ &\qquad \cdot \|R_\tau(\Delta_{k,h} y - \partial y)\|_{\mathcal{Y}^k[0,\tau)} \\ &\leq \gamma(h)\Lambda_F(\alpha R, \gamma(h)\alpha R)\|R_\tau y\|_{\mathcal{Y}[0,\tau)} \\ &\leq \gamma(h)\Lambda_F(\alpha R, \gamma(h)\alpha R) \left\| R_\tau \begin{pmatrix} F\left(\partial y\right) \\ y \end{pmatrix} \right\|_{\mathcal{W}[0,\tau)}. \end{aligned} \tag{4.6}$$

We claim $R_\tau(\Phi_h - I)\big|_{\mathcal{G}_{C_F}\cap\mathcal{B}_{\alpha R,\mathcal{W}}(0)}$ is compact for all $\tau > 0$. Let $\{x_n\}_{n\geq 1}$ be a convergent sequence in $\mathcal{G}_{C_F}$. Let $x_n = (F(\partial y_n), y_n)^T$ and observe that $y_n \to y \in \mathcal{Y}$. Since the composition of the continuous operators R_τ, F, $\Delta_{k,h} - \partial$ is continuous, it follows that $R_\tau F(\Delta_{k,h} - \partial)y_n$ converges, and hence $R_\tau(\Phi_h - I)x_n$ converges, hence $R_\tau(\Phi_h - I)\big|_{\mathcal{G}_{C_F}\cap\mathcal{B}_{\alpha R,\mathcal{W}}(0)}$ is continuous. It then suffices to show that $R_\tau F(\Delta_{k,h} - \partial)$ maps bounded subsets of $\mathcal{B}_{\alpha R,\mathcal{Y}}(0)$ into relatively compact sets of $\mathcal{Y}^k[0,\tau)$. Let $Q = (Q_0^\tau, \ldots, Q_{r-1}^\tau)$, where $Q_i^\tau = R_\tau(\Delta_h^i - \partial^i)$ is as in Definition 3.5. Let $\Omega \subset \mathcal{B}_{\alpha R,\mathcal{W}}(0)$ be bounded and consider a sequence $\{x_i\}_{i\geq 1}$ where $x_i \in \Omega$. Since $\Delta_{k,h}$ is regular, Q is compact, so there exists a subsequence $\{x_{i_j}\}_{j\geq 1}$ such that $\{Qx_{i_j}\}_{j\geq 1}$ converges to a point $y \in \overline{Q(\Omega)}$. Then by the local Lipschitz condition (4.2),

$$\begin{aligned}
\|R_\tau F(\Delta_{k,h} - \partial)(x_{i_j}) - R_\tau F y\|_{\mathcal{U}[0,\tau)} &\leq \Lambda_F(\alpha R, \gamma(h)\alpha R + \alpha R)\|R_\tau(\Delta_{k,h} - \partial)x_{i_j} - R_\tau y\|_{\mathcal{Y}^k[0,\tau)} \\
&= \Lambda_F(\alpha R, \gamma(h)\alpha R + \alpha R)\|Qx_{i_j} - y\|_{\mathcal{Y}^k} \\
&\to 0 \quad \text{as } j \to \infty. \qquad (4.7)
\end{aligned}$$

Hence $R_\tau F(\Delta_{k,h} - \partial)(x_{i_j})$ converges to $R_\tau F y \in \overline{R_\tau F(\Delta_{k,h} - \partial)(\Omega)} \subset \mathcal{Y}^k[0,\tau)$ as $j \to \infty$, and we have established the claim.

Since $\Phi_h \in \mathcal{O}^{\mathcal{W},\alpha R}_{C_F, C_F^{\text{Euler}}[h]}$ it follows from (4.6) that $\vec{\delta}_{\mathcal{W},\alpha R}(C_F, C_F^{\text{Euler}}[h]) \leq \gamma(h)\Lambda_F(\alpha R, \gamma(h)\alpha R)$ where the gap distance $\vec{\delta}$ is as given by equation (9.1). Since $F(0) = 0$, $C_F(0) = C_F^{\text{Euler}}[h](0) = 0$, the result follows from Theorem 9.1 with $\varepsilon = \frac{\varrho}{R}$, since $\mathcal{W}$ is truncation complete. □

We now highlight the important features of the above result.

- In the examples of Euler operators given in section 3, the Euler approximation constants have the property that $\gamma(h) \to 0$ as $h \to 0$. This means that inequality (4.3) can always be met for appropriate choices of $h > 0$.
- In the case whereby controllers are specified via a locally Lipschitz memoryless feedback (3.3), (3.4), it is straightforward to see that $\Lambda_F = G[f]$ for $1 \leq p \leq \infty$, where $G\colon \mathbb{R}_+^2 \to \mathbb{R}_+$ is the growth function:

$$G[f](\varrho, \epsilon) = \sup_{\substack{|x|\leq\varrho \\ |x-y|\leq\epsilon}} |f(x) - f(y)|. \qquad (4.8)$$

- The above result has a requirement that the Euler operators are regular, this is due to the compactness requirement in the regional robust stability result, Theorem 9.1. The requirement of regularity will be removed in the later global result, Theorem 4.5 below.

We now give two corollaries and an example to further illustrate the utility of Theorem 4.1. To highlight the nature of the results, we only state the qualitative versions of these results, but the analogues of the constructive bound of Theorem 4.1 can be obtained straightforwardly.

Corollary 4.2 *Let $1 \leq p \leq \infty$, $1 \leq r \leq \infty$, $M, N \geq 1$ and let $\mathcal{W} := \mathcal{U} \times \mathcal{Y}$ where $\mathcal{U} = L^p(\mathbb{R}_+, \mathbb{R}^M)$, $\mathcal{Y} = W_0^{r,p}(\mathbb{R}_+, \mathbb{R}^N)$ or $\mathcal{Y} = CW_0^{r,\infty}(\mathbb{R}_+, \mathbb{R}^N)$. Let P be a causal operator $P : \mathcal{U}_a \to \mathcal{Y}_a$, with $P(0) = 0$ and $C_f : \mathcal{Y}_a \to \mathcal{U}_a$ a locally Lipschitz continuous feedback given by (3.3). Suppose the closed loop system $[P, C_f]$ is both locally gain stable and globally gain-function stable. Then $[P, C_f^{\text{Euler}}[\cdot]]$ is semi-globally gain stable.*

Example 4.3 Let $p = \infty$, $r = 1$, $M = N = 1$. Let the plant $P\colon \mathcal{U}_a \to \mathcal{Y}_a$ and controller $C_f\colon \mathcal{Y}_a \to \mathcal{U}_a$ be given by:

$$\begin{aligned} P\ :\ \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1^2 + u_1, \quad x(0) = 0 \\ y &= x_1 \\ C_f\ :\ u_2 &= f(y_2, \dot{y}_2) = -2y_2^2 + y_2 + \dot{y}_2 \end{aligned}$$

Suppose $\|(u_0, y_0)^T\| \le r$. Since $\dot{x}_2 = -x_1 - x_2 - y_1^2 + 4y_0y_1 - 2y_0^2 + y_0 + \dot{y}_0 + u_0$ and $-y_1^2 + 4y_0y_1 \le 4y_0^2$, we can write $\dot{x} = Ax + B\varphi$ where $A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\|\varphi\|_{L^\infty} \le 6r^2 + 3r$. Hence $g[\Pi_{P//C_f}](r) \le c(r + r^2)$, for some constant $c > 0$ and the closed loop system $[P, C_f]$ is both locally gain stable and globally gain-function stable. The closed loop $[P, C_f^{\text{Euler}}[h]]$ for $h > 0$ has the uniqueness property since the closed loop is an ordinary differential delay system. Then $[P, C_f^{\text{Euler}}[\cdot]]$ is semi-globally gain stable.

Corollary 4.4 *Let $1 \le p \le \infty$, $1 \le r \le \infty$ and let $\mathcal{W} := \mathcal{U} \times \mathcal{Y}$ where $\mathcal{U} = L^p(\mathbb{R}_+, \mathbb{R}^M)$, $\mathcal{Y} = W_0^{r,p}(\mathbb{R}_+, \mathbb{R}^N)$ or $\mathcal{Y} = CW_0^{r,\infty}(\mathbb{R}_+, \mathbb{R}^N)$. Let P be a causal operator $P : \mathcal{U}_a \to \mathcal{Y}_a$, with $P(0) = 0$ and $\{C_{f[\lambda]} : \mathcal{Y}_a \to \mathcal{U}_a\}_{\lambda > 0}$ be a one parameter family of locally Lipschitz continuous feedbacks given by (3.3). Suppose $[P, C_{f[\cdot]}]$ is semi-globally gain stable. Then $[P, C_{f[\cdot]}^{\text{Euler}}[\cdot]]$ is semi-globally gain stable.*

4.2 Global stability

In the special case whereby gain stability holds globally and the controller F satisfies a global Lipschitz condition, with Lipschitz constant L_F satisfying:

$$\|F(x) - F(y)\|_{\mathcal{U}} \le L_F \|x - y\|_{\mathcal{Y}^k} \quad x, y \in \mathcal{Y}^k, \tag{4.9}$$

then a version of Theorem 4.1 holds globally ($R = \infty$), under a stabilizability assumption, without the requirement that the Euler operator is regular and without the restriction to truncation complete signal spaces, see [11, Th. 3.1] for the particular case of a linear feedback. This is because under global stability assumptions, it is not necessary to utilize Schauder fixed point theory as in Theorem 9.1, and the result follows from the simpler robust stability theorem [11, Th. 2.1], which is based on [15, Th. 1]. Here we state a generalisation of [11, Th. 3.1]. The proof is omitted as it follows similarly to 4.1 above using [11, Th. 2.1] in place of Theorem 9.1.

Theorem 4.5 *Let $1 \le p \le \infty$, $1 \le r \le \infty$, $M, N \ge 1$ and let $\mathcal{W} := \mathcal{U} \times \mathcal{Y}$ where $\mathcal{W} = L^p(\mathbb{R}_+, \mathbb{R}^M) \times W_0^{r,p}(\mathbb{R}_+, \mathbb{R}^N)$ or $\mathcal{W} = L^p(\mathbb{R}_+, \mathbb{R}^M) \times CW_0^{r,p}(\mathbb{R}_+, \mathbb{R}^N)$. Suppose $F\colon \mathcal{Y}_a^k \to \mathcal{U}_a$ is causal and globally Lipschitz with $F(0) = 0$ and is such that the controller $C_F : \mathcal{Y}_a \to \mathcal{U}_a$ given by (3.1) applied to some causal operator $P : \mathcal{U}_a \to \mathcal{Y}_a$, with $P(0) = 0$, yields a closed-loop system $[P, C_F]$ which is gain stable on $\mathcal{W}$. Let $0 < \alpha := \|\Pi_{C_F//P}\|_{\mathcal{W},\mathcal{W}} < \infty$. Suppose $h > 0$ is such that $C_F^{\text{Euler}}[h]$ is stabilizable,*

$$\lambda := \gamma(h) L_F < \frac{1}{\alpha},$$

and that $[P, C_F^{\text{Euler}}[h]]$ *is either globally or regularly well posed on* $\mathcal{W}$*, where* $C_F^{\text{Euler}}[h]\colon \mathcal{Y}_a \to \mathcal{U}_a$ *is given by* (3.2) *and* $\Delta_{k,h}$ *is a regular* $\mathcal{Y}$ *Euler operator. Then the closed-loop system* $[P, C_F^{\text{Euler}}[h]]$ *is gain stable on* $\mathcal{W}$ *and*

$$\left\| \Pi_{C_F^{\text{Euler}}[h]//P} \right\|_{\mathcal{W},\mathcal{W}} \leq \frac{\alpha(1+\lambda)}{1-\alpha\lambda}. \tag{4.10}$$

As regularity is not required, this result holds for both zero and first order hold sampled Euler operators.

5 Output feedback stabilization via state feedback under a relative degree condition

Let us consider the case of the following nonlinear SISO plant:

$$\begin{aligned} P \quad : \quad \dot{x} &= f(x) + g(x)u_1 \\ y_1 &= h(x), \qquad\qquad x(0) = 0 \in \mathbb{R}^n, \end{aligned} \tag{5.1}$$

where $f, g \in C^\infty$ and P is of relative degree $r \leq n$, that is:

$$\begin{aligned} L_g L_f^i h(x) &= 0, \quad \text{for all } x \in \mathbb{R}^n,\ 1 \leq i \leq r-1, \\ L_g L_f^{n-1} h(x) &\neq 0, \quad \text{for all } x \in \mathbb{R}^n. \end{aligned}$$

Here L denotes the Lie derivative. It is well known [19, Prop. 9.1.1] that for any input u_1, if $\partial_{r-1} y_1 := (y_1, \partial^1 y_1, \ldots, \partial^{r-1} y_1)^T$, then

$$(\partial_{r-1} y_1)(t) = (y_1(t), L_f h(x(t)), \ldots L_f^{r-1} h(x(t)))^T,$$

and that under appropriate completeness assumptions $\partial_{r-1} y_1$ forms a partial state. Thus with appropriate signal domains and co-domains, a plant of the form $\partial_{r-1} \circ P$ can be thought of as an input to (partial) state operator.

The following result then establishes that a solution to the standard (partial) state feedback disturbance attenuation problem in an L^p sense implies a solution to the derivative output feedback disturbance attenuation problem in the $L^p, W^{r,p}$ sense, as considered in this paper. In the context of $p = 2$, this shows that in the case of full relative degree, the gain stability conditions in Theorem 4.1 are met by solving the standard (partial) state feedback nonlinear $\mathcal{H}_\infty$ problem.

Proposition 5.1 *Let* $1 \leq p \leq \infty$, $1 \leq r \leq n$ *and let* $\mathcal{X} = L^p(\mathbb{R}_+, \mathbb{R}^n)$, $\mathcal{U} = L^p(\mathbb{R}_+, \mathbb{R})$, $\mathcal{Y} = W^{r,p}(\mathbb{R}_+, \mathbb{R})$*. Let* $R > 0$*. Suppose* $P\colon \mathcal{U}_a \to \mathcal{Y}_a$ *is given by* (5.1) *and has relative degree* r*. Suppose further that there exists a controller* $C\colon \mathcal{X}_a \to \mathcal{U}_a$ *such that* $[\partial_n \circ P, C]$ *is* $\mathcal{U} \times \mathcal{X}$ *gain stable on* $\mathcal{B}_{R,\mathcal{U}\times\mathcal{X}}(0)$*. Suppose* $[P, C \circ \partial_n]$ *has the uniqueness property. Then* $[P, C \circ \partial_n]$ *is* $\mathcal{U} \times \mathcal{Y}$ *gain stable on* $\mathcal{B}_{R,\mathcal{U}\times\mathcal{Y}}(0)$ *and*

$$\|\Pi_{C\circ\partial_n//P}|_{\mathcal{B}_{R,\mathcal{U}\times\mathcal{Y}}(0)}\| \leq \|\Pi_{C//\partial_n\circ P}|_{\mathcal{B}_{R,\mathcal{U}\times\mathcal{X}}(0)}\|.$$

Proof. Let $(u_0, y_0) \in \mathcal{U} \times \mathcal{Y}$ and suppose $\|(u_0, y_0)\|_{\mathcal{U}\times\mathcal{Y}} \leq R$. Let $x_0 = \partial_n y_0$, hence $\|(u_0, x_0)\|_{\mathcal{U}\times\mathcal{X}} = \|(u_0, y_0)\|_{\mathcal{U}\times\mathcal{Y}} \leq R$. Let $H_{\partial_n \circ P//C}(u_0, x_0)^T = ((u_1, x_1)^T, (u_2, x_2)^T)$, thus

$$u_2 = Cx_2 = C(x_0 - x_1) = C(x_0 - \partial_n P u_1) = C(x_0 - \partial_n P(u_0 - u_2)). \tag{5.2}$$

Any solution $((u_0, y_0)^T, (\tilde{u}_1, y_1)^T, (\tilde{u}_2, y_2)^T)$ of $[P, C \circ \partial_n]$ satisfies:

$$\tilde{u}_2 = C \circ \partial_n \tilde{y}_2 = C(\partial_n y_0 - \partial_n P \tilde{u}_1) = C(x_0 - \partial_n P(u_0 - \tilde{u}_2)) \tag{5.3}$$

Therefore by equation (5.2) it follows that equation (5.3) has a solution $\tilde{u}_2 = u_2$. Consequently $\tilde{u}_1 = u_1$ and hence also $\partial_n y_1 = \partial_n P \tilde{u}_1 = \partial_n P u_1 = x_1$ and $\partial_n \tilde{y}_2 = x_2$ is a solution for $[P, C \circ \partial_n]$. By the uniqueness property for $[P, C \circ \partial_n]$ it follows that this solution is unique, hence $[P, C \circ \partial_n]$ is globally well posed, and $\Pi_{C \circ \partial_n // P}$ is defined. Since $[\partial_n \circ P, C]$ is $\mathcal{U} \times \mathcal{X}$ gain stable on $\mathcal{B}_{R,\mathcal{U}\times\mathcal{X}}(0)$, by letting $\gamma = \|\Pi_{C//\partial_n \circ P}|_{\mathcal{B}_{R,\mathcal{U}\times\mathcal{X}}(0)}\| < \infty$ it follows that

$$\left\| \begin{pmatrix} \tilde{u}_2 \\ y_2 \end{pmatrix} \right\|_{\mathcal{U}\times\mathcal{Y}} = \left\| \begin{pmatrix} u_2 \\ \partial y_2 \end{pmatrix} \right\|_{\mathcal{U}\times\mathcal{X}} = \left\| \begin{pmatrix} u_2 \\ x_2 \end{pmatrix} \right\|_{\mathcal{U}\times\mathcal{X}} \leq \gamma \left\| \begin{pmatrix} u_0 \\ x_0 \end{pmatrix} \right\|_{\mathcal{U}\times\mathcal{X}} = \gamma \left\| \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \right\|_{\mathcal{U}\times\mathcal{Y}}$$

as required. □

We remark that since in the case of $p < \infty$ the Euler constructions based on numerical differentiation require $k \leq r - 2$, it follows that these constructions are applicable to systems of relative degree $r \geq 2$ stabilizable by a function of the partial state $(y, y^{(1)}, \ldots, y^{(r-2)})$. In the case of $p = \infty$ the numerical differentiation based Euler controllers can be applied to feedbacks of a (partial) state $(y, y^{(1)}, \ldots, y^{(r-1)})$ (which in the case of systems with $r = n$ corresponds to full state feedback). Importantly, the high gain observer construction is applicable to full state feedback case of $r = n$ if $p = 2$ or $p = \infty$.

6 Fast sampling theorems

Up to this point we have been concerned with the reconstruction of output derivatives from measurement of the output only. To avoid the need for infinite dimensional storage as in the case of the standard Euler operator, we reconstructed the output derivatives via a process of zero or first order hold sampling of the output to give the perfectly and integrally sampled Euler reconstructions. However, in all our examples of Euler operators to date, the output measurement has been assumed to be available for feedback in continuous time, that is we have chosen $\Delta^0_{k,h} = I$. We will now show that the sampling machinery developed can also be directly applied to output measurement channel itself, thus giving rise to fast sampling theorems (possibly including derivative reconstruction). In this case we choose:

$$\Delta^0_{k,h} = H^i_h \circ S^j_h, \qquad i, j = 0, 1. \tag{6.1}$$

where $\Delta^0_{k,h}$ is given by either the zero order holds $i = 0, j = 0, 1$, or the first order holds $i = 1, j = 0, 1$ (the case where $\Delta^0_{k,h} = D_h$ as in (3.13) corresponds to the insertion of a pure delay in the closed loop, and the main results thus will establish robustness to delay perturbations). If $r > k > 1$, then $\Delta_{k,h} = (\Delta^0_{k,h}, \Delta^1_{k,h}, \ldots, \Delta^k_{k,h})$

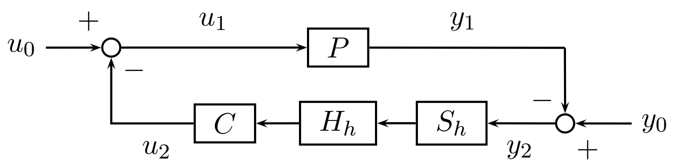

Figure 2: Sample and hold feedback configuration: $\delta_h = H_h \circ S_h$.

corresponds to a fully sampled output and output derivative reconstruction from sampled output measurements based on either perfect samples or integrally reconstructed samples at the sample times $\{t \geq 0 \mid t = nh,\ n \in \mathbb{N}\}$. Theorems 4.1 and 4.5 apply directly with the Euler approximation constants given in Proposition 6.1 below. Of special interest is the particular case where $r = 1$, ($\Delta_{k,h} = \Delta^0_{k,h}$), as it corresponds to the sampled data version of an output feedback controller (without derivative reconstruction), as studied in [2, 4, 8, 24, 25, 26, 27, 28, 32]. We consider causal controllers of the form

$$C \colon \mathcal{Y}_a \to \mathcal{U}_a. \tag{6.2}$$

The sampled data controller is defined to be:

$$C^{\text{sampled}}[h] \colon \mathcal{Y}_a \to \mathcal{U}_a \quad : \quad C^{\text{sampled}} = C \circ \Delta^0_{k,h}. \tag{6.3}$$

We summarize the Euler approximation properties in the following result:

$y \mapsto \Delta^0_{k,h}(y)$, $y \mapsto \delta_h(y)$	Euler approximation constants for $CW^{r,p}_0(\mathbb{R}_+, \mathbb{R}^N)$ and $W^{r,p}_0(\mathbb{R}_+, \mathbb{R}^N)$, $r \geq 1$
$\Delta^0_{k,h} = \delta_h = H^0_h \circ S^0_h, \quad rp > 1$	$\gamma_0(h) = 2hN\,(2(1 + 2hp))^{\frac{1}{p}}$ (6.4)
$\Delta^0_{k,h} = \delta_h = H^1_h \circ S^0_h, \quad rp > 1$	$\gamma_0(h) = 3hN\,(2(1 + 3hp))^{\frac{1}{p}}$ (6.5)
$\Delta^0_{k,h} = \delta_h = H^0_h \circ S^1_h$	$\gamma_0(h) = 3hN\,(2(1 + 3hp))^{\frac{1}{p}}$ (6.6)
$\Delta^0_{k,h} = \delta_h = H^1_h \circ S^1_h$	$\gamma_0(h) = 4hN\,(2(1 + 4hp))^{\frac{1}{p}}$ (6.7)

Table 2: Sample and hold operators and their Euler approximation constants.

Proposition 6.1 *Let* $1 \leq p \leq \infty$, $1 \leq k < r \leq \infty$, $h > 0$ *and let* $\Delta_{k,h}$ *be defined by* (3.5), (3.7), (6.1), (3.9) *and table 2. Then* $\Delta_{k,h}$ *is an* $\mathcal{Y}$ *Euler operator, and the Euler approximation constants for* $\mathcal{Y} = W^{r,p}(\mathbb{R}_+, \mathbb{R}^N)$ *are given by table 1, equations* (3.15), (3.16), (3.17) *or* (3.18) *and table 2, equations* (6.4), (6.5), (6.6) *or* (6.7). *Additionally if* $\delta^0_h, \Delta^0_{k,h} = H^1_h \circ S^1_h$ *or if* $rp > 1$ *and* $\delta^0_h, \Delta^0_{k,h} = H^1_h \circ S^0_h$, *then* $\Delta_{k,h}$ *is an regular* $\mathcal{Y}$ *Euler operator.*

Proof. It is straightforward to establish that $\rho(h) = c(h) = 2h$, $\rho(h) = c(h) = 3h$, $\rho(h) = c(h) = 3h$, $\rho(h) = c(h) = 4h$ for $\Delta^0_{k,h} = H^0_h \circ S^0_h$, $\Delta^0_{k,h} = H^1_h \circ S^0_h$, $\Delta^0_{k,h} = H^0_h \circ S^1_h$, $\Delta^0_{k,h} = H^1_h \circ S^1_h$ respectively and the Euler approximation constants then follow from Proposition 3.2. Regularity for $\Delta^0_{k,h} = H^1_h \circ S^0_h$ ($rp > 1$), $\Delta^0_{k,h} = H^1_h \circ S^1_h$ follows from the proof of Proposition 3.7 and Proposition 3.9. □

We now specialize Theorem 4.1 to the important case of sampled data output feedback control as in Figure 2. Due to the semi-global nature of the result, regularity of the sample and hold operation is required; hence this result applies to the case of first order holds.

Theorem 6.2 *Let* $1 \leq p \leq \infty$, $M, N \geq 1$ *and let* $\mathcal{W} := \mathcal{U} \times \mathcal{Y}$ *where* $\mathcal{W} = L^p(\mathbb{R}_+, \mathbb{R}^M) \times W_0^{1,p}(\mathbb{R}_+, \mathbb{R}^N)$ *or* $\mathcal{W} = L^\infty(\mathbb{R}_+, \mathbb{R}^M) \times CW_0^{1,\infty}(\mathbb{R}_+, \mathbb{R}^N)$*. Let* $R > \varrho > 0$*. Suppose* $P : \mathcal{U}_a \to \mathcal{Y}_a$ *is a causal operator with* $P(0) = 0$ *and* $C \colon \mathcal{Y} \to \mathcal{U}$ *is causal and locally Lipschitz, with Lipschitz function* Λ_C *given by* (4.2)*, and with* $C(0) = 0$*. Suppose the closed-loop system* $[P, C]$ *is gain stable on* $\mathcal{B}_{R,\mathcal{W}}(0)$*. Let* $0 < \alpha := \|\Pi_{C//P}|_{\mathcal{B}_{R,\mathcal{W}}(0)}\|_{\mathcal{W},\mathcal{W}} < \infty$ *and suppose* $h > 0$ *is such that*

$$\lambda := \gamma_0(h)\Lambda_C\left(\alpha R, \gamma_0(h)\alpha R\right) \leq \frac{R - \varrho}{R\alpha}. \tag{6.8}$$

Suppose $[P, C^{\text{sampled}}[h]]$ *has the uniqueness property, where* $C^{\text{sampled}}[h] \colon \mathcal{Y}_a \to \mathcal{U}_a$ *is given by* (6.3) *and where* $\Delta^0_{k,h} = H^1_h \circ S^0_h$ *(*$rp > 1$*) or* $\Delta^0_{k,h} = H^1_h \circ S^1_h$*. Then the closed-loop system* $[P, C^{\text{sampled}}[h]]$ *is gain stable on* $\mathcal{B}_{\varrho,\mathcal{W}}(0)$ *and*

$$\left\|\Pi_{C^{\text{sampled}}[h]//P}\Big|_{\mathcal{B}_{\varrho,\mathcal{W}}(0)}\right\|_{\mathcal{W},\mathcal{W}} \leq \frac{\alpha R(1 + \lambda)}{\varrho}. \tag{6.9}$$

Proof. Follows from Theorem 4.1. □

The global version of the result is equally applicable to either the zero or first order holds.

Theorem 6.3 *Let* $1 \leq p \leq \infty$, $M, N \geq 1$ *and let* $\mathcal{W} := \mathcal{U} \times \mathcal{Y}$ *where* $\mathcal{W} = L^p(\mathbb{R}_+, \mathbb{R}^M) \times W_0^{1,p}(\mathbb{R}_+, \mathbb{R}^N)$ *or* $\mathcal{W} = L^p(\mathbb{R}_+, \mathbb{R}^M) \times CW_0^{1,p}(\mathbb{R}_+, \mathbb{R}^N)$*. Let* $R > \varrho > 0$*. Suppose* $P : \mathcal{U}_a \to \mathcal{Y}_a$ *is a causal operator with* $P(0) = 0$ *and* $C \colon \mathcal{Y} \to \mathcal{U}$ *is causal and globally Lipschitz, with Lipschitz constant* L_C *given by* (4.9)*, and with* $C(0) = 0$*. Suppose the closed-loop system* $[P, C]$ *is gain stable. Let* $0 < \alpha := \|\Pi_{C//P}\|_{\mathcal{W},\mathcal{W}} < \infty$*. Suppose* $h > 0$ *is such that* $C^{\text{sampled}}[h]$ *is stabilizable,*

$$\lambda := \gamma(h)L_C < \frac{1}{\alpha},$$

and that $[P, C^{\text{sampled}}[h]]$ *is either globally or regularly well posed on* $\mathcal{W}$*, where* $C^{\text{sampled}}[h] \colon \mathcal{Y}_a \to \mathcal{U}_a$ *is given by* (6.3) *and where* $\Delta^0_{k,h} = H^0_h \circ S^1_h$ *or* $\Delta^0_{k,h} = H^1_h \circ S^1_h$ *or* $\Delta^0_{k,h} = H^0_h \circ S^0_h$*,* $\Delta^0_{k,h} = H^1_h \circ S^0_h$ *(*$rp > 1$*). Then the closed-loop system* $[P, C^{\text{sampled}}[h]]$ *is gain stable and*

$$\left\|\Pi_{C^{\text{sampled}}[h]//P}\right\|_{\mathcal{W},\mathcal{W}} \leq \frac{\alpha(1 + \lambda)}{1 - \alpha\lambda}. \tag{6.10}$$

Proof. Follows from Theorem 4.5. □

7 Initial conditions

Up to this point we have implicitly required zero initial conditions: our analysis has started at time $t = 0$, and we have required that $P(0) = C(0) = 0$ (if $0 \in \Omega$ then gain stability over Ω implies $H_{P,C}(0) = 0$ which in turn implies $P(0) = C(0) = 0$). In the case of the Euler controllers, this has been enforced by defining delay and samples of signals at negative times to be zero or in the case of the observer based controller initial conditions were set to zero, see equations (3.10), (3.19), (3.20) and (3.29). We now consider the case where non-zero initial conditions are present. We specify a non zero plant and controller initial condition by removing the requirement that $C(0) = P(0) = 0$. This includes permitting non-zero initialisations of the Euler controllers.

The key observation is that if the closed loop system is time-invariant then the closed loop generates bounded signals from any initial condition which corresponds to a closed loop state which is reachable in finite time, simply by considering an appropriately time shifted version of the system. Since there are disturbances acting at both the input and output channels, this means that closed loop stability is maintained from any initial condition corresponding to a controller state and a plant state which are reachable from the open loop controller input and from the open loop plant input respectively. We formalise this below.

Let $w_0 \in \mathcal{W}$ and let desired plant and controller initial conditions x_P^0, x_C^0 be given. The initial conditions have the same type as the corresponding state: for the Euler controllers based on numerical differentiation the state domain involves the delay line; that is if C is defined via the composition of an Euler operator Δ and a memoryless nonlinear operator F, then a controller $C\colon \mathcal{Y}_a \to \mathcal{U}_a$, $u_2 = Cy_2$ with state $x_C^\tau \in \mathcal{Y}[0, rh]$ at time $\tau > rh$ is given by:

$$(x_C^\tau)(s) = y_2(\tau - rh + s), \quad s \in [0, rh].$$

For high gain observers based controllers the states are simply the observer states.

Let $w_0 \in \mathcal{W}$. Suppose $\tilde{w}_0 = \tilde{w}_0(x_P^0, x_C^0, w_0)$ satisfies: (i) $v_0 = \begin{cases} \tilde{w}_0(t) & \text{if } t < \tau \\ w_0(t-\tau) & \text{if } t \geq \tau \end{cases} \in \mathcal{W}$, (ii) $\tilde{w}_0 = \begin{pmatrix} \tilde{u}_1 + C\tilde{y}_2 \\ \tilde{y}_2 + P\tilde{u}_1 \end{pmatrix}$ for some $\tilde{u}_1 \in \mathcal{U}$, $\tilde{y}_2 \in \mathcal{Y}$, and (iii) $X_P^\tau \tilde{u}_1 = x_P^0$, $X_C^\tau \tilde{y}_2 = x_C^0$, where $X_P^\tau u$ is a state vector for the plant P at time τ given input u and zero initial conditions; and $X_C^\tau y$ is a state for the controller C at time τ given input y and zero initial conditions.

Note that the condition $v_0 \in \mathcal{W}$ is a compatibility condition; informally it is the requirement that $\tilde{w}_0$ and w_0 can be concatenated in the space $\mathcal{W}$, i.e. the concatenated signal is appropriately smooth at the point of concatenation, e.g. if $\mathcal{W} = W^{r,p}(\mathbb{R}_+, \mathbb{R})$ then the requirement is that $\tilde{w}_0^{(i)}(\tau) = w_0^{(i)}(0)$, $1 \leq i \leq r$ and $v_0^{(r)}$ is uniformly continuous at τ.

Noting that if $((u_1, y_1)^T, (u_2, y_2)^T) = H_{P,C}\tilde{w}_0$, then $u_1 = \tilde{u}_1$, $y_2 = \tilde{y}_2$, we can see that the time invariance of P and C gives:

$$\begin{aligned} \left\|\Pi_{C(x_C^0)//P(x_P^0)} w_0\right\|_{\mathcal{W}} &= \left\|(I - T_\tau)\Pi_{C//P} v_0\right\|_{\mathcal{W}} \\ &\leq \left\|\Pi_{C//P} v_0\right\|_{\mathcal{W}} \\ &\leq g[\Pi_{C//P}] \left(\|R_\tau \tilde{w}_0\|_{\mathcal{W}[0,\tau)} + \|w_0\|_{\mathcal{W}}\right) \end{aligned} \tag{7.1}$$

We let χ be defined as follows:

$$\chi = \chi(x_C^0, x_P^0, w_0) = \inf_{\substack{\tau \geq 0, \\ \tilde{u}_1 \in \mathcal{U},\, \tilde{y}_2 \in \mathcal{Y}}} \left\{ r \geq 0 \mid r = \|R_\tau \tilde{w}_0\|_{\mathcal{W}[0,\tau]},\ \tilde{w}_0 \text{ satisfies (i), (ii), (iii)} \right\}. \tag{7.2}$$

Then:

$$\left\|\Pi_{C(x_C^0)//P(x_P^0)} w_0\right\|_{\mathcal{W}} \leq g[\Pi_{C//P}] \left(\chi(x_C^0, x_P^0, w_0) + \|w_0\|_{\mathcal{W}}\right) \tag{7.3}$$

Hence any gain or gain-function bound for the initial condition free case, e.g. inequalities (4.4), (4.10), (6.9) or (6.10), (possibly over a finite set $0 \leq \|w_0\|_{\mathcal{W}} \leq R$) implies a similar bound for the case of non-zero initial conditions (over the finite set $0 \leq \|w_0\|_{\mathcal{W}} \leq R - \chi$ if $R < \infty$), thus giving closed loop signal bounds in terms of initial conditions and output reconstruction initialisation errors.

We also remark that in the case of linear systems, a further coherent approach to non-zero initial conditions in this context has been given in [11].

8 Conclusions

The results in this paper first establish nonlinear separation principles in the setting of a general nonlinear input-output theory. The existence of certain closed loop gain properties with a nonlinear plant and a controller based on measurements of derivatives of the output (for example a state measurement) is used to guarantee the existence of a controller based on direct measurement of the output only. A variety of constructions of such a controller is given, based on (sampled) numerical differentiation or high gain observers. The proofs are fully constructive and the results give conditions under which semi-global and global stability can be attained in a variety of signal space settings. Stability is achieved in the sense of disturbance attenuation; disturbances are present at both the input and output channels; consequently robust stability theory [15] gives automatic guarantees on the robustness to unmodelled dynamics.

The sampling procedures utilized to develop finite dimensional realizations of the derivative reconstructions based on numerical differentiation are then also applied directly to the output measurement channel itself, thus establishing fully sampled versions of the results. The results are also specialized to the case where no derivative construction is required, thus establishing conditions under which measurement feedback controllers can be replaced by sampled data controllers under a process of zero or first order hold sampling; once again in the context of disturbance attenuation.

The approach taken in this paper is strongly distinct to previous approaches to nonlinear separation principles and to classical approaches to fast sampling theorems; we do not use a state space approach or a singular perturbation analysis at all to generate the core results: state space calculations are used only for specific computations on explicit examples. One important consequence of this approach is generality: we make no assumptions on the underlying realisation of the plant or controller; assumptions are only made on key stability or continuity properties with respect to both input and output disturbances. Therefore, the underlying systems may be generated from e.g. lumped, distributed or delay models, or even defy any form of model representation, and the results are applicable to systems with initial conditions. Finally we emphasize that the notions of gain stability which we consider are natural performance measures in the context of robust stability and guarantee robustness to unmodelled dynamics. We therefore consider the required conditions (or similar) to be a natural goal of controller synthesis.

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9 Appendix A

The semi-global robust stability result given below is based on [15, Th. 5], but accounts for technicalities arising from the choices of signal space considered in this paper (in particular due to the fact that $\mathcal{W}$ may not be closed under the action of T_τ), and differences arising from the treatment of the concepts of well-posedness. A key difference between this result and [15, Th. 5] is that no a-priori assumption of existence of solutions is made on the perturbed closed loop $[P_1, C]$, there is just a requirement that solutions are unique where defined; the gap construction itself is used to guarantee global well posedness. This is much more than a mathematical nicety; the uniqueness property is a substantively weaker property than well posedness, and can typically be verified much more easily. We also observe that the proof below is simplified by the application of the Schauder fixed point theorem [31] as opposed to the treatment in [15, Th. 5] which uses an argument based on the Leray-Schauder degree (see also [5]).

Observe that we require the signal spaces to be truncation complete spaces, where a space $\mathcal{V}$ is said to be truncation complete if $V[0,\tau)$ is complete for all $0 < \tau < \infty$.

We first define a directed gap distance appropriate for semi-global applications,

$$\vec{\delta}_{\mathcal{W},r}(P_1,P_2) := \begin{cases} \inf_{\Phi\in\mathcal{O}^{\mathcal{W},r}_{P_1,P_2}} \sup_{\substack{x\in\mathcal{G}_{P_1}\setminus\{0\},\\ \|R_\tau x\|_{\mathcal{W}[0,\tau)}\le r,\ \tau>0}} \left(\frac{\|R_\tau(\Phi-I)|_{\mathcal{G}_{P_1}}x\|_{\mathcal{W}[0,\tau)}}{\|R_\tau x\|_{\mathcal{W}[0,\tau)}}\right), & \text{if } \mathcal{O}^{\mathcal{W},r}_{P_1,P_2}\neq\emptyset \\ \infty, & \text{if } \mathcal{O}^{\mathcal{W},r}_{P_1,P_2}=\emptyset \end{cases} \tag{9.1}$$

where

$$\mathcal{O}^{\mathcal{W},r}_{P_1,P_2} := \left\{\Phi\colon \mathcal{G}_{P_1}\cap\{x\in\mathcal{W}\mid \|R_\tau x\|_{\mathcal{W}[0,\tau)}\le r,\ \tau>0\}\to\mathcal{G}_{P_2} \;\middle|\; \begin{array}{l}\Phi \text{ is causal, surjective, } \Phi(0)=0 \text{ and}\\ R_\tau(\Phi-I) \text{ is compact for all } \tau>0\end{array}\right\}.$$

We now give the semi-global robust stability result, formulated in the usual setting of perturbations to the plant P. In the context of this paper, we are interested in perturbations of the controller C, to which the theorem also applies by interchanging the roles of P and C.

Theorem 9.1 *Let $\mathcal{U}$, $\mathcal{Y}$ be truncation complete signal spaces, and let $\mathcal{W}=\mathcal{U}\times\mathcal{Y}$. Consider $P\colon\mathcal{U}_a\to\mathcal{Y}_a$, $P_1\colon\mathcal{U}_a\to\mathcal{Y}_a$ and $C\colon\mathcal{Y}_a\to\mathcal{U}_a$. Let $R>0$ and suppose $[P,C]$ is gain stable on $\mathcal{B}_R\subset\mathcal{W}$, and $[P_1,C]$ has the uniqueness property. Let $0<\varepsilon<1$. If $\gamma=\|\Pi_{P//C}|_{\mathcal{B}_R}\|_{\mathcal{W},\mathcal{W}}$ and*

$$\vec{\delta}_{\mathcal{W},\gamma R}(P,P_1)\le\frac{1-\varepsilon}{\gamma}$$

then the closed-loop system $[P_1,C]$ is gain stable on $\mathcal{B}_{\varepsilon R}$ with

$$\|\Pi_{P_1//C}|_{\mathcal{B}_{\varepsilon R}}\|_{\mathcal{W},\mathcal{W}}\le\|\Pi_{P//C}|_{\mathcal{B}_R}\|_{\mathcal{W},\mathcal{W}}\left(\frac{1+\vec{\delta}_{\mathcal{W},\gamma R}(P,P_1)}{\varepsilon}\right).$$

Proof. Let $0<\tau<\infty$. Since $\left\|\Pi_{P//C}|_{\mathcal{B}_R}\right\|_{\mathcal{W},\mathcal{W}}\ge 1$, it follows that $\vec{\delta}_{\mathcal{W},\gamma R}(P,P_1)<\infty$ and hence there exists a causal mapping $\Phi\colon\mathcal{G}_P\cap\mathcal{B}_{\gamma R}\to\mathcal{G}_{P_1}$ such that $R_\tau(\Phi-I)$ is compact and the following inequality holds for all $v\in\mathcal{W}$ such that $\|R_\tau v\|_\tau\le\gamma R$:

$$\|R_\tau(\Phi-I)v\|_\tau\le\vec{\delta}_{\mathcal{W},\gamma R}(P,P_1)\cdot\|R_\tau v\|_\tau. \tag{9.2}$$

Suppose $w \in \mathcal{W}$, $\|R_\tau w\|_\tau \leq \varepsilon R$, and consider the equation

$$R_\tau w \;=\; R_\tau(I + (\Phi - I)\Pi_{P//C})\bar{x} = R_\tau(\Pi_{C//P} + \Phi\Pi_{P//C})\bar{x}. \tag{9.3}$$

We claim that equation (9.3) has a solution $\bar{x} \in \mathcal{W}$, with $x = R_\tau\bar{x} \in V$ where:

$$V = \left\{ x \in \mathcal{W}[0,\tau) \;\middle|\; \|x\|_\tau \leq \frac{\|R_\tau w\|_\tau}{\varepsilon} \right\}.$$

By definition of $\mathcal{W}[0,\tau)$, for every $x \in V$, there exists $\bar{x} \in \mathcal{W}$ such that $x = R_\tau\bar{x}$. Hence since $(I - \Phi)\Pi_{P//C}$ is causal, the following operator is well-defined:

$$Q_w \colon V \to \mathcal{W}[0,\tau) \quad : \quad x \mapsto R_\tau w + R_\tau(I - \Phi)\Pi_{P//C}\bar{x}.$$

Then for all $x \in V$, and for any choice $\bar{x} \in \mathcal{W}$ s.t. $x = R_\tau\bar{x}$, it follows that $\|R_\tau\bar{x}\|_\tau = \|x\|_\tau \leq \frac{\|R_\tau w\|_\tau}{\varepsilon} \leq R$. Since $\|R_\tau\Pi_{P//C}\bar{x}\|_\tau \leq \|\Pi_{P//C}|_{\mathcal{B}_R}\|_{\mathcal{W},\mathcal{W}}\|R_\tau\bar{x}\|_\tau \leq \gamma R$ it follows from (9.2) with $v = \Pi_{P//C}\bar{x}$ that:

$$\begin{aligned}
\|R_\tau(\Phi - I)\Pi_{P//C}\bar{x}\|_\tau &\leq \vec{\delta}_{\mathcal{W},\gamma R}(P,P_1)\cdot\|R_\tau\Pi_{P//C}\bar{x}\|_\tau \\
&\leq \vec{\delta}_{\mathcal{W},\gamma R}(P,P_1)\cdot\|\Pi_{P//C}|_{\mathcal{B}_R}\|_{\mathcal{W},\mathcal{W}}\|R_\tau\bar{x}\|_\tau \\
&\leq (1-\varepsilon)\frac{\|R_\tau w\|_\tau}{\varepsilon} \qquad (9.4)
\end{aligned}$$

Then:

$$\|Q_w x\|_\tau \leq \|R_\tau w\|_\tau + \|R_\tau(I - \Phi)\Pi_{P//C}\bar{x}\|_\tau \leq \frac{\|R_\tau w\|_\tau}{\varepsilon}. \tag{9.5}$$

Therefore $Q_w(V) \subset V$. Since $R_\tau(I - \Phi)$ is compact and $\Pi_{P//C}$ is bounded, it follows that Q_w is compact. Since $\mathcal{W}$ is truncation complete, $\mathcal{W}[0,\tau)$ is a Banach space. Hence since $V \subset \mathcal{W}[0,\tau)$ is non-empty, closed, bounded and convex, it follows by Schauder's fixed point theorem that Q_w has a fixed point in V. Hence equation (9.3) has a solution $\bar{x} \in \mathcal{W}$, with $x = R_\tau\bar{x} \in V$ as claimed.

By the uniqueness property for $[P_1, C]$, $\Pi_{P_1//C} \colon \mathcal{W} \to \mathcal{W}_a$ is defined. Let $x \in V$ and suppose $\bar{x} \in \mathcal{W}$ be a solution of (9.3) with $x = R_\tau\bar{x}$. Since $w_1 = \Phi\Pi_{P//C}\bar{x} \in \mathcal{G}_{P_1}$, $w_2 = \Pi_{C//P}\bar{x} \in \mathcal{G}_C$ and Φ, $\Pi_{P_1//C}$, $\Pi_{P//C}$, $\Pi_{C//P}$ are causal, it follows from equation (9.3) that $(w, R_\tau w_1, R_\tau w_2) = (w, R_\tau\Phi\Pi_{P//C}\bar{x}, R_\tau\Pi_{C//P}\bar{x})$ is a solution for $[P_1, C]$. Since this holds for all $\tau > 0$, it follows that $\omega_w = \infty$ for $[P_1, C]$. Consequently $\mathrm{dom}\Pi_{P_1//C}w = [0,\infty)$ and thus $[P_1, C]$ is globally well posed. Since $x \in V$ and $R_\tau\Pi_{P_1//C}w = R_\tau\Phi\Pi_{P//C}\bar{x}$, the following inequality holds for all $\tau > 0$, $w \in \mathcal{W}$, $\|R_\tau w\|_\tau \leq \varepsilon R$:

$$\begin{aligned}
\|R_\tau\Pi_{P_1//C}w\|_\tau &= \|R_\tau\Phi\Pi_{P//C}\bar{x}\|_\tau \\
&\leq \|R_\tau\Pi_{P//C}\bar{x}\|_\tau + \|R_\tau(\Phi - I)\Pi_{P//C}\bar{x}\|_\tau \\
&\leq (1 + \vec{\delta}_{\mathcal{W},\gamma R}(P,P_1))\big\|\Pi_{P//C}|_{\mathcal{B}_R}\big\|_{\mathcal{W},\mathcal{W}}\|x\|_\tau \\
&\leq (1 + \vec{\delta}_{\mathcal{W},\gamma R}(P,P_1))\big\|\Pi_{P//C}|_{\mathcal{B}_R}\big\|_{\mathcal{W},\mathcal{W}}\frac{\|R_\tau w\|_\tau}{\varepsilon}
\end{aligned}$$

hence $[P_1, C]$ is gain stable on $\mathcal{B}_{\varepsilon R}$ and:

$$\|\Pi_{P_1//C}|_{\mathcal{B}_{\varepsilon R}}\|_{\mathcal{W},\mathcal{W}} \leq \|\Pi_{P//C}|_{\mathcal{B}_R}\|_{\mathcal{W},\mathcal{W}}\left(\frac{1 + \vec{\delta}_{\mathcal{W},\gamma R}(P,P_1)}{\varepsilon}\right)$$

as required. □