Modular Games for Coalgebraic Fixed Point Logics

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Abstract

We build on existing work on finitary modular coalgebraic logics [3,4], which we extend with general fixed points, including CTL- and PDL-like fixed points, and modular evaluation games. These results are generalisations of their correspondents in the modal μ-calculus, as described e.g. in [19]. Inspired by recent work of Venema [21], we provide our logics with evaluation games that come equipped with a modular way of building the game boards. We also study a specific class of modular coalgebraic logics that allow for the introduction of an implicit negation operator.

Keywords: coalgebra, modal logic, fixed point logic, parity games

1 Introduction

Modular coalgebraic logics were introduced in [3,4], where it was shown that the syntax and semantics of logics for $\mathbb{T}$-coalgebras, with $\mathbb{T}$ an ω-accessible Set functor, can be defined modularly by exploiting the structure of $\mathbb{T}$, and moreover, that expressiveness of the resulting logics w.r.t. behavioural equivalence, as well as sound and complete proof systems for these logics, can also be derived modularly. In terms of expressivity, these logics are more expressive than logics induced by monadic predicate liftings, as considered in [16], but are as expressive as logics induced by finitary polyadic predicate liftings, as defined in [17].

Coalgebraic fixed point logics were first considered in the work of Venema [21], where a finitary version of the coalgebraic logic of Moss [14] was used as the underlying modal language. Our motivation for considering fixed point logics over different

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modal languages is rooted in our interest in verification techniques for systems modelled as coalgebras. In this setting, the logics obtained through the modular techniques described in [3] appear to be better suited as specification logics.

The syntax of modular coalgebraic logics is based on the notion of *syntax constructor* [3], while their semantics uses a notion of *one-step semantics* for a syntax constructor [3], which generalises the *predicate liftings* of [16]. The logics obtained from syntax constructors are originally boolean, but in order to ensure that fixed points have a well-defined semantics, we leave out negation from these languages. However, for the specific class of syntax constructors which are *closed under duals* (that is, for each modality they specify, a semantically dual modality is also specified), a safe negation becomes definable in the language, and thus the expressivity of the logic stays as before. For this class of syntax constructors, we also introduce a general way of defining CTL- and PDL-like fixed points, and illustrate their applicability via examples. For instance, we obtain the fixed points of Dynamic Epistemic Logic [2] via the coalgebraic semantics for this logic described in [5]. In standard model checking terminology, these fixed points are referred to as 'alternation-free', and enjoy a linear-time model checking algorithm based on *parity games* [8].

The results concerning the implicit negation and the alternation-free fragments of our logics make use of the notion of an $\mathcal{S}$*-modality* (of some finite arity), with $\mathcal{S}$ a syntax constructor with an associated one-step semantics. This notion also allows us to relate logics induced by sets of polyadic predicate liftings, as considered in [17], with logics induced by syntax constructors. As a result, we obtain a way to add fixed points to logics of the former type.

In [21], deciding about the satisfaction of formulae by states of a coalgebra is achieved through deciding the winner of so-called *evaluation games*. These are parity games that generalize those for the modal μ-calculus [15,7,10,19,20,22], by replacing the usual single moves of either the verifier or the refuter in positions that correspond to modal formulae by two consecutive moves: a move of the verifier, who has to exhibit a relation between sub-formulae of the original formula and states of the coalgebra, that witnesses the satisfaction of the given modal formula by a state of the coalgebra, and a move of the refuter, who has to choose an element of this relation. These two consecutive moves are, in turn, inspired by similar moves in the bisimulation game of Baltag [1].

We introduce a variant of the evaluation games of [21] tailored to our fixed point logics, and prove their adequacy w.r.t. the standard coalgebraic semantics. The only difference w.r.t. [21] is in the moves corresponding to modal positions, where the one-step semantics for the syntax constructor defining the underlying modal language is used to define the valid moves. The distinctive feature of our games, however, is that they come equipped with *one-step games*. These adequately replace the two consecutive moves, of the verifier followed by the refuter, in modal positions, by an equivalent sub-game played between the verifier and the refuter. The use of one-step games has some advantages: on the one hand, it provides a way to *construct* the board of the evaluation games *by induction on the structure of the signature functor*; on the other hand, only witnessing relations that are relevant to deciding the winner of an evaluation game are accounted for in the one-step games, thus significantly reducing the size of the resulting parity games.

2 Preliminaries

Existing work [3] shows how to modularly derive coalgebraic modal logics for inductively-defined classes of endofunctors, including the class of so-called *polynomial functors* (that is, functors defined inductively on the identity, constant, powerset and discrete probability distribution functors, using products, coproducts, exponentiation and functor composition). A modal language with finitary modalities can be defined using a *syntax constructor* [3], that is, an inclusion-preserving, ω-accessible set endofunctor $\mathcal{S} : \mathsf{Set} \to \mathsf{Set}$. Given a syntax constructor $\mathcal{S}$, the (negation-free) *modal language* $\mathcal{L}^{\mathcal{S}}$ it induces is the least set of formulae which is closed under finite (including empty) conjunctions and disjunctions and under the application of $\mathcal{S}$. Equivalently (using the ω-accessibility of $\mathcal{S}$), $\mathcal{L}^{\mathcal{S}}$ is defined inductively by:

$$\mathcal{L}^{\mathcal{S}} \ni \phi := \mathsf{ff} \mid \mathsf{tt} \mid \phi \vee \phi \mid \phi \wedge \psi \mid \Psi$$

where $\Psi \in \mathcal{S}(F)$ with $F \subseteq \mathcal{L}^{\mathcal{S}}$ finite.

Simple syntax constructors can be used to define modal languages for (coalgebras of) the constant, identity, powerset and discrete probability distribution functors, as follows:

$$\begin{aligned} \mathcal{S}_A(L) &= \{a, \neg a \mid a \in A\} \\ \mathcal{S}_{\mathbb{I}\mathsf{d}}(L) &= \{\bigcirc\phi \mid \phi \in L\} \\ \mathcal{S}_{\mathbb{P}}(L) &= \{\Box\phi,\ \Diamond\phi \mid \phi \in L\} \\ \mathcal{S}_{\mathbb{D}}(L) &= \{L_p\phi,\ G_p\phi \mid \phi \in L,\ p \in [0,1] \cap \mathbb{Q}\} \end{aligned}$$

In the definition of $\mathcal{S}_{\mathbb{D}}$, $L_p\phi$ is to be read as "ϕ holds in the next state with probability *at least* p", whereas $G_p\phi$ is to be read as "ϕ holds in the next state with probability *greater than* p".

Syntax constructors can be combined to obtain modal languages for (coalgebras of) functors structured using products, coproducts, exponentiation with constant exponent E and functor composition, as follows:

$$\begin{aligned} (\mathcal{S}_1 \otimes \mathcal{S}_2)(L) &= \{[\pi_i]\phi \mid \phi \in \overline{\mathcal{S}_i(L)}, i = 1, 2\} \\ (\mathcal{S}_1 \oplus \mathcal{S}_2)(L) &= \{[\kappa_i]\phi,\ \langle\kappa_i\rangle\phi \mid \phi \in \overline{\mathcal{S}_i(L)}, i = 1, 2\} \\ (\mathcal{S} \odot E)(L) &= \{[e]\phi \mid e \in E, \phi \in \overline{\mathcal{S}(L)}\} \\ (\mathcal{S}_1 \circledcirc \mathcal{S}_2)(L) &= \mathcal{S}_1(\overline{\mathcal{S}_2(L)}) \end{aligned}$$

where $\overline{\cdot}$ denotes closure under finite (including empty) conjunctions and disjunctions. For a polynomial functor, the resulting syntax can be expressed using a BNF with multiple levels, one for each ingredient of the functor [3,4].

Example 2.1 An alternative way of associating a syntax constructor with each ω-accessible, weak pullback preserving endofunctor $\mathbb{T} : \mathsf{Set} \to \mathsf{Set}$ is to take $\mathcal{S} = \mathbb{T}$. This is the approach followed in [14] [4].

We note that all of the above definitions are negation-free variants of the definitions in [3]. The restriction to negation-free fragments is a common way to ensure

[4] The restriction regarding the ω-accessibility of $\mathbb{T}$ is not present in [14]. Here it is required since we are concerned with languages with finitary modalities.

that fixed point logical operators defined on top of these languages have a well-defined (fixed point) semantics, see e.g. [19].

Given a functor $\mathbb{T}$ and a syntax constructor $\mathcal{S}$, a semantics for the modal language $\mathcal{L}^{\mathcal{S}}$ w.r.t. $\mathbb{T}$-coalgebras can be obtained from a *one-step semantics for* $\mathcal{S}$ *w.r.t.* $\mathbb{T}$ [3]. Given a set L (of formulae) and a set X (of points), a *one-step semantics* $[\![\mathcal{S}]\!]$ *for* $\mathcal{S}$ *w.r.t.* $\mathbb{T}$ maps interpretations of the formulae in L over the elements of X (given by functions $d : L \to \mathcal{P}X$, or equivalently, by relations $\models \subseteq X \times L$) to interpretations of the formulae in $\mathcal{S}L$ over the elements of $\mathbb{T}X$ (given by functions $d' : \mathcal{S}L \to \mathcal{P}\mathbb{T}X$, or by relations $\models' \subseteq \mathbb{T}X \times \mathcal{S}L$), see [3] for details. (Here, $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$ also denotes the powerset functor, but a different notation is employed when this functor is not used as a signature functor.) The interpretation of formulae in $\mathcal{L}^{\mathcal{S}}$ over the states of a $\mathbb{T}$-coalgebra (C, γ) is then defined inductively on the structure of formulae, by

$$c \models \Psi \qquad \text{iff} \qquad \gamma(c)\,([\![\mathcal{S}]\!] \models_{Base(\Psi)})\,\Psi$$

and the usual definitions for finite conjunctions and disjunctions, where for $\Psi \in \mathcal{S}(F)$ with $F \subseteq \mathcal{L}^{\mathcal{S}}$, $Base(\Psi)$ is the smallest F with this property, while $\models_{Base(\Psi)}$ is the restriction of the relation $\models\, \subseteq C \times \mathcal{L}^{\mathcal{S}}$ to $C \times Base(\Psi)$ (and thus $([\![\mathcal{S}]\!] \models_{Base(\Psi)}) \subseteq \mathbb{T}C \times \mathcal{S}(Base(\Psi))$).

One-step semantics for the syntax constructors $\mathcal{S}_A$, $\mathcal{S}_{\mathbb{I}\mathsf{d}}$, $\mathcal{S}_{\mathbb{P}}$ and $\mathcal{S}_{\mathbb{D}}$ can be defined in a natural way. Specifically, they map a relation $\models \subseteq X \times L$ to the relations $[\![\mathcal{S}_A]\!] \models$, $[\![\mathcal{S}_{\mathbb{I}\mathsf{d}}]\!] \models$, $[\![\mathcal{S}_{\mathbb{P}}]\!] \models$, $[\![\mathcal{S}_{\mathbb{D}}]\!] \models$ defined as follows:

$$\begin{array}{rcl}
b\,([\![\mathcal{S}_A]\!] \models)\, a & \text{iff} & b = a \\
b\,([\![\mathcal{S}_A]\!] \models)\, \neg a & \text{iff} & b \neq a \\
x\,([\![\mathcal{S}_{\mathbb{I}\mathsf{d}}]\!] \models)\, \bigcirc \phi & \text{iff} & x \models \phi \\
Y\,([\![\mathcal{S}_{\mathbb{P}}]\!] \models)\, \Box\phi & \text{iff} & x \models \phi \text{ for all } x \in Y \\
Y\,([\![\mathcal{S}_{\mathbb{P}}]\!] \models)\, \Diamond\phi & \text{iff} & x \models \phi \text{ for some } x \in Y \\
\mu\,([\![\mathcal{S}_{\mathbb{D}}]\!] \models)\, L_p\phi & \text{iff} & \sum\limits_{x \models \phi} \mu(x) \geq p \\
\mu\,([\![\mathcal{S}_{\mathbb{D}}]\!] \models)\, G_p\phi & \text{iff} & \sum\limits_{x \models \phi} \mu(x) > p
\end{array}$$

Also, one-step semantics for syntax constructors built using $\_ \otimes \_$, $\_ \oplus \_$, $\_ \odot E$ and $\_ \circledcirc \_$, w.r.t. functors built using products, coproducts, exponentiation with constant exponent E and respectively functor composition, can be modularly derived from one-step semantics for the ingredient syntax constructors [3]. Concretely, if $[\![\mathcal{S}_i]\!]$ is a one-step semantics for $\mathcal{S}_i$ w.r.t. $\mathbb{T}_i$, with $i = 1, 2$, then the one-step semantics $[\![\mathcal{S}_1 \otimes \mathcal{S}_2]\!]$ for $\mathcal{S}_1 \otimes \mathcal{S}_2$ w.r.t. $\mathbb{T}_1 \times \mathbb{T}_2$, $[\![\mathcal{S}_1 \oplus \mathcal{S}_2]\!]$ for $\mathcal{S}_1 \oplus \mathcal{S}_2$ w.r.t. $\mathbb{T}_1 + \mathbb{T}_2$, $[\![\mathcal{S}_1 \odot E]\!]$ for $\mathcal{S}_1 \odot E$ w.r.t. ${\mathbb{T}_1}^E$ and $[\![\mathcal{S}_1 \circledcirc \mathcal{S}_2]\!]$ for $\mathcal{S}_1 \circledcirc \mathcal{S}_2$ w.r.t. $\mathbb{T}_1 \circ \mathbb{T}_2$ are defined by:

$$\begin{array}{rcl}
(t_1, t_2)\,([\![\mathcal{S}_1 \otimes \mathcal{S}_2]\!] \models)\,[\pi_i]\phi_i & \text{iff} & t_i([\![\mathcal{S}_i]\!] \models)\phi_i \\
t\,([\![\mathcal{S}_1 \oplus \mathcal{S}_2]\!] \models)\,[\kappa_i]\phi_i & \text{iff} & t = \iota_i(z) \in \iota_i(\mathbb{T}_i X) \text{ implies } z\,([\![\mathcal{S}_i]\!] \models)\,\phi_i \\
t\,([\![\mathcal{S}_1 \oplus \mathcal{S}_2]\!] \models)\,\langle\kappa_i\rangle\phi_i & \text{iff} & t = \iota_i(z) \in \iota_i(\mathbb{T}_i X) \text{ and } z\,([\![\mathcal{S}_i]\!] \models)\,\phi_i \\
f\,([\![\mathcal{S}_1 \odot E]\!] \models)\,[e]\phi_1 & \text{iff} & f(e)\,([\![\mathcal{S}_1]\!] \models)\,\phi_1
\end{array}$$

where $\iota_i : \mathbb{T}_i X \to \mathbb{T}_1 X + \mathbb{T}_2 X$ are the coproduct injections, and

$$([\![\mathcal{S}_1 \circledcirc \mathcal{S}_2]\!] \models) \;=\; ([\![\mathcal{S}_1]\!] \overline{[\![\mathcal{S}_2]\!] \models})$$

for each $\models \subseteq X \times L$, where $\overline{[\![\mathcal{S}_2]\!] \models} \subseteq \mathbb{T}_2 X \times \overline{\mathcal{S}_2 L}$ denotes the natural extension of the relation $([\![\mathcal{S}_2]\!] \models) \subseteq \mathbb{T}_2 X \times \mathcal{S}_2 L$ to formulae containing finite conjunctions and disjunctions. (See [3] for further details.)

Example 2.2 Given an ω-accessible endofunctor $\mathbb{T}$, a one-step semantics for the syntax constructor $\mathcal{S} = \mathbb{T}$ w.r.t. the functor $\mathbb{T}$ can be defined by mapping each relation $\models \subseteq X \times L$ to the relation $\models^{\mathbb{T}} \subseteq \mathbb{T}X \times \mathbb{T}L$ defined by $t \models^{\mathbb{T}} \Phi$ iff there exists $w \in (\mathbb{T} \models)$ such that $\mathbb{T}\pi_1(w) = t$ and $\mathbb{T}\pi_2(w) = \Phi$. (Here, $\pi_1 : X \times L \to X$ and $\pi_2 : X \times L \to L$ denote the canonical projections.)

Example 2.3 *Transition systems with spatial structure* were proposed in [13] as a general model for spatial logic. They are defined as coalgebras of the functor $\mathbb{T} = \mathbb{P} \times (1 + \mathbb{P} \circ (\mathbb{I}\mathsf{d} \times \mathbb{I}\mathsf{d}))$, where 1 denotes the constant functor induced by a one-element set. Thus, $\mathbb{T}$-coalgebras incorporate non-deterministic structure, through the first component of $\mathbb{T}$, as well as spatial structure, through the second component of $\mathbb{T}$. As far as the spatial structure is concerned, a one-step observation of $0 \in 1$ corresponds to *inactive states*, whereas a one-step observation in $\mathbb{P} \circ (\mathbb{I}\mathsf{d} \times \mathbb{I}\mathsf{d})$ describes the spatial structure of *active states*; for the latter, the empty set describes states which are *local*, i.e. have no spatial structure. By applying the modular techniques described in [?] to this functor, one arrives at a modal language with the following (multi-sorted) syntax, where the formula sort of interest is $\mathcal{L}_1$ and the remaining sorts are merely used to define this sort:

$$\begin{array}{lll}
\mathcal{L}_1 \ni \phi ::= \mathsf{ff} \mid \mathsf{tt} \mid \phi_1 \vee \phi_2 \mid \phi_1 \wedge \phi_2 \mid [\pi_1]\psi \mid [\pi_2]\chi & & (\psi \in \mathcal{L}_2, \chi \in \mathcal{L}_3) \\
\mathcal{L}_2 \ni \psi ::= \mathsf{ff} \mid \mathsf{tt} \mid \psi_1 \vee \psi_2 \mid \psi_1 \wedge \psi_2 \mid \Box\phi \mid \Diamond\phi & & (\phi \in \mathcal{L}_1) \\
\mathcal{L}_3 \ni \chi ::= \mathsf{ff} \mid \mathsf{tt} \mid \chi_1 \vee \chi_2 \mid \chi_1 \wedge \chi_2 \mid [\kappa_1]\xi \mid [\kappa_2]\zeta \mid \langle\kappa_1\rangle\xi \mid \langle\kappa_2\rangle\zeta & & (\xi \in \mathcal{L}_4, \zeta \in \mathcal{L}_5) \\
\mathcal{L}_4 \ni \xi ::= \mathsf{ff} \mid \mathsf{tt} \mid \xi_1 \vee \xi_2 \mid \xi_1 \wedge \xi_2 \mid 0 \mid \neg 0 & & \\
\mathcal{L}_5 \ni \zeta ::= \mathsf{ff} \mid \mathsf{tt} \mid \zeta_1 \vee \zeta_2 \mid \zeta_1 \wedge \zeta_2 \mid [\pi_1]\phi \mid [\pi_2]\phi & & (\phi \in \mathcal{L}_1)
\end{array}$$

Both the temporal and the spatial modalities defined in [13] can now be recovered, namely by defining: $\Box\phi := [\pi_1]\Box\phi$, $\Diamond\phi := [\pi_1]\Diamond\phi$, $0 := [\pi_2]\langle\kappa_1\rangle 0$, $\phi_1 \mid \phi_2 := [\pi_2]\langle\kappa_2\rangle\Diamond([\pi_1] \bigcirc \phi_1 \wedge [\pi_2] \bigcirc \phi_2)$ (where the notation for the $\Box$, $\Diamond$ and 0 modalities has been overloaded in order to maintain the notation of [13]). We note that the above language is a negation-free version of the language of [13], but, as we will see in Section 3.3, this does not lead to a loss in expressivity.

Example 2.4 *Simple Segala systems* [18] can be modelled as coalgebras of the functor $(\mathbb{P} \circ \mathbb{D})^E$. The one-step observations one can make about the states of such systems consist of non-deterministic transitions into discrete probability distributions over states. This mirrors the original definition of simple Segala systems, which divides the states into non-deterministic and probabilistic ones, the former being observed through non-deterministic transitions into the latter, and the latter being observed through probabilistic transitions back to the former. The language induced by $(\mathcal{S}_{\mathbb{P}} \circledcirc \mathcal{S}_{\mathbb{D}}) \odot E$ is a negation-free variant of the language considered in [11], and contains modalities of the form $[e]\Box L_{p^-}$ and $[e]\Box G_{p^-}$ with $e \in E$.

3 Modular Coalgebraic Fixed Point Logics

From now on we restrict our attention to syntax constructors with an associated one-step semantics that is *monotonic* in the following sense.

Definition 3.1 (Monotonic one-step semantics) *Given interpretations $d, d' : L \to \mathcal{P}X$, we write $d \subseteq d'$ if $d(\phi) \subseteq d'(\phi)$ for each $\phi \in L$. A one-step semantics $[\![\mathcal{S}]\!]$ for a syntax constructor $\mathcal{S}$ is said to be* monotonic *if, for each $d, d' : L \to \mathcal{P}X$, we have that $d \subseteq d'$ implies $[\![\mathcal{S}]\!](d) \subseteq [\![\mathcal{S}]\!](d')$.*

It is easy to check that all syntax constructors considered in Section 2 are monotonic:

Proposition 3.2 *$\mathcal{S}_A$, $\mathcal{S}_{\mathbb{Id}}$, $\mathcal{S}_{\mathbb{P}}$ and $\mathcal{S}_{\mathbb{D}}$ are monotonic. Moreover, if $\mathcal{S}_1$ and $\mathcal{S}_2$ are monotonic, so are $\mathcal{S}_1 \otimes \mathcal{S}_2$, $\mathcal{S}_1 \oplus \mathcal{S}_2$, $\mathcal{S}_1 \odot E$ and $\mathcal{S}_1 \circledcirc \mathcal{S}_2$.*

3.1 Syntax

By adding fixed point formulae to the language $\mathcal{L}^{\mathcal{S}}$, we obtain the following language:

$$\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V}) \ni \phi := \mathsf{ff} \mid \mathsf{tt} \mid \phi \vee \phi \mid \phi \wedge \psi \mid \Psi \mid x \mid \mu x.\phi \mid \nu x.\phi$$

where $\mathcal{V}$ is a set of variables, $x \in \mathcal{V}$, and $\Psi \in \mathcal{S}(F)$ with $F \subseteq \mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$ finite.

Example 3.3 The fixed point languages of $\mathcal{S}_{\mathbb{P}}$ and $\mathcal{S}_{\mathbb{P}} \odot E$ are the mono- and multi-modal propositional μ-calculi (of e.g. [19]), respectively.

3.2 Semantics

The interpretation of formulae in $\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$ over the states of a $\mathbb{T}$-coalgebra (C, γ) is defined w.r.t. a valuation $V : \mathcal{V} \to \mathcal{P}(C)$, as follows:

$$\begin{aligned}
c \models_V \Psi \in \mathcal{S}(F) \quad &\text{iff} \quad \gamma(c)\,([\![\mathcal{S}]\!]{\models_{V,Base(\Psi)}})\,\Psi \\
c \models_V x \quad &\text{iff} \quad c \in V(x) \\
c \models_V \mu x.\phi \quad &\text{iff} \quad c \in \bigcap \{B \subseteq C \mid [\![\phi]\!]_{V[B/x]} \subseteq B\} \\
c \models_V \nu x.\phi \quad &\text{iff} \quad c \in \bigcup \{B \subseteq C \mid B \subseteq [\![\phi]\!]_{V[B/x]}\}
\end{aligned}$$

where

$$[\![\phi]\!]_V = \{c \in C \mid c \models_V \phi\}$$

$$V[B/x](y) = \begin{cases} B & y = x \\ V(y) & o.w. \end{cases}$$

and the relation $\models_{V,Base(\Psi)} \subseteq C \times Base(\Psi)$ gives the interpretation of formulae in $Base(\Psi)$ over the states of (C, γ).

The absence of negation in the language and the monotonicity of the one-step semantics $[\![\mathcal{S}]\!]$ ensure that the semantics for fixed point formulae is well-defined:

Lemma 3.4 (Monotonicity) *Let (C, γ) be a $\mathbb{T}$-coalgebra, $\phi \in \mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$, and $V, V' : \mathcal{V} \to \mathcal{P}(C)$ two valuations such that $V(x) \subseteq V'(x)$ for all $x \in \mathcal{V}$. Then, for $c \in C$, we have: $c \models_V \phi$ implies $c \models_{V'} \phi$.*

Proof (Sketch) The statement is proved by structural induction on ϕ. The case where $\phi \in \mathcal{S}F$ with $F \subseteq \mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$ finite uses the monotonicity of $[\![\mathcal{S}]\!]$. $\square$

The previous lemma ensures that, for a $\mathbb{T}$-coalgebra (C, γ), a valuation $V : \mathcal{V} \to \mathcal{P}(C)$ and a formula $\phi \in \mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$, the map $X \in \mathcal{P}(C) \longmapsto [\![\phi]\!]_{V[X/x]} \in \mathcal{P}(C)$ is a monotone map on the complete lattice $\mathcal{P}(C)$, and therefore by the Knaster-Tarski theorem, this map has a least and a greatest fixed point; these fixed points can be computed as the intersection of all pre-fixed points and the union of all post-fixed points, respectively.

Example 3.5 $[\![\mathcal{S}_{\mathbb{P}}]\!]$ and $[\![\mathcal{S}_{\mathbb{P}} \odot E]\!]$ provide semantics for the languages of mono- and multi-modal propositional μ-calculus (of e.g. [19]), interpreted over transition systems and labelled transition systems, respectively.

3.3 Simulating Negation

It is worth noting that, for languages where the semantic dual of each modal operator is also in the language, we do not lose any expressivity by leaving out the negation operator. We will show that, in this case, negation can be implicitly defined. To this end, we introduce the notions of an $\mathcal{S}$*-modality*, and of a *dual modality*.

Definition 3.6 ($\mathcal{S}$-modality) *For a syntax constructor $\mathcal{S}$ and $n \in \omega$, an $\mathcal{S}$-*modality of arity *n is an element of $\mathcal{S}(n)$ which does not belong to any of $\mathcal{S}(1), \ldots, \mathcal{S}(n-1)$* [5] *.*

Thus, an $\mathcal{S}$-modality a of arity n is a modal operator which takes n arguments; the set $n = \{0, \ldots, n-1\}$ is used to define the placeholders for the arguments of a.

Next, we define what it means to apply an $\mathcal{S}$-modality of arity n to a set of n formulae.

Definition 3.7 *If a is an $\mathcal{S}$-modality of arity n and $\phi_1, \ldots, \phi_n \in \mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$, we define $a(\phi_1, \ldots, \phi_n)$ as $\mathcal{S}([\phi_1, \ldots, \phi_n])(a) \in \mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$, where $[\phi_1, \ldots, \phi_n] : n \to \{\phi_1, \ldots, \phi_n\}$ maps i to ϕ_{i+1} for $i = 0, \ldots, n-1$.*

We also note that a one-step semantics $[\![\mathcal{S}]\!]$ for $\mathcal{S}$ w.r.t. $\mathbb{T}$ automatically provides a (coalgebraic) semantics for each $\mathcal{S}$-modality.

Example 3.8 The $\Box$ and $\Diamond$ operators specified by the syntax constructor $\mathcal{S}_{\mathbb{P}}$ are unary $\mathcal{S}_{\mathbb{P}}$-modalities, when identified with the two elements of $\mathcal{S}_{\mathbb{P}}(1)$.

Example 3.9 For the spatial transition systems of Example 2.3, the modal operator 0 is an $\mathcal{S}$-modality of arity 0, the temporal operators $\Box\_$ and $\Diamond\_$ are unary $\mathcal{S}$-modalities, while the spatial operator $\_ \mid \_$ is a binary $\mathcal{S}$-modality.

Incidentally, the notion of $\mathcal{S}$-modality allows us to relate languages induced by finitary polyadic predicate liftings, as considered in [17], on the one hand, and languages induced by syntax constructors with associated one-step semantics on the other. To this end, we write $\hat{\mathcal{P}} : \mathsf{Set} \to \mathsf{Set}$ for the contravariant powerset functor. Given a set Λ of *finitary polyadic predicate liftings* for a functor $\mathbb{T}$ (that is, natural transformations $\lambda : \hat{\mathcal{P}}^n \to \hat{\mathcal{P}}\mathbb{T}$ with $n \in \omega$), a syntax constructor $\mathcal{S}_\Lambda : \mathsf{Set} \to \mathsf{Set}$ can be defined by

$$\mathcal{S}_\Lambda(L) = \{\lambda(\phi_1, \ldots, \phi_n) \mid \lambda \in \Lambda \text{ has arity } n,\ \phi_i \in L \text{ for } i = 1, \ldots, n\}$$

[5] Recall that $\mathcal{S}$ is inclusion-preserving.

while a one-step semantics $[\![\mathcal{S}_\Lambda]\!]$ for $\mathcal{S}$ w.r.t. $\mathbb{T}$ can be defined by mapping an interpretation $d : L \to \mathcal{P}X$ to the interpretation $d' : \mathcal{S}_\Lambda L \to \mathcal{P}\mathbb{T}X$ given by

$$d'(\lambda(\phi_1, \ldots, \phi_n)) = \lambda_X(d(\phi_1), \ldots, d(\phi_n)) \quad \text{for} \quad \phi_1, \ldots, \phi_n \in L$$

Thus, $\mathcal{S}_\Lambda$-modalities of arity n are exactly the n-ary predicate liftings of Λ and, as expected, the semantics of these modal operators agree with each other. Now monotonicity of the one-step semantics $[\![\mathcal{S}_\Lambda]\!]$ amounts to the predicate liftings $\lambda : (\hat{\mathcal{P}})^n \to \hat{\mathcal{P}}\mathbb{T}$ being monotonic in all arguments. Consequently, our approach can be used to add fixed points to logics induced by sets of monotonic predicate liftings.

We now return to the issue of simulating negation in the language $\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$. For a given set of formulae L, we introduce a syntactic negation for the formulae of L via the set $L^c := \{\phi^c \mid \phi \in L\}$. Now for a given set of points X, an interpretation relation $\models \subseteq X \times L$ is extended to negated formulae in L^c via the relation $\models^c \subseteq X \times L^c$ given by

$$x \models^c \phi^c \quad \text{iff} \quad x \not\models \phi \qquad \text{for } x \in X \text{ and } \phi \in L$$

Definition 3.10 (Closure under duals) *A syntax constructor $\mathcal{S}$ with a one-step semantics $[\![\mathcal{S}]\!]$ is said to be* closed under duals *if, for each $\mathcal{S}$-modality a, there exists an $\mathcal{S}$-modality $\overline{a}$ of arity n, called the* dual of a, *such that for each relation $\models \subseteq X \times L$, the relation $\models' = [\![\mathcal{S}]\!](\models \cup \models^c) \subseteq \mathbb{T}X \times \mathcal{S}(L \cup L^c)$ satisfies*

$$t \models' \overline{a}(\phi_1, \ldots, \phi_n) \quad \textit{iff} \quad t \not\models' a(\phi_1^c, \ldots, \phi_n^c)$$

Thus, $\mathcal{S}$ is closed under duals if, whenever it specifies a modality, then it also specifies its semantic dual.

Example 3.11 The syntax constructors $\mathcal{S}_A$, $\mathcal{S}_{\mathbb{Id}}$, $\mathcal{S}_{\mathbb{P}}$ and $\mathcal{S}_{\mathbb{D}}$ with their associated one-step semantics are closed under duals. In particular, we have $\overline{a} = \neg a$, $\overline{\bigcirc} = \bigcirc$, $\overline{\Box} = \Diamond$ and $\overline{L_p} = G_{1-p}$ for $0 \leq p \leq 1$.

Proposition 3.12 *If the syntax constructors for all the ingredients of a polynomial functor $\mathbb{T}$ (with their respective one-step semantics) are closed under duals, then so is the combined syntax constructor for $\mathbb{T}$ (with the one-step semantics defined modularly from the one-step semantics w.r.t. its ingredients).*

Proof (Sketch) The statement follows from the definitions of $\mathcal{S}_1 \otimes \mathcal{S}_2$, $\mathcal{S}_1 \oplus \mathcal{S}_2$, $\mathcal{S}_1 \odot E$ and $\mathcal{S}_1 \circledcirc \mathcal{S}_2$ and of the associated one-step semantics, together with the observations that, if a_i is an $\mathcal{S}_i$-modality, for $i = 1, 2$, then $\overline{[\pi_i]a_i} = [\pi_i]\overline{a_i}$, $\overline{[\kappa_i]a_i} = \langle\kappa_i\rangle\overline{a_i}$ and $\overline{[e]a_i} = [e]\overline{a_i}$, and that the dual of an $\mathcal{S}_1 \circledcirc \mathcal{S}_2$-modality can be defined in terms of the duals of $\mathcal{S}_1$- and $\mathcal{S}_2$-modalities; for example, if a_i is a unary $\mathcal{S}_i$-modality, with $i = 1, 2$, then $\overline{a_1 a_2} = \overline{a_1}\ \overline{a_2}$. $\square$

As a consequence of Proposition 3.12, the modal language for spatial transition systems described in Example 2.3 is closed under duals.

We now observe that any $\Psi \in \mathcal{S}(F)$ with F finite is of the form $a(\phi_1, \ldots, \phi_n)$, with a an $\mathcal{S}$-modality. If $Base(\Psi) = \{\phi_1, \ldots, \phi_n\}$, then since $\mathcal{S}(Base(\Psi))$ is isomorphic to $\mathcal{S}(n)$ (via $\mathcal{S}([\phi_1, \ldots, \phi_n])$), there must exist an $a \in \mathcal{S}(n)$ such that

$\Psi = \mathcal{S}([\phi_1, \ldots, \phi_n])(a)$, that is, $\Psi = a(\phi_1, \ldots, \phi_n)$. Moreover, by the minimality of $Base(\Psi)$, it follows that a can not come from any of $\mathcal{S}(1), \ldots, \mathcal{S}(n-1)$. Thus, a is an $\mathcal{S}$-modality of arity n.

We note in passing that the above argument also applies to the case $\mathcal{S} = \mathbb{T}$, that is, to the finitary version of Moss' coalgebraic logic [14]. Thus, when $\mathbb{T}$ is ω-accessible, the finitary version of the ∇ modality, considered in [21], can be regarded as a shorthand for an infinite number of modalities, each with a specific finite arity.

The previous observation is used in the following definition.

Definition 3.13 (Negation) *For a syntax constructor $\mathcal{S}$ which is closed under duals, the* negations *of formulae in $\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$ are defined inductively as follows:*

$$\mathsf{ff}^c := \mathsf{tt} \qquad \mathsf{tt}^c := \mathsf{ff}$$
$$(\phi \vee \psi)^c := \phi^c \wedge \psi^c \qquad (\phi \wedge \psi)^c := \phi^c \vee \psi^c$$
$$(a(\phi_1, \ldots, \phi_n))^c := \overline{a}(\phi_1^c, \ldots, \phi_n^c) \qquad x^c := x$$
$$(\mu x.\phi)^c := \nu x.\phi^c \qquad (\nu x.\phi)^c := \mu x.\phi^c$$

Proposition 3.14 *For a state c of a $\mathbb{T}$-coalgebra (C, γ), a valuation $V : \mathcal{V} \to \mathcal{P}(C)$, and a formula $\phi \in \mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$ we have*

$$c \models_V \phi \quad \textit{iff} \quad c \not\models_{V^c} \phi^c$$

where $V^c : \mathcal{V} \to \mathcal{P}(C)$ is given by $V^c(x) = C \setminus V(x)$ for $x \in \mathcal{V}$.

Proof. The statement follows from Definitions 3.10 and 3.13 and the definition of $\models_V$. $\square$

In order to account for some interesting examples, we now introduce the notion of a *derived $\mathcal{S}$-modality*. Intuitively, a derived modality involves applications of $\mathcal{S}$-modalities as well as of boolean operators, with nested applications of $\mathcal{S}$-modalities not being allowed.

Definition 3.15 (Derived $\mathcal{S}$-modality) *For a syntax constructor $\mathcal{S}$ and $n \in \omega$, a* derived $\mathcal{S}$-modality *of arity n is an element of $\overline{\mathcal{S}(\overline{n})}$, which does not belong to any of $\overline{\mathcal{S}(\overline{1})}, \ldots, \overline{\mathcal{S}(\overline{n-1})}$.*

It follows easily that, when $\mathcal{S}$ is closed under duals, for each derived $\mathcal{S}$-modality one can also define a semantic dual, again as a derived $\mathcal{S}$-modality.

Example 3.16 In the case of spatial transition systems, that is, coalgebras of the functor $\mathbb{T} = \mathbb{P} \times (1 + \mathbb{P} \circ (\mathbb{I}\mathsf{d} \times \mathbb{I}\mathsf{d}))$, $\_ \mid \mathsf{tt}$, $\mathsf{tt} \mid \_ \in \mathcal{S}_{\mathbb{T}}(\overline{1}) \subseteq \overline{\mathcal{S}_{\mathbb{T}}(\overline{1})}$ are derived $\mathcal{S}_{\mathbb{T}}$-modalities of arity 1 (where the binary modality $\_ \mid \_$ was defined in Example 2.3). Their duals are (semantically equivalent to) $[\pi_2][\kappa_2]\Box[\pi_1] \bigcirc \_$ and $[\pi_2][\kappa_2]\Box[\pi_2] \bigcirc \_$, respectively.

We conclude this section by looking at conjunction-preserving modalities. These will play a rôle when defining until- and dynamic-like fixed points in the next section.

Proposition 3.17 *For a syntax constructor $\mathcal{S}$ with an associated one-step semantics $[\![\mathcal{S}]\!]$, a (derived) $\mathcal{S}$-modality of arity n preserves conjunctions if and only if its* *dual preserves disjunctions.*

Proof. Follows from Definitions 3.10 and 3.13 and Proposition 3.14. $\square$

Example 3.18 $\bigcirc$ is both conjunction- and disjunction-preserving, whereas $\square$ is conjunction-preserving. The $\mathcal{S}_{\mathbb{D}}$-modalities are neither conjunction- nor disjunction-preserving, with the exception of $\overline{L_1} \equiv G_0$, defined as

$$\begin{array}{lll} c \models_V L_1\phi & \text{iff} & \forall c' \in C \text{ s.t. } \gamma(c)(c') \neq 0,\ c' \models_V \phi \\ c \models_V G_0\phi & \text{iff} & \exists c' \in C \text{ s.t. } \gamma(c)(c') \neq 0 \text{ and } c' \models_V \phi \end{array}$$

where L_1 is conjunction-preserving and its dual G_0 is disjunction-preserving.

Example 3.19 The modalities $\_\,|\,\mathsf{tt}$, $\mathsf{tt}\,|\,\_$ of Example 3.16 are disjunction-preserving, whereas their duals are conjunction-preserving.

Conjunction-preserving $\mathcal{S}_1 \otimes \mathcal{S}_2$-, $\mathcal{S}_1 \oplus \mathcal{S}_2$-, $\mathcal{S}_1 \odot E$- and $\mathcal{S}_1 \circledcirc \mathcal{S}_2$-modalities can be derived from conjunction-preserving $\mathcal{S}_1$- and $\mathcal{S}_2$-modalities, as shown next.

Proposition 3.20 *Let $\mathcal{S}_i$ be a syntax constructor with one-step semantics $[\![\mathcal{S}_i]\!]$, for $i = 1, 2$. If a_i is a conjunction-preserving (derived) $\mathcal{S}_i$-modality, for $i = 1, 2$, then so are $[\pi_i]a_i$ (as an $\mathcal{S}_1 \otimes \mathcal{S}_2$-modality), $[\kappa_i]a_i$ (as an $\mathcal{S}_1 \oplus \mathcal{S}_2$-modality), $[e]a_1$ (as an $\mathcal{S}_1 \odot E$-modality) and, in the case of modalities of arity 1, also a_1a_2 (as an $\mathcal{S}_1 \circledcirc \mathcal{S}_2$-modality).*

Proof (Sketch) The conclusion follows by noting that $[\pi_i]$, $[\kappa_i]$ and $[e]$ are conjunction-preserving (by the definitions of $[\![\mathcal{S}_1 \otimes \mathcal{S}_2]\!]$, $[\![\mathcal{S}_1 \oplus \mathcal{S}_2]\!]$ and $[\![\mathcal{S}_1 \odot E]\!]$), and that the successive application of conjunction-preserving modalities is itself conjunction-preserving. $\square$

3.4 Macros

As in the propositional μ-calculus, one can distinguish fragments of our coalgebraic fixed point logics that have desirable properties, for example the *independent fixed point fragment* $I\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$ and the *alternation-free fragment* $A\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$. For $\sigma_1, \sigma_2 \in \{\mu, \nu\}$ and writing $\mathit{Subf}(\phi)$ for the set of sub-formulae of a formula $\phi \in \mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$, these fragments are defined as follows:

- $\phi \in I\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$ iff $\sigma_1\, x.\phi_1 \in \mathit{Subf}(\phi)$ and $\sigma_2\, y.\phi_2 \in \mathit{Subf}(\phi)$ implies that x is not free in ϕ_2 and y is not free in ϕ_1;
- $\phi \in A\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$ iff $\mu\, x.\phi_1 \in \mathit{Subf}(\phi)$ and $\nu\, y.\phi_2 \in \mathit{Subf}(\phi)$ implies that x is not free in ϕ_2 and y is not free in ϕ_1.

While the $I\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$-fragment does not allow any dependency among the fixed point sub-formulae of a formula, the $A\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$-fragment allows dependency as long as the fixed points are of the same type. These fragments relate to each other as follows:

$$I\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V}) \subseteq A\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V}) \subseteq \mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$$

The two well-known independent fixed point fragments of the propositional μ-calculus are CTL with its until fixed points, and PDL with its dynamic fixed points, see [19] for details. The definitions of these special fixed point formulae can be extended to our general coalgebraic fixed point logic $\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$.

Definition 3.21 *Given a functor $\mathbb{T}$, a syntax constructor $\mathcal{S}$ with a one-step semantics $[\![\mathcal{S}]\!]$ w.r.t. $\mathbb{T}$ such that $\mathcal{S}$ is closed under duals, and a finite set α of conjunction-preserving, unary $\mathcal{S}$-modalities*[6], until *and* dynamic fixed point formulae *are defined as follows:*

$$A(\phi\, U_\alpha\, \psi) := \mu x.(\psi \vee (\phi \wedge \bigvee_{a\in\alpha} \overline{a}\,\mathsf{tt} \wedge \bigwedge_{a\in\alpha} a\, x))$$

$$E(\phi\, U_\alpha\, \psi) := \mu x.(\psi \vee (\phi \wedge \bigvee_{a\in\alpha} \overline{a}\, x))$$

$$[\ ]^*_\alpha\, \phi := \nu x.(\phi \wedge \bigwedge_{a\in\alpha} a\, x)$$

$$\langle\ \rangle^*_\alpha \phi := \mu x.(\phi \vee \bigvee_{a\in\alpha} \overline{a}\, x)$$

where $x \in \mathcal{V}$, $\phi, \psi \in \mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$, x does not occur free in ϕ, ψ, and for $a \in \alpha$, $\overline{a}$ is the dual of a.

We motivate our conjunction-preservation condition on the set α of $\mathcal{S}$-modalities in Definition 3.21 by noting that, under this condition, the $\mathcal{S}$-modality defined by $\Box_\alpha x := \bigwedge_{a\in\alpha} a\, x$ is itself conjunction-preserving, and thus can be regarded as a generalisation of the $\Box$-modality used in the standard definitions of the until and dynamic modalities.

Proposition 3.22 *The until and dynamic fixed point formulae defined above belong to the $I\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$-fragment of $\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$.*

Proof. Since x does not occur free in ϕ and ψ, neither will it occur free in any fixed point formulae that might occur in $\mathit{Subf}(\phi)$ or $\mathit{Subf}(\psi)$. $\Box$

Intuitively, the formula $E(\phi\, U_\alpha\, \psi)$ is read as "there exists a route described by modalities in α along which ϕ holds until ψ holds", whereas the formula $A(\phi\, U_\alpha\, \psi)$ is read as "along all routes described by modalities in α, ϕ holds until ψ holds". Particular choices for the set α can be obtained using Example 3.18 and Proposition 3.20. Below we mention some choices for α which give us known fixed points.

Example 3.23 In the case of labelled transition systems, that is, coalgebras of the functor $\mathbb{P}^E$, the until operators of CTL (as defined e.g. in [19]) are recovered as $A(\phi\, U_\alpha\, \psi)$ and $E(\phi\, U_\alpha\, \psi)$ with $\alpha = \{[e]\Box \mid e \in E\}$.

Example 3.24 In the case of spatial transition systems, taking α to consist of the duals of the two derived modalities of Example 3.16 yields the *somewhere* modality of spatial logic (as used e.g. in [6]):

$$\Diamond\phi \;=\; E(\mathsf{tt}\, U_\alpha\, \phi) \;:=\; \mu\, x.\,(\phi \vee (\mathsf{tt} \mid x) \vee (x \mid \mathsf{tt}))$$

Also, by taking $\alpha = \{\Diamond\}$, where $\Diamond$ was defined in Example 2.3, one recovers the *sometime* modality of spatial logic:

$$\blacklozenge\phi \;=\; E(\mathsf{tt}\, U_\alpha\, \phi) \;:=\; \mu\, x.\,(\phi \vee \Diamond x)$$

[6] Derived $\mathcal{S}$-modalities can also be considered here.

The duals of the above two modalities, defined using Definition 3.13 as $(E(\mathsf{tt}\, U_\alpha\, \phi^c))^c$ for the respective choices of α, are the *everywhere* and respectively *every time* modalities of [6].

Example 3.25 As shown in previous work [5], coalgebras of the functor $\mathbb{T} = \mathbb{P}^E \times (1+\mathbb{I}\mathsf{d})^{E'} \times C$ (subject to additional axioms) provide semantics for epistemic update. The language $\mathcal{L}^{\mathcal{S}}$ induced by $\mathcal{S} = (\mathcal{S}_{\mathbb{P}} \odot E) \otimes ((1 \oplus \mathcal{S}_{\mathbb{I}\mathsf{d}}) \odot E') \otimes \mathcal{S}_C$ gives rise to a modular coalgebraic logic for $\mathbb{T}$, which is shown in [5] to be equivalent to the Dynamic Epistemic Logic (DEL) of [2]. By extending this language to $\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$ and taking $\alpha = \{[\pi_1][e]\Box \mid e \in E\}$, one obtains a dynamic fixed point $[\ ]^*_\alpha\phi$, which is equivalent to the *common knowledge* fixed point of DEL. At the same time, taking $\alpha' = \{[\pi_2][e'][\kappa_2] \mid e' \in E'\}$ provides us with the *update* fixed point of DEL. Moreover, taking $\alpha \cup \alpha'$ provides us with a new dynamic modality that quantifies over both knowledge and update transitions.

Example 3.26 In the case of simple Segala systems, by taking $\alpha = \{[e]\Box L_1 \mid e \in E\}$, the resulting until operator $A(\phi\, U_\alpha\, \psi)$ requires that along every path (alternating between non-deterministic and probabilistic transitions), ϕ holds until ψ holds (in the non-deterministic states reached along the path). In contrast, $E(\phi\, U_\alpha\, \psi)$ requires the existence of a path along which ϕ holds until ψ holds.

4 Games

In this section we present a game-theoretic approach to deciding satisfaction between states of coalgebras and formulae of our fixed point logics.

4.1 Evaluation Games

We first recall the main definitions from the theory of two-player infinite games (see e.g. [8]). A *graph game* played between two players, here referred to as $\exists$ and $\forall$, is defined by:

- a set *Pos* of positions, with each position belonging to exactly one player,
- for each position of the game, a set of possible moves from that position,
- an initial position.

A *play* in a graph game is a (finite or infinite) sequence of positions, such that the first position is the initial position, and each subsequent position is obtained by a valid move from the position immediately preceding it. A *full play* is either an infinite play or a finite play where there are no possible moves from the last position. A *winning condition* for a graph game associates, to each infinite play, a winner and a loser. (Finite plays are always lost by the player who can not move.) The winner of an infinite play can be defined e.g. via a *parity winning condition* – this involves defining a *parity map* $\Omega : Pos \to \omega$ with finite range, and letting $\exists$ win exactly those infinite plays for which the maximum of those values $\Omega(p)$ that occur infinitely often in that play is even. A *strategy* for a player in a graph game maps partial plays ending in positions associated to that player to next moves for that player. A strategy is *history-free* if it only depends on the current position. A player is said to *use a strategy* in a play if all of his moves in that play obey the rules in the strategy. A strategy is *winning* for a player P from a position $p \in Pos$ if P wins all plays starting in p by using the strategy.

Following [21], we now define a (parity) graph game for evaluating a formula of $\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$ in a state of a $\mathbb{T}$-coalgebra.

Definition 4.1 (Evaluation game) *Given a pointed $\mathbb{T}$-coalgebra $C = (C, \gamma, c_0)$, a valuation $V : \mathcal{V} \to \mathcal{P}(C)$ and a* clean[7] *formula $\phi_0 \in \mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$, the* evaluation game *$\mathcal{E}^{C,V}_{\phi_0}$ is an infinite two-player game, played between $\exists$ (who aims to verify the statement $c_0 \models_V \phi$) and $\forall$ (who aims to refute this statement) as follows:*

- *The positions of the game are elements of the set $Pos = (C \times Subf(\phi_0)) \sqcup (\mathbb{T}C \times Subf(\phi_0)) \sqcup \mathcal{P}(C \times Subf(\phi_0))$. We use the superscript $(\_)^o$ (for "observations") for positions in $\mathbb{T}C \times Subf(\phi_0)$, whenever we need to distinguish such positions from positions in $C \times Subf(\phi_0)$[8].*
- *The possible moves are as follows[9]:*

Position	*Player*	*Moves*
(c, ff)	$\exists$	$\emptyset$
(c, tt)	$\forall$	$\emptyset$
$(c, \phi \vee \psi)$	$\exists$	$\{ (c, \phi), (c, \psi) \}$
$(c, \phi \wedge \psi)$	$\forall$	$\{ (c, \phi), (c, \psi) \}$
$(c, \sigma x.\phi_x)$	–	$\{ (c, \phi_x) \}$
(c, x), $x \in BVar(\phi_0)$	–	$\{ (c, \phi_x) \}$
(c, x), $x \notin BVar(\phi_0)$, $c \in V(x)$	$\forall$	$\emptyset$
(c, x), $x \notin BVar(\phi_0)$, $c \notin V(x)$	$\exists$	$\emptyset$
$(c, \psi) \in C \times \mathcal{S}(\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V}))$	–	$\{ (\gamma(c), \psi)^o \}$
$(t, \psi)^o \in \mathbb{T}C \times \mathcal{S}(\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V}))$	$\exists$	$\{ Z \subseteq C \times Subf(\phi_0) \mid (t, \psi) \in [\![\mathcal{S}]\!](Z) \}$
$Z \subseteq C \times \mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$	$\forall$	Z

where $\sigma \in \{\mu, \nu\}$ and for a variable x, $\mu x.\phi_x$ or $\nu x.\phi_x$ is the subformula of ϕ which binds x.

- *The winning conditions of the game are as follows:*
  - · *finite plays are lost by the player who can not move,*
  - · *infinite plays are won by $\exists$ (respectively $\forall$) if the outermost variable that is unfolded infinitely often in that play[10] is a ν-variable (μ-variable).*

[7] A formula ϕ is called *clean* if no variable occurs both free and bound in ϕ, and if different occurrences of fixed point operators do not bind the same variable.

[8] As noted by one of our referees, this distinction is needed in the case when $\mathbb{T} = \mathbb{I}\mathsf{d}$, to prevent several unfoldings of the coalgebra map in consecutive moves.

[9] Whenever no player is associated to a position, this is because there is only one possible move from that position, and thus it does not matter which player moves in such a position.

[10] Since ϕ is clean, this variable is uniquely defined; see e.g. [21] for the proof of a similar result.

In what follows, we will write $\mathcal{E}^{c_0}_{\phi_0}$ for $\mathcal{E}^{C,V}_{\phi_0}$ whenever C and V are clear from the context.

The only difference w.r.t. the evaluation games of [21] is the set of moves of $\exists$ in positions of type $(t, \psi) \in \mathbb{T}C \times \mathcal{S}(\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V}))$ – here, the one-step semantics of $\mathcal{S}$ w.r.t. $\mathbb{T}$ is used to determine when a relation $Z \subseteq C \times Subf(\phi_0)$ can be regarded as a witness for (t, ψ). Indeed, the evaluation game of [21] can be obtained as a particular case, namely by taking $\mathcal{S} = \mathbb{T}$ and $[\![\mathcal{S}]\!]$ as in Example 2.2.

We note that the winning condition of $\mathcal{E}^{c_0}_{\phi_0}$ for infinite plays can be reformulated as a *parity condition*. This is done by first defining a map $\Omega : Subf(\phi_0) \to \omega$ subject to the following constraints:

- $\Omega(\phi) = 0$ unless $\phi = x$ with $x \in BVar(\phi_0)$,
- for $x \in BVar(\phi_0)$, $\Omega(x)$ is odd if x is a μ-variable, and even if x is a ν-variable,
- $\Omega(x) \leq \Omega(y)$ whenever the formula binding x is a subformula of the formula binding y.

A map $\Omega' : Pos \to \omega$ can then be defined by letting $\Omega'(c, \phi) = \Omega(\phi)$. It can easily be seen that the winning condition of $\mathcal{E}^{c_0}_{\phi_0}$ for infinite plays is equivalent to the parity winning condition induced by Ω'; that is, the outermost variable that is unfolded infinitely often in a play is a ν-variable iff the maximum of the values $\Omega'(c, x)$ which occur infinitely often in that play is even.

According to general results, see e.g. [7,15,19], and since the above evaluation games are parity games, they enjoy the *history-free determinacy* property, that is, in each position of the game, either $\exists$ or $\forall$ has a history-free winning strategy.

We now prove an adequacy result (of the evaluation game w.r.t. the semantics of fixed point formulae), which generalises a similar result in [21].

Theorem 4.2 (Adequacy of evaluation game) *For a pointed $\mathbb{T}$-coalgebra (C, γ, c), a valuation $V : \mathcal{V} \to \mathcal{P}(C)$, and a clean formula $\phi \in \mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$ we have*

(i) $c \models_V \phi$ *iff* $\exists$ *has a history-free winning strategy in* $\mathcal{E}^c_\phi$ *from position* (c, ϕ),

(ii) $c \not\models_V \phi$ *iff* $\forall$ *has a history-free winning strategy in* $\mathcal{E}^c_\phi$ *from position* (c, ϕ).

Proof. The proof of the "only if" direction of the first statement is done by constructing history-free winning strategies for $\exists$ by induction on the structure of ϕ. The construction for non-modal formulae is as for the modal μ-calculus, and follows the same line as the proof of the adequacy result in [21]. For formulae in $\mathcal{S}(\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V}))$, assume we are at position (c, Ψ) with $\Psi \in \mathcal{S}F$ and $F \subseteq \mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$ finite. Since $c \models_V \Psi$, we have $t\,([\![\mathcal{S}]\!] \models_{V,Base(\Psi)})\,\Psi$, where $t = \gamma(c)$. A strategy for $\exists$ in the game starting at position (t, Ψ) is obtained by extending the strategy coming from the induction hypothesis with the rule 'at (t, Ψ) choose $\models_{V,Base(\Psi)}$ as Z'. This is a legitimate move, since $t\,([\![\mathcal{S}]\!] \models_{V,Base(\Psi)})\,\Psi$ and therefore $(t, \Psi) \in [\![\mathcal{S}]\!](Z)$. To show that the resulting strategy is a winning strategy for $\exists$ in the game starting at (c, Ψ), we show that it is impossible for $\forall$ to win if $\exists$ follows this strategy. Assume that, at position Z, $\forall$ chooses $(c', \psi) \in Z$ as the next position. (If Z is empty, then $\forall$ loses immediately.) By the choice of Z, we have $c' \models_V \psi$. Now by the induction hypothesis, $\exists$ has a winning strategy starting from (c', ψ). We have thus proved that $\exists$ wins in the game starting at (c, Ψ).

The proof of the "if" direction is also done by induction. For the modal case, assume we are at position (c, Ψ) with $\Psi \in \mathcal{S}(\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V}))$ and $\exists$ has a winning strategy in the game starting at (c, Ψ). This strategy provides a certain $Z \subseteq C \times \mathit{Subf}(\phi_0)$ such that $(t, \Psi) \in [\![\mathcal{S}]\!]Z$, where $t = \gamma(c)$. By the induction hypothesis, $c' \models_V \psi$ for all $(c', \psi) \in Z$, and thus $Z \subseteq \models_V$. Now by the monotonicity of $[\![\mathcal{S}]\!]$ we have $[\![\mathcal{S}]\!]Z \subseteq [\![\mathcal{S}]\!] \models_V$, and hence $(t, \Psi) \in [\![\mathcal{S}]\!] \models_V$. We have thus proved that $t\,([\![\mathcal{S}]\!] \models_V)\,\Psi$, and therefore $c \models_V \Psi$.

The second statement follows easily, since if $c \not\models \phi$, then by the first statement $\exists$ does not have a (history-free) winning strategy in (c, ϕ), and by the determinacy property of parity games, $\forall$ has a history-free winning strategy in (c, ϕ). $\square$

As a consequence of Theorem 4.2, we obtain a result about the implicit negation of a fixed point formula. Before stating this result, we define the complement of an evaluation game.

Definition 4.3 (Complement game) *For a $\mathbb{T}$-coalgebra $C = (C, \gamma, c_0)$, a valuation $V : \mathcal{V} \to \mathcal{P}(C)$ and a clean formula $\phi_0 \in \mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$, the* complement of the evaluation game *$\mathcal{E}^{C,V}_{\phi_0}$ is the evaluation game $\mathcal{E}^{C,V^c}_{\phi_0^c}$, where ϕ_0^c is as in Definition 3.13 and V^c is as in Proposition 3.14.*

Analysing the definition of the complement of $\mathcal{E}^{C,V}_{\phi_0}$, we see that this game is obtained from $\mathcal{E}^{C,V}_{\phi_0}$ by complementing the formulae defining the positions of $\mathcal{E}^{C,V}_{\phi_0}$, complementing the valuation V, and reversing the roles of $\exists$ and $\forall$.

Corollary 4.4 *For a syntax constructor $\mathcal{S}$ with a one-step semantics $[\![\mathcal{S}]\!]$ such that $\mathcal{S}$ is closed under duals, a pointed $\mathbb{T}$-coalgebra (C, γ, c_0), a valuation $V : \mathcal{V} \to \mathcal{P}(C)$ and a clean formula $\phi_0 \in \mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$, a player does not have a history-free winning strategy in $\mathcal{E}^{C,V}_{\phi_0}$ iff he has a history-free winning strategy in its* complement *$\mathcal{E}^{C,V^c}_{\phi_0^c}$.*

Proof. From Proposition 3.14 it follows that $c_0 \models_V \phi_0$ iff $c_0 \not\models_{V^c} \phi_0^c$. Then, by Theorem 4.2 and respectively the determinacy property, $\exists$ has a history-free winning strategy at (c_0, ϕ_0) in $\mathcal{E}^{C,V}_{\phi_0}$ iff $\forall$ has a history-free winning strategy at (c_0, ϕ_0^c) in $\mathcal{E}^{C,V^c}_{\phi_0^c}$ iff $\exists$ does not have a history-free wining strategy at (c_0, ϕ_0^c) in $\mathcal{E}^{C,V}_{\phi_0^c}$. The case for $\forall$ is proved similarly. $\square$

4.2 One-Step Games

The evaluation game $\mathcal{E}^{c_0}_{\phi_0}$ has the drawback that in a position of type $(t, \psi) \in \mathbb{T}C \times \mathcal{S}(\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V}))$, some of the possible moves of $\exists$ are not relevant when it comes to deciding the winner of the game. Indeed, only relations Z which are minimal among those with the property that $(t, \psi) \in [\![\mathcal{S}]\!](Z)$ are relevant, as shown next.

Definition 4.5 *Given a position $(t, \psi) \in \mathbb{T}C \times \mathcal{S}\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$ in the game $\mathcal{E}^{c_0}_{\phi_0}$, a relation $Z \subseteq C \times \mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$ with the property that $(t, \psi) \in [\![\mathcal{S}]\!](Z)$ is said to be* minimal relative to *(t, ψ) if there is no $Z' \subseteq C \times \mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$ such that $Z' \subsetneq Z$ and $(t, \psi) \in [\![\mathcal{S}]\!](Z')$.*

Lemma 4.6 *Let $\tilde{\mathcal{E}}^{c_0}_{\phi_0}$ be the game obtained from $\mathcal{E}^{c_0}_{\phi_0}$ by only allowing relations Z which are minimal relative to (t, ψ) as possible moves of $\exists$ in positions of type $(t, \psi) \in \mathbb{T}C \times \mathcal{S}\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$. Then $\exists$ has a winning strategy in $\mathcal{E}^{c_0}_{\phi_0}$ iff he has a winning strategy in $\tilde{\mathcal{E}}^{c_0}_{\phi_0}$.*

Proof. Assume first that $\exists$ has a winning strategy in $\mathcal{E}_{\phi_0}^{c_0}$. This strategy provides, for each position of type $(t, \psi) \in \mathbb{T}C \times \mathcal{S}\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$, a relation $Z \subseteq C \times \mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$. Then, there exists $Z' \subseteq Z$ (not necessarily unique) such that $(t, \psi) \in [\![\mathcal{S}]\!](Z')$ and Z' is minimal relative to (t, ψ). Thus, Z' is a legitimate move in the game $\tilde{\mathcal{E}}_{\phi_0}^{c_0}$. Since $\forall$'s choices in Z' are a subset of his choices in the position Z of the game $\mathcal{E}_{\phi_0}^{c_0}$, and since $\exists$ had a winning strategy from each $z \in Z$ in $\mathcal{E}_{\phi_0}^{c_0}$, it follows that $\exists$ also has a winning strategy from each $z \in Z'$ in $\tilde{\mathcal{E}}_{\phi_0}^{c_0}$. We have thus proved that $\exists$ has a winning strategy from (t, ψ) in $\tilde{\mathcal{E}}_{\phi_0}^{c_0}$. Now assume that $\exists$ has a winning strategy in $\tilde{\mathcal{E}}_{\phi_0}^{c_0}$. Since by always using this strategy when playing in $\mathcal{E}_{\phi_0}^{c_0}$, the game stays inside $\tilde{\mathcal{E}}_{\phi_0}^{c_0}$, it follows that $\exists$ can also win in $\mathcal{E}_{\phi_0}^{c_0}$ with this strategy. □

However, even if $\exists$'s moves are limited to the minimal Zs, it is not straightforward to identify these relations in the case of a complex functor $\mathbb{T}$ with an associated syntax constructor $\mathcal{S}$ and a one-step semantics $[\![\mathcal{S}]\!]$ for $\mathcal{S}$ w.r.t. $\mathbb{T}$. To overcome this, we replace the two moves (of $\exists$ followed by $\forall$) from a position of type $(t, \psi) \in \mathbb{T}C \times \mathcal{S}(\mu\mathcal{L}^{\mathcal{S}}(\mathcal{V}))$ to a position of type $(c, \phi) \in C \times \mu\mathcal{L}^{\mathcal{S}}(\mathcal{V})$, by a sequence of moves in a "sub-game" played by $\exists$ and $\forall$. This sequence of moves essentially constructs the minimal relations Z by induction on the structure of the functor $\mathbb{T}$. Moreover, $\exists$ has a winning strategy in the modified game if and only if he has a winning strategy in the original one. The concept of a *one-step game* is used to define the above-mentioned sequence of moves.

Definition 4.7 (One-step game) *A* one-step game *w.r.t. a functor $\mathbb{T}$ and a syntax constructor $\mathcal{S}$ is a graph game between $\exists$ and $\forall$, whose positions include positions of type $(t, \psi) \in \mathbb{T}X \times \mathcal{S}L$ and of type $(x, \phi) \in X \times L$, with X and L being arbitrary sets, and such that positions of type $(x, \phi) \in X \times L$ are* terminal*, that is, they are not associated with either $\exists$ or $\forall$ and there are no moves defined for these positions.*

For each simple polynomial functor with corresponding syntax constructor and one-step semantics, we associate a one-step game which, when played instead of the two moves, of $\exists$ followed by $\forall$, in positions of type $(t, \psi) \in \mathbb{T}X \times \mathcal{S}L$ of the game $\tilde{\mathcal{E}}_{\phi_0}^{c_0}$, has the same effect as these two moves in terms of the positions being reached. Moreover, we show how to obtain one-step games for functors built using products, coproducts, exponentiation and functor composition by combining one-step games for the ingredient functors, and that these combinations preserve the adequacy property w.r.t. the original evaluation games.

Example 4.8 (i) A one-step game $\mathcal{G}_A$ w.r.t. A and $\mathcal{S}_A$ is given by:

Position	Player	Moves
$(a, a) \in A \times \mathcal{S}_A L$	$\forall$	$\emptyset$
$(b, a) \in A \times \mathcal{S}_A L$ with $b \neq a$	$\exists$	$\emptyset$
$(a, \neg a) \in A \times \mathcal{S}_A L$	$\exists$	$\emptyset$
$(b, \neg a) \in A \times \mathcal{S}_A L$ with $b \neq a$	$\forall$	$\emptyset$

(ii) A one-step game $\mathcal{G}_{\mathbb{Id}}$ w.r.t. $\mathbb{Id}$ and $\mathcal{S}_{\mathbb{Id}}$ is given by:

Position	Player	Moves
$(x, \bigcirc\phi) \in X \times \mathcal{S}_{\mathbb{Id}} L$	–	$\{ (x, \phi) \}$

(iii) A one-step game $\mathcal{G}_{\mathbb{P}}$ w.r.t. $\mathbb{P}$ and $\mathcal{S}_{\mathbb{P}}$ is given by:

Position	Player	Moves
$(t, \Box\phi) \in \mathbb{P}X \times \mathcal{S}_{\mathbb{P}} L$	$\forall$	$\{ (x, \phi) \mid x \in t \}$
$(t, \Diamond\phi) \in \mathbb{P}X \times \mathcal{S}_{\mathbb{P}} L$	$\exists$	$\{ (x, \phi) \mid x \in t \}$

(iv) A one-step game $\mathcal{G}_{\mathbb{D}}$ w.r.t. $\mathbb{D}$ and $\mathcal{S}_{\mathbb{D}}$ is given by:

Position	Player	Moves
$(\mu, L_p\phi) \in \mathbb{D}X \times \mathcal{S}_{\mathbb{D}} L$	$\exists$	$\{ \{(x_1, \phi), \ldots, (x_n, \phi)\} \mid \mu(x_i) \neq 0, \sum_{i=1}^{n} \mu(x_i) \geq p\,,$ $\sum_{i=1}^{n} \mu(x_i) - \mu(x_j) < p$ for each $j \in \{1, \ldots, n\} \}$
$(\mu, G_p\phi) \in \mathbb{D}X \times \mathcal{S}_{\mathbb{D}} L$	$\exists$	$\{ \{(x_1, \phi), \ldots, (x_n, \phi)\} \mid \mu(x_i) \neq 0, \sum_{i=1}^{n} \mu(x_i) > p\,,$ $\sum_{i=1}^{n} \mu(x_i) - \mu(x_j) \leq p$ for each $j \in \{1, \ldots, n\} \}$
$Z \subseteq X \times L$	$\forall$	Z

The above one-step games have been obtained by unfolding the definitions of $[\![\mathcal{S}_A]\!]$, $[\![\mathcal{S}_{\mathbb{Id}}]\!]$, $[\![\mathcal{S}_{\mathbb{P}}]\!]$ and $[\![\mathcal{S}_{\mathbb{D}}]\!]$, requiring minimality of the Zs in $\exists$'s moves, and simplifying the resulting one-step games by making implicit those steps where the sets of possible moves are singletons. For example, at position $(t, \Box\phi)$ in $\mathcal{G}_{\mathbb{P}}$, the only player that can move is $\forall$, since in the original game $\exists$ has no choice of a minimal relation but to move to $\{(t, \phi) \mid \phi \in \Phi\}$. Similarly, at position $(t, \Diamond\phi)$, only $\exists$ can move, since all the minimal relations he can choose are singletons and thus $\forall$ is left with no choice. It is also worth noting that, for $\mathbb{T} = \mathbb{P}$, the moves of the one-step game are similar to the moves corresponding to modal positions in the games for the modal μ-calculus, see e.g. [19]. Finally, the game $\mathcal{G}_{\mathbb{D}}$ still requires two moves, one of $\exists$ and one of $\forall$, to go from positions of type $(\mu, \psi) \in \mathbb{D}X \times \mathcal{S}_{\mathbb{D}} L$ to positions of type $(x, \phi) \in X \times L$. The underlying reason for this is that the modalities L_p and G_p are neither conjunction- nor disjunction-preserving, and thus both $\exists$ and $\forall$ have a real choice to make.

We now show how to obtain one-step games for complex endofunctors, by combining the one-step games for their ingredients.

Definition 4.9 (Combining one-step games) *For $i = 1, 2$, let $\mathcal{G}_i$ be a one-step game w.r.t. $\mathbb{T}_i$ and $\mathcal{S}_i$.*

(i) *A one-step game $\mathcal{G}_1 \otimes \mathcal{G}_2$ w.r.t. $\mathbb{T}_1 \times \mathbb{T}_2$ and $\mathcal{S}_1 \otimes \mathcal{S}_2$ is obtained by adding the following moves to the union of the moves of $\mathcal{G}_1$ and $\mathcal{G}_2$:*

Position	*Player*	*Moves*
$((t_1, t_2), [\pi_i]\phi) \in (\mathbb{T}_1 \times \mathbb{T}_2)X \times (\mathcal{S}_1 \otimes \mathcal{S}_2)L$	–	$\{ (t_i, \phi) \}$
$(t, \bigvee \Phi) \in \mathbb{T}_i X \times \overline{\mathcal{S}_i L}$	$\exists$	$\{ (t, \phi) \mid \phi \in \Phi \}$
$(t, \bigwedge \Phi) \in \mathbb{T}_i X \times \overline{\mathcal{S}_i L}$	$\forall$	$\{ (t, \phi) \mid \phi \in \Phi \}$

(ii) *A one-step game $\mathcal{G}_1 \oplus \mathcal{G}_2$ w.r.t. $\mathbb{T}_1 + \mathbb{T}_2$ and $\mathcal{S}_1 \oplus \mathcal{S}_2$ is obtained by adding the following moves to the union of the moves of $\mathcal{G}_1$ and $\mathcal{G}_2$:*

Position	*Player*	*Moves*
$(\iota_i(t_i), [\kappa_j]\phi) \in (\mathbb{T}_1 + \mathbb{T}_2)X \times (\mathcal{S}_1 \oplus \mathcal{S}_2)L$ *with* $i \neq j$	$\forall$	$\emptyset$
$(\iota_i(t_i), [\kappa_i]\phi) \in (\mathbb{T}_1 + \mathbb{T}_2)X \times (\mathcal{S}_1 \oplus \mathcal{S}_2)L$	–	$\{ (t_i, \phi) \}$
$(\iota_i(t_i), \langle\kappa_j\rangle\phi) \in (\mathbb{T}_1 + \mathbb{T}_2)X \times (\mathcal{S}_1 \oplus \mathcal{S}_2)L$ *with* $i \neq j$	$\exists$	$\emptyset$
$(\iota_i(t_i), \langle\kappa_i\rangle\phi) \in (\mathbb{T}_1 + \mathbb{T}_2)X \times (\mathcal{S}_1 \oplus \mathcal{S}_2)L$	–	$\{ (t_i, \phi) \}$
$(t, \bigvee \Phi) \in \mathbb{T}_i X \times \overline{\mathcal{S}_i L}$	$\exists$	$\{ (t, \phi) \mid \phi \in \Phi \}$
$(t, \bigwedge \Phi) \in \mathbb{T}_i X \times \overline{\mathcal{S}_i L}$	$\forall$	$\{ (t, \phi) \mid \phi \in \Phi \}$

(iii) *A one-step game $\mathcal{G} \odot E$ w.r.t. $\mathbb{T}^E$ and $\mathcal{S} \odot E$ is obtained by adding the following moves to the moves of $\mathcal{G}$:*

Position	*Player*	*Moves*
$(f, [e]\phi) \in (\mathbb{T}X)^E \times (\mathcal{S} \odot E)(L)$	–	$\{ (f(e), \phi) \}$
$(t, \bigvee \Phi) \in \mathbb{T}X \times \overline{\mathcal{S}L}$	$\exists$	$\{ (t, \phi) \mid \phi \in \Phi \}$
$(t, \bigwedge \Phi) \in \mathbb{T}X \times \overline{\mathcal{S}L}$	$\forall$	$\{ (t, \phi) \mid \phi \in \Phi \}$

(iv) *A one-step game $\mathcal{G}_1 \circledcirc \mathcal{G}_2$ w.r.t. $\mathbb{T}_1 \circ \mathbb{T}_2$ and $\mathcal{S}_1 \circledcirc \mathcal{S}_2$ is given by the union of the moves of $\mathcal{G}_1$ and $\mathcal{G}_2$ and the following moves:*

Position	*Player*	*Moves*
$(t, \bigvee \Phi) \in \mathbb{T}_2 X \times \overline{\mathcal{S}_2 L}$	$\exists$	$\{ (t, \phi) \mid \phi \in \Phi \}$
$(t, \bigwedge \Phi) \in \mathbb{T}_2 X \times \overline{\mathcal{S}_2 L}$	$\forall$	$\{ (t, \phi) \mid \phi \in \Phi \}$

The moves corresponding to finite conjunctions and disjunctions in the definitions of $\mathcal{G}_1 \otimes \mathcal{G}_2$, $\mathcal{G}_1 \oplus \mathcal{G}_2$, $\mathcal{G}_1 \odot E$ and $\mathcal{G}_1 \circledcirc \mathcal{G}_2$ have the rôle of dealing with conjunctions and disjunctions occurring at inner levels in the structure of formulae in $\mathcal{L}(\mathcal{S})$. For example, if $\mathbb{T} = \mathbb{P} \times (1 + \mathbb{P} \circ (\mathbb{I}\mathsf{d} \times \mathbb{I}\mathsf{d}))$, (and thus $\mathbb{T}$-coalgebras are spatial transition systems), the language induced by $\mathcal{S}_{\mathbb{P}} \otimes (\mathcal{S}_1 + \mathcal{S}_{\mathbb{P}} \circledcirc (\mathcal{S}_{\mathbb{I}\mathsf{d}} \otimes \mathcal{S}_{\mathbb{I}\mathsf{d}}))$ contains formulae of form $[\pi_2]\langle\kappa_2\rangle\Diamond([\pi_1] \bigcirc \phi_1 \wedge [\pi_2] \bigcirc \phi_2)$. The binary conjunction in this formula will be dealt with in a move from a position of type $\mathbb{P}(X \times X) \times \overline{\mathcal{S}_{\mathbb{I}\mathsf{d}} L \otimes \mathcal{S}_{\mathbb{I}\mathsf{d}} L}$. This is accounted for by the additional moves of $\forall$ in the game $\mathcal{G}_{\mathbb{I}\mathsf{d}} \otimes \mathcal{G}_{\mathbb{I}\mathsf{d}}$.

In what follows, we make formal the relationship between one-step games and minimal relations relative to specific positions in the game $\mathcal{E}^{c_0}_{\phi_0}$.

Definition 4.10 (Play tree) *Let $\mathcal{G}$ be a one-step game w.r.t. $\mathbb{T}$ and $\mathcal{S}$. A* play tree *in $\mathcal{G}$ is a tree labelled by nodes of $\mathcal{G}$ with the following properties:*

(i) *the root of the tree is some $(t, \psi) \in \mathbb{T}X \times \mathcal{S}L$, and the leaves of the tree are (terminal) nodes of type $(x, \phi) \in X \times L$;*

(ii) *each $\exists$ node has only one successor, taken from the set of $\mathcal{G}$-moves of $\exists$ in that node;*

(iii) *the successors of a $\forall$ node are all $\mathcal{G}$-successors of that node.*

The notion of adequacy of a one-step game now captures the necessary conditions for the recovery of minimal relations via one-step games.

Definition 4.11 (Adequacy of one-step game) *Given a one-step semantics $[\![\mathcal{S}]\!]$ for $\mathcal{S}$ w.r.t. $\mathbb{T}$, a one-step game $\mathcal{G}$ w.r.t. $\mathbb{T}$ and $\mathcal{S}$ is called* adequate for $[\![\mathcal{S}]\!]$ *if for any $t \in \mathbb{T}X$ and $\psi \in \mathcal{S}L$, minimal relations $Z \subseteq X \times L$ relative to (t, ψ) are in one-to-one correspondence with sets of leaves of play trees in $\mathcal{G}$.*

Example 4.12 Let $\mathbb{T}$ be the signature functor for spatial transition systems, that is, $\mathbb{T} = \mathbb{P} \times (1 + \mathbb{P} \circ (\mathbb{Id} \times \mathbb{Id}))$. Consider the position $\big((X, \iota_2(Y)), [\pi_2]\langle\kappa_2\rangle\Diamond([\pi_1] \bigcirc \phi_1 \wedge [\pi_2] \bigcirc \phi_2)\big)$ in the one-step game $\mathcal{G}_{\mathbb{P}} \otimes (\mathcal{G}_1 \oplus \mathcal{G}_{\mathbb{P}} \circledcirc (\mathcal{G}_{\mathbb{Id}} \otimes \mathcal{G}_{\mathbb{Id}}))$. Here, $X \in \mathbb{P}C$, $Y \in \mathbb{P}(C \times C)$, whereas the formula corresponds to $\phi_1 \mid \phi_2$, as defined in Example 2.3. The play trees starting from this position are of the form:

$$
\begin{tikzcd}[column sep=small]
 & \big((X, \iota_2(Y)), [\pi_2]\langle\kappa_2\rangle\Diamond([\pi_1] \bigcirc \phi_1 \wedge [\pi_2] \bigcirc \phi_2)\big) \arrow[d] & \\
 & \big(\iota_2(Y), \langle\kappa_2\rangle\Diamond([\pi_1] \bigcirc \phi_1 \wedge [\pi_2] \bigcirc \phi_2)\big) \arrow[d] & \\
 & \big(Y, \Diamond([\pi_1] \bigcirc \phi_1 \wedge [\pi_2] \bigcirc \phi_2)\big) \arrow[d, "\exists"] & \\
 & (y, [\pi_1] \bigcirc \phi_1 \ \wedge \ [\pi_2] \bigcirc \phi_2) \arrow[dl, "\forall"'] \arrow[dr, "\forall"] & \\
(y, [\pi_1] \bigcirc \phi_1) \arrow[d] & & (y, [\pi_2] \bigcirc \phi_2) \arrow[d] \\
(\pi_1(y), \bigcirc\phi_1) \arrow[d] & & (\pi_2(y), \bigcirc\phi_2) \arrow[d] \\
(\pi_1(y), \phi_1) & & (\pi_2(y), \phi_2)
\end{tikzcd}
$$

with $y \in Y$. The first two moves in such play trees are uniquely determined. Following these, $\exists$ chooses an element $y \in Y$ to witness $\big(Y, \Diamond([\pi_1]\bigcirc\phi_1 \wedge [\pi_2] \bigcirc \phi_2)\big)$. Once this choice has been made, both of $\forall$'s possible next moves have to be taken into account when defining a minimal relation relative to $\big((X, \iota_2(Y)), \phi_1 \mid \phi_2\big)$. This corresponds to the intuition that, in order to provide a witness for $\big((X, \iota_2(Y)), \phi_1 \mid \phi_2\big)$, $\exists$ has to choose an element $y \in Y$ such that *both* ϕ_1 holds in $\pi_1(y)$ *and* ϕ_2 holds in $\pi_2(y)$.

Proposition 4.13 (i) *$\mathcal{G}_A$, $\mathcal{G}_{\mathbb{Id}}$, $\mathcal{G}_{\mathbb{P}}$ and $\mathcal{G}_{\mathbb{D}}$ are adequate for $[\![\mathcal{S}_A]\!]$, $[\![\mathcal{S}_{\mathbb{Id}}]\!]$, $[\![\mathcal{S}_{\mathbb{P}}]\!]$ and $[\![\mathcal{S}_{\mathbb{D}}]\!]$ respectively.*

(ii) *If $\mathcal{G}_i$ is adequate for $[\![\mathcal{S}_i]\!]$, for $i = 1, 2$, then $\mathcal{G}_1 \otimes \mathcal{G}_2$, $\mathcal{G}_1 \oplus \mathcal{G}_2$ and $\mathcal{G}_1 \circledcirc \mathcal{G}_2$ are adequate for $[\![\mathcal{S}_1 \otimes \mathcal{S}_2]\!]$, $[\![\mathcal{S}_1 \oplus \mathcal{S}_2]\!]$ and $[\![\mathcal{S}_1 \circledcirc \mathcal{S}_2]\!]$, respectively.*

Proof (Sketch) Follows directly from the definitions of the corresponding one-step semantics. □

Theorem 4.14 (One-step adequacy) *If $\mathcal{G}$ is adequate for $[\![\mathcal{S}]\!]$, then $\exists$ has a winning strategy in $\mathcal{E}^{c_0}_{\phi_0}$ if and only if he has a winning strategy in the game obtained from $\mathcal{E}^{c_0}_{\phi_0}$ by replacing the last two moves in Definition 4.1 by the moves of $\mathcal{G}$.*

Proof (Sketch) Assume first that $\exists$ has a winning strategy in the original game. This strategy provides, for each position of type $(t, \psi) \in \mathbb{T}C \times \mathcal{S}L$, a relation $Z \subseteq C \times L$ s.t. $(t, \psi) \in [\![\mathcal{S}]\!](Z)$. By Lemma 4.6, we can assume that Z is minimal relative to (t, ψ). We construct a winning strategy for $\exists$ in the modified game by replacing $\exists$'s move given by the winning strategy in a position of type $(t, \psi) \in \mathbb{T}C \times \mathcal{S}L$ with moves in the modified game. By adequacy of $\mathcal{G}$ for $[\![\mathcal{S}]\!]$, for each minimal $Z \subseteq C \times L$ relative to $(t, \psi) \in [\![\mathcal{S}]\!](Z)$, there exists a corresponding play tree starting in (t, ψ). The moves of $\exists$ in the modified game are obtained directly from this play tree. Specifically, in each $\exists$ position that belongs to the play tree, $\exists$ chooses the only move that keeps the play inside the play tree. Now since the play tree contains all possible $\forall$ moves in the modified game, a move of $\forall$ will itself keep the play inside the play tree. Since the Z move of $\exists$ in the original game was part of a winning strategy, so is the newly built strategy in the modified game.

Now assume that $\exists$ has a winning strategy in the modified game. This strategy can be used to define, for each position of type $(t, \psi) \in \mathbb{T}C \times \mathcal{S}L$, a play tree in $\mathcal{G}$ – this is done by using $\exists$'s strategy in each $\exists$ position, and collecting all of $\forall$'s $\mathcal{G}$-moves in each $\forall$ position, repeatedly until a terminal position in $\mathcal{G}$ is reached. By adequacy of $\mathcal{G}$ for $[\![\mathcal{S}]\!]$, to each such play tree in $\mathcal{G}$ there corresponds a minimal Z relative to (t, ψ). The resulting relations Z and the winning strategy of $\exists$ in the modified game can now be used to define a winning strategy for $\exists$ in the original game – this is done by replacing $\exists$'s moves in positions of type $(t, \psi) \in \mathbb{T}C \times \mathcal{S}L$ by the moves resulting from the play trees. □

5 Summary and Future Work

We have extended the modular coalgebraic logics of [**?**,**?**] with general fixed points, of which until- and dynamic-like fixed points are an instance. Following [21], we have provided the resulting fixed point logics with a game semantics (by defining evaluation games for formulae and states of coalgebras), and have shown the adequacy of this semantics w.r.t. the standard fixed point semantics. Furthermore, we have shown that the moves corresponding to modal positions in these evaluation games can be replaced by so-called one-step games, whose boards can be built inductively on the structure of the underlying signature functors, and whose moves simulate exactly those moves of the evaluation games which are relevant to deciding the existence of winning strategies for $\exists$ (and thus the satisfaction of formulae by states of coalgebras).

Existing temporal logics for probabilistic systems (as described e.g. in [9]) allow the formalisation of properties of the kind "with probability at least p, ϕ holds until ψ holds". Such languages, interpreted over Markov chains (which are exactly the $\mathbb{D}$-coalgebras), are not recovered as fragments of our fixed point logic for the functor

$\mathbb{D}$. We believe that this is due to our choice of modalities L_p and G_p and of their semantics. Ongoing work aims to address this by changing the underlying modal language.

Developing proof systems for coalgebraic fixed point logics is a natural and routine extension of this paper, but proving completeness of these proof systems requires more work. Proof systems are obtained in two steps: (1) the proof system constructors of [**?**] are used to derive a complete set of axioms and rules for the underlying modal language, and (2) a generalization of Kozen's induction rule for fixed points [12] is added to the proof system. The subtlety of the first step is that proof system constructors are functors operating on a category of boolean theories, which for the purpose of well-definedness of our fixed points have to be restricted to theories closed under conjunction and disjunction.

Our adequacy theorem shows that deciding about the satisfaction of a formula by a pointed coalgebra is equivalent to deciding whether $\exists$ has a history-free winning strategy in the evaluation game. The time complexity of the latter is exponential in the size of the game board [19]. However, it is well known that if one restricts the fixed points to the alternation-free fragment, the complexity reduces to polynomial time [19]. Our game boards are generalizations of those for the propositional μ-calculus, and the exact impact this has on complexity deserves further study. However, we conjecture that by only using minimal relations (Definition 4.5) as possible moves of $\exists$, similar complexity results to those for the propositional μ-calculus can be obtained.

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