l^p Gain Bounds for Switched Adaptive Controllers

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***Abstract*— A class of discrete plants controlled by a switching adaptive strategy is considered, and l^p bounds, $1 \le p \le \infty$, are obtained for the closed loop gain relating input and output disturbances to internal signals.**

I. Introduction

It is well known that the gain of the closed loop operator mapping the external input and output disturbances to the input and output of the plant plays a key role in the study of robust stability [1]. Obtaining good bounds for this quantity is therefore of high importance in adaptive control, where the study of performance and robustness of adaptive controllers is still in its infancy. Switched adaptive controllers are an attractive class of controllers to consider in this context.

In the context of continuous time systems, L^2 bounds can be found in the work of Morse and co-workers, see e.g. [2], [3]. Related work in both the discrete and continuous settings can also be found in [4], [5], [6], [7], and related results have also been obtained by Megretski. The recent bounds obtained by Vinnicombe [6] achieve rather tight l^2 bounds for pairs of discrete first order plants of the form $y_{k+1} = ay_k + bu_k$. The results in this paper extend Vinnicombe's results in several directions. Namely, 1. the gain bounds are from disturbances in both the input and output channels to the internal signals, 2. the bounds are constructed for any signal space l^p, $1 \le p \le \infty$, 3. the class of plants considered is broader (to include some higher order plants) and 4. the switching controller is defined for a finite set of plants within the given class. If the analysis is restricted to the special case considered by Vinnicombe, we obtain bounds which are asymptotically equivalent.

We remark that the results we obtain are restricted in various ways, e.g. the underlying controllers are assumed to be dead-beat and the class of plants considered is not very wide; the generalization of these preliminary results is the topic of current research.

We close this introduction with some notation: Throughout this paper V is the considered signal space, here it is the space l^p for $1 \le p \le \infty$ where $l^p := \left\{ a \in \operatorname{map}(\mathbb{N},\mathbb{R}) \;\middle|\; \sum_i |a_i|^p < \infty \right\}$ and $\operatorname{map}(A,B)$ is the set of all functions from A to B. The signal space V is equipped with the corresponding p-norm, i.e. $\|\cdot\| := \|\cdot\|_p$. Note that the p-norm will be used for vectors in $\mathbb{R}^n$ as well, we will write $\|\cdot\|$ for these norms as well. Finally the extended signal space V_e is $\left\{ a : \mathbb{N} \to \mathbb{R} \;\middle|\; \forall n \in \mathbb{N} : a|_{[0,n]} \in V \right\} = \operatorname{map}(\mathbb{N},\mathbb{R})$.

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II. Definition of switched control and system class

Consider the closed loop system $[P,C]$

$$u_0 = u_1 + u_2,$$
$$y_0 = y_1 + y_2,$$
$$y_1 = Pu_1,$$
$$u_2 = Cy_2,$$

which is illustrated in Figure 1.

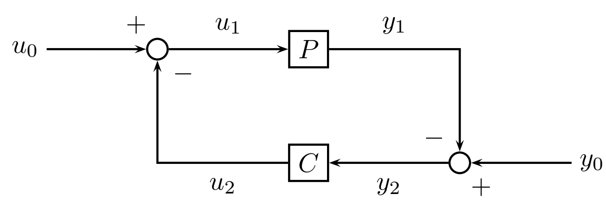

Fig. 1. The closed loop system $[P,C]$

Assume that the disturbances $(u_0, y_0) \in V \times V$. Let $\mathcal{P}$ be a finite parameter set, e.g. $\mathcal{P} = \{1,\dots,N\}$, $N \in \mathbb{N}$. For every $p \in \mathcal{P}$ the operator

$$P_p : V_e \to V_e, \quad u_1 \mapsto P_p\{u_1\} = y_1,$$

is described by the equations:

$$\begin{aligned} y_1(k) &= \sum_{i=1}^{\sigma} a_p^i y_1(k-i) + b_p u_1(k-1), \\ y_1(-k) &= 0 \quad \forall k \in \mathbb{N}, \end{aligned} \tag{1}$$

where $a_p^1, a_p^2, \dots, a_p^\sigma, b_p \in \mathbb{R}$ are known and $\sigma \in \mathbb{N}$ is the maximum order of all plants P_p, $p \in \mathcal{P}$. It will be assumed that

$$b_p \neq 0 \quad \forall p \in \mathcal{P}.$$

It is known that $P = P_{p*}$ for an unknown $p^* \in \mathcal{P}$. Taking the disturbances into account (1) becomes for $k \in \mathbb{N}$

$$\begin{aligned} y_2(k) = &\sum_{i=1}^{\sigma} a_p^i y_2(k-i) + b_p u_2(k-1) \\ &+ y_0(k) - \sum_{i=1}^{\sigma} a_p^i y_0(k-i) - b_p u_0(k-1). \end{aligned} \tag{2}$$

The aim is to construct a causal operator

$$C : V_e \to V_e, \quad y_2 \mapsto C\{y_2\} = u_2,$$

such that the internal signals (u_2, y_2) of the closed loop $[P, C]$ fulfill, for every $(u_0, y_0) \in V \times V$, the property $(u_2, y_2) \in V \times V$. In particular the controller C should stabilize P. Furthermore we are aiming for performance results.

For each $p \in P$, define $C_p : V_e \to V_e$ to be the dead beat controller:

$$y_2 \to \left(k \mapsto u_2(k) = -\frac{1}{b_p} \sum_{i=1}^{\sigma} a_p^i y_2(k-i+1) \right). \quad (3)$$

The desired switched controller C uses a switching strategy, which chooses at every time an active controller C_q, $q \in \mathcal{P}$. The switching strategy has to ensure that the closed loop $[P, C]$ remains stable. The structure of C is illustrated in Figure 2.

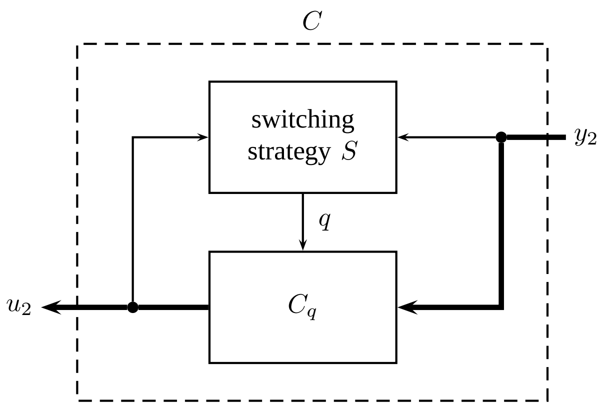

Fig. 2. The structure of the controller C

The switching strategy S is a causal operator of the form

$$S : V_e \times V_e \to \text{map}(\mathbb{N}, \mathcal{P}), \quad (u_2, y_2) \mapsto q,$$

with the property

$$S\{u_2, y_2\}\big|_{[0,k]} = S\big\{u_2|_{[0,k-1]}, y_2|_{[0,k]}\big\}\big|_{[0,k]}.$$

With $q(k) = S\{u_2, y_2\}(k)$ the controller C can then be described by

$$C\{y_2\}(k) = C_{q(k)}\{y_2\}(k) \quad \forall\, k \in \mathbb{N}. \quad (4)$$

The structure of the switching strategy S is indicated by Figure 3.

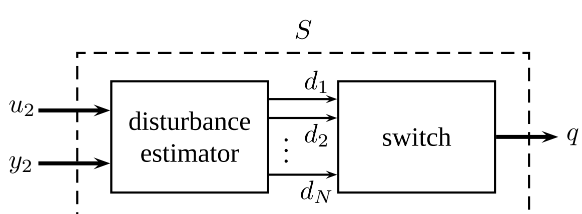

Fig. 3. The structure of the switching strategy S

For $p \in \mathcal{P}$ the estimate d_p^k of the disturbance at time $k \in \mathbb{N}$ is a vector of length $k+1$, i.e.

$$d_p^k = \left(d_p^k(0), d_p^k(1), d_p^k(2), \ldots, d_p^k(k)\right) \in \left(\mathbb{R}^h\right)^{k+1},$$

where $h \in \mathbb{N}$ is a number depending on the specificly chosen estimator and the order σ. For a given disturbance estimation procedure, the strategy chooses as actual controller $C_{q(k)}$, where $q(k)$ is such that the corresponding disturbance estimator is minimal, i.e.

$$S\{u_2, y_2\}(k) = q(k) = \underset{p \in \mathcal{P}}{\text{argmin}}\ \|d_p^k\|. \quad (5)$$

No specific estimation procedure is required in the statement of our final result, instead our results are based on some required general properties of estimators. These properties and two exemplar estimation procedures satisfying these properties are given in the following section.

III. Properties of the disturbance estimations

The results will be based on the following assumptions on the disturbance estimators.

Assumptions

1) *For every $k \in \mathbb{N}$ and $p \in \mathcal{P}$ the estimator d_p^k is independent of $y_2|_{[k+1,\infty)}$ and $u_2|_{[k,\infty)}$.*
2) *There exists a constant $c_1 > 0$ such that for all $k \in \mathbb{N}$: $\|d_{p^*}^k\| \leq c_1 \|u_0, y_0\|$.*
3) *There exists a constant $c_2 > 0$ such that for all $k \in \mathbb{N}$: $|y_2(k)| \leq c_2 \left\| d_{q(k-1)}^k \big|_{[k-\sigma,k]} \right\|$.*
4) *For all $p \in \mathcal{P}$ and for all $k, k' \in \mathbb{N}$ with $0 \leq k < k'$: $\|d_p^k\| \leq \|d_p^{k'}|_{[0,k]}\|$.*

Property 1 states that the disturbance estimator only utilizes information which is available at time k, i.e. that the disturbance estimation is causal. Assumption 2 ensures that for the estimator of the real plant the estimated disturbances are bounded by the real disturbances. The third assumption is technical and the idea is that if at time $k-1$ the controller q was chosen then the "memory" of P_q was cancelled by the corresponding dead-beat controller C_q. The next output $y_2(k)$ is therefore independent of the past history of the estimated plant and hence only arise from the current estimated disturbances. Assumption 4 reflects a kind of minimality of every disturbance estimation.

In particular consider the following two estimation procedures:

Estimator A
Let, for $p \in \mathcal{P}$ and $k \in \mathbb{N}$,

$$S_p^A(k) := \Big\{ \left(u^p_{0\,[0,k-1]}, y^p_{0\,[0,k]}\right) \in \mathbb{R}^k \times \mathbb{R}^{k+1} \;\Big|\; P_p\{u^p_{0\,[0,k-1]} - u_2|_{[0,k-1]}\}|_{[0,k]} = y^p_{0\,[0,k]} - y_2|_{[0,k]} \Big\},$$

i.e. the set of all truncated disturbance signals $\left(u^p_{0\,[0,k-1]}, y^p_{0\,[0,k]}\right) \in \mathbb{R}^k \times \mathbb{R}^{k+1}$ which are consistent with the signals $(u_2|_{[0,k-1]}, y_2|_{[0,k]})$ and the plant P_p. Then

$$d_p^k := \underset{x \in S_p^A(k)}{\text{argmin}} \{\|x\|\}.$$

Consider d_p^k as element of $(\mathbb{R}^2)^{k+1}$ by writing $d_p^k(k) = \Big(0, y^p_{0\,[0,k]}(k)\Big)$.
Note that although this estimation is intuitive it is not recursively generated in the sense that in general for $k' > k$

$$d_p^{k'}|_{[0,k]} \neq d_p^k.$$

Estimator B
Let, for $p \in \mathcal{P}$ and $k \in \mathbb{N}$,

$$\begin{aligned} S_p^B(k) := \Big\{ \Big(u^p_{0\,[k-1]}, y^p_{0\,[k-\sigma,k]}\Big) \in \mathbb{R} \times \mathbb{R}^{\sigma+1} \Big| \\ y_2(k) + \sum_{i=1}^{\sigma} a_p^i y_2(k-i) + b_p u_2(k-1) = \\ y^p_{0\,[k-\sigma,k]}(k) - \sum_{i=1}^{\sigma} a_p^i y^p_{0\,[k-\sigma,k]}(k-i) + b_p u^p_{0\,[k-1]} \Big\}, \end{aligned}$$

i.e. the set of all disturbance values $\Big(u^p_{0\,[k-1]}, y^p_{0\,[k-\sigma,k]}\Big) \in \mathbb{R} \times \mathbb{R}^{\sigma+1}$, which are consistent with the current signals $\big(u_2(k-1), y_2|_{[k-\sigma,k]}\big)$ and plant P_p, i.e. fulfill (2). Then

$$d_p^k(k) := \operatorname*{argmin}_{x \in S_p^B(k)} \|x\|.$$

and $d_p^k(i) = d_p^i(i)$ for all $0 \leq i < k$. The vector d_p^k is therefore an element of $\big(\mathbb{R} \times \mathbb{R}^{\sigma+1}\big)^{k+1}$.

Proposition 1: Both estimators A and B fulfill Assumptions 1-4.

Proof: By definition the sets $S_p^A(k)$ and $S_p^B(k)$ does not depend on $u_2|_{[k,\infty)}$ and $y_2|_{[k+1,\infty)}$ which yields Assumption 1 for both estimations.

Note that for any disturbance signals $u_0, y_0 \in V$

$$\big(u_0|_{[0,k-1]}, y_0|_{[0,k]}\big) \in S_{p^*}^A(k) \quad \forall k \in \mathbb{N}$$

and

$$\big(u_0(k-1), y_0|_{[k-\sigma,k]}\big) \in S_{p^*}^B(k) \quad \forall k \in \mathbb{N}.$$

Therefore, for estimator A,

$$\big\|d_{p^*}^k\big\| \leq \big\|u_0|_{[0,k-1]}, y_0|_{[0,k]}\big\| \leq \|u_0, y_0\| \quad \forall k \in \mathbb{N}.$$

For estimator B observe that

$$d_{p^*}^k(k) \leq \big\|u_0(k-1), y_0|_{[k-\sigma,k]}\big\| \quad \forall k \in \mathbb{N}$$

and hence

$$\begin{aligned} &\big\|d_{p^*}^k\big\| = \big\|d_{p^*}^0(0), d_{p^*}^1(1), \ldots, d_{p^*}^k(k)\big\| \\ &\leq \left\| \begin{matrix} \begin{pmatrix} u_0(-1) \\ y_0(-\sigma) \end{pmatrix}, & \begin{pmatrix} u_0(0) \\ y_0(1-\sigma) \end{pmatrix}, & \ldots, & \begin{pmatrix} u_0(k-1) \\ y_0(k-\sigma) \end{pmatrix} \\ y_0(1-\sigma), & y_0(2-\sigma), & \ldots, & y_0(k+1-\sigma) \\ & \vdots & & \\ y_0(0), & y_0(1), & \ldots, & y_0(k) \end{matrix} \right\| \\ &\leq \|\underbrace{1, 1, \ldots, 1}_{\sigma+1}\| \|u_0, y_0\|. \end{aligned}$$

This shows that Assumption 2 holds for both estimators with $c_1 = 1$ and $c_1 = \|1, 1, \ldots, 1\|$, resp. To show that Assumption 3 holds for estimators A and B, fix $k \in \mathbb{N}$ and let $q := q(k-1)$. Since

$$u_2(k-1) = -\frac{1}{b_q} \sum_{i=1}^{\sigma} a_q^i y_2(k-i)$$

it follows from (2) that, for any disturbance estimation (u_0^q, y_0^q),

$$y_2(k) = b_q u_0^q(k-1) + y_0^q(k) - \sum_{i=1}^{\sigma} a_q^i y_0^q(k-i).$$

Note that q depends on k. Hence

$$\begin{aligned} &|y_2(k)| \leq \\ &\quad c_2 \left\|u_0^q(k-1),\, y_0^q(k),\, y_0^q(k-1),\, \ldots,\, y_0^q(k-\sigma)\right\|, \end{aligned} \tag{6}$$

where

$$c_2 := (\sigma+2) \max_{p \in \mathcal{P}} \left(\max\{|b_p|, 1, |a_p^1|, |a_p^2|, \ldots, |a_p^\sigma|\}\right).$$

Observe that

$$\begin{aligned} &\left\|u_0^q(k-1),\, y_0^q(k),\, y_0^q(k-1),\, \ldots,\, y_0^q(k-\sigma)\right\| \\ &\leq \left\| \begin{pmatrix} 0 \\ y_0^q(k) \end{pmatrix}, \begin{pmatrix} u_0^q(k-1) \\ y_0^q(k-1) \end{pmatrix}, \ldots, \begin{pmatrix} u_0^q(k-\sigma) \\ y_0^q(k-\sigma) \end{pmatrix} \right\| \end{aligned}$$

and therefore Assumption 3 holds for estimator A.

Estimator B fulfills

$$\left\|d_q^k(k)\right\| = \left\|u_0^q(k-1),\, y_0^q(k),\, y_0^q(k-1), \ldots,\, y_0^q(k-\sigma)\right\|$$

and it is by (6) obvious that

$$\begin{aligned} |y_2(k)| &\leq c_2 \left\|d_q^k(k)\right\| \\ &\leq c_2 \left\|d_q^k(k), d_q^k(k-1), \ldots, d_q^k(k-\sigma)\right\|. \end{aligned}$$

Assumption 4 is clear, because for all $k < k'$ and $p \in \mathcal{P}$ the definition of estimator A yields

$$d_p^{k'}|_{[0,k]} \in S_p^A(k)$$

and the definition of estimator B yields

$$d_p^{k'}|_{[0,k]} = d_p^k.$$

qed

IV. Main result

Theorem 2: Let $V = l^r$ for $r \in [1, \infty]$ and let $\{\ P_p : V_e \to V_e \ |\ p \in \mathcal{P}\ \}$ be the set of given plants which are defined by (1) and where $\mathcal{P}$ is a finite parameter set. Let the controller $C : V_e \to V_e$ be defined as in (4) with a switching strategy defined by (5) and let the disturbance estimators fulfill Assumptions 1-4. Then for all $p^* \in \mathcal{P}$ the closed-loop system $[P_{p^*}, C]$ has the following properties:

1) $(u_0, y_0) \in V \times V \quad \Rightarrow \quad (u_2, y_2) \in V \times V$
2) There exists $\gamma = \gamma(p^*) > 0$ such that, for all $(u_0, y_0) \in V \times V$,

$$\|(u_2, y_2)\| \leq \gamma \|u_0, y_0\|.$$

Before we give the proof we would like to highlight that the proof is constructive; explicit upper bounds for the gain γ can be obtained by following the steps in the proof.

Proof: It is clear that the second assertion implies the first one and therefore only the existence of $\gamma > 0$ such that $\|(u_2, y_2)\| \leq \gamma\|u_0, y_0\|$ for all $(u_0, y_0) \in V \times V$ will be shown.

The definition (3) and (4), yields that there exists $c_0 > 0$ such that

$$|u_2(k)| \leq c_0\|y_2\|_{[k-\sigma+1,k]}. \tag{7}$$

Hence

$$\|u_2\| \leq c_0\|\underbrace{1, 1, \ldots, 1}_{\sigma}\| \; \|y_2\|,$$

and therefore it is sufficient to show existence of $\tilde{\gamma} > 0$ such that for all $(u_0, y_0) \in V \times V$

$$\|y_2\| \leq \tilde{\gamma}\|u_0, y_0\|.$$

Let $(u_0, y_0) \in V \times V$. Since the plant P_{p^*} is strictly causal and the controller C is causal by Assumption 1 and by causality of C_q, there exist a unique solutions $(u_2, y_2) \in V_e^2$ of the closed-loop system $[P, C]$. There exists therefore unique disturbance estimations d_p^k for $p \in \mathcal{P}$ and for $k \in \mathbb{N}$. Write $q(k) = S\{y_2, u_2\}(k)$ for the switching-signal and let $Q = \{k_0 = 0, k_1, k_2, \ldots\}$ be the set of switching times with $k_i < k_{i+1}$ for all $i \in \mathbb{N}$, i.e.

$$q_i := q(k_i) = q(k_i + l) \neq q(k_{i+1}) \quad 0 \leq l < k_{i+1} - k_i.$$

If Q is a finite set then define $k_{|Q|} = \infty$ and ignore in the following all k_is with $i > |Q|$. Write for $i \in \mathbb{N}$

$$y_2^i := y_2|_{[k_i+1,k_{i+1}-1]}$$

and observe that

$$\begin{aligned}&\|y_2\| \\ &= \big\|\|y_2^0, y_2^1, y_2^2, \ldots\|, \|y_2(k_0), y_2(k_1), y_2(k_2), \ldots\|\big\|.\end{aligned}$$

It will be shown that there exist $\gamma_1 > 0$ and $\gamma_2 > 0$ such that

$$\left\|y_2^0, y_2^1, y_2^2, \ldots\right\| \leq \gamma_1\|u_0, y_0\|$$

and

$$\|y_2(k_0), y_2(k_1), y_2(k_2), \ldots\| \leq \gamma_2\|u_0, y_0\|.$$

The proof of the theorem would then with $\tilde{\gamma} = \|\gamma_1, \gamma_2\|$ be complete.

STEP 1: It will be shown that

$$\exists\gamma_1 > 0 : \left\|y_2^0, y_2^1, y_2^2, \ldots\right\| \leq \gamma_1\|u_0, y_0\|.$$

It is first shown inductively that for every $n \in \mathbb{N}$

$$\left\|y_2^0, y_2^1, \ldots, y_2^n\right\| \leq c_2 \left\| \begin{matrix} d_{q_n}^{k_{n+1}-1} \\ d_{q_n}^{k_{n+1}-2} \\ \vdots \\ d_{q_n}^{k_{n+1}-(\sigma+1)} \end{matrix} \right\|. \tag{8}$$

If for any $n \in \mathbb{N}$ the difference $k_{n+1} - k_n$ is smaller then $\sigma+1$ then the last entry would be $d_{q_n}^{k_n}$ instead of $d_{q_n}^{k_{n+1}-(\sigma+1)}$.

Introduce the notation

$$d_q^a|_{[-b]} := d_q^a|_{[a-b,a]} \quad \text{for } q \in \mathcal{P}, \; a, b \in \mathbb{N}.$$

Starting with $n = 0$ observe that by Assumption 3

$$\begin{aligned}\|y_2^0\| &= \|y_2(k_0+1), y_2(k_0+2), \ldots, y_2(k_1-1)\| \\ &\leq c_2 \left\|\left\|d_{q_0}^{k_0+1}|_{[-\sigma]}\right\|, , \ldots, \left\|d_{q_0}^{k_1-1}|_{[-\sigma]}\right\|\right\| \\ &= c_2 \left\| \begin{matrix} \left(d_{q_0}^{k_0+1}|_{[-\sigma]}, \; d_{q_0}^{k_0+\sigma+2}|_{[-\sigma]}, \ldots\right) \\ \left(d_{q_0}^{k_0+2}|_{[-\sigma]}, \; d_{q_0}^{k_0+\sigma+3}|_{[-\sigma]}, \ldots\right) \\ \vdots \\ \left(d_{q_0}^{k_0+\sigma+1}|_{[-\sigma]}, \; d_{q_0}^{k_0+2\sigma+2}|_{[-\sigma]}, \ldots\right) \end{matrix} \right\|.\end{aligned}$$

Note that every row is finite and the last entries are

$$d_{q_0}^{k_1-1}|_{[-\sigma]}, d_{q_0}^{k_1-2}|_{[-\sigma]}, \ldots, d_{q_0}^{k_1-(\sigma+1)}|_{[-\sigma]},$$

but not necessarily in this order. Using now successively Assumption 4 and the simple general fact that for any sequence s and $a_1 < a < b < b_1$

$$\left\|s|_{[a,b]}\right\| \leq \left\|s|_{[a_1,b_2]}\right\|$$

one arrives at

$$\|y_2^0\| \leq \left\| \begin{matrix} d_{q_0}^{k_1-1} \\ d_{q_0}^{k_1-2} \\ \vdots \\ d_{q_0}^{k_1-(\sigma+1)} \end{matrix} \right\|.$$

Therefore $n = 0$ is shown.
For $n > 0$ observe first that

$$\begin{aligned}\left\| \begin{matrix} d_{q_{n-1}}^{k_n-1} \\ d_{q_{n-1}}^{k_n-2} \\ \vdots \\ d_{q_{n-1}}^{k_n-(\sigma+1)} \end{matrix} \right\| &= \left\| \begin{matrix} \|d_{q_{n-1}}^{k_n-1}\| \\ \|d_{q_{n-1}}^{k_n-2}\| \\ \vdots \\ \|d_{q_{n-1}}^{k_n-(\sigma+1)}\| \end{matrix} \right\| \\ &\overset{(5)}{\leq} \left\| \begin{matrix} \|d_{q_n}^{k_n-1}\| \\ \|d_{q_n}^{k_n-2}\| \\ \vdots \\ \|d_{q_n}^{k_n-(\sigma+1)}\| \end{matrix} \right\| \leq \left\| \begin{matrix} d_{q_n}^{k_n} \\ d_{q_n}^{k_n-1} \\ \vdots \\ d_{q_n}^{k_n-\sigma} \end{matrix} \right\|.\end{aligned}$$

Now

$$\|y_2^0, y_2^1, \dots y_2^n\| = \left\| \|y_2^0, y_2^1, \dots, y_2^{n-1}\|, y_2^n \right\|$$

$$\le c_2 \left\| \left\| \begin{matrix} d_{q_n}^{k_n} \\ d_{q_n}^{k_n-1} \\ \vdots \\ d_{q_n}^{k_n-\sigma} \end{matrix} \right\|, \left\| \begin{matrix} d_{q_n}^{k_n+1}\big|_{[-\sigma]} \\ d_{q_n}^{k_n+2}\big|_{[-\sigma]} \\ \vdots \\ d_{q_n}^{k_{n+1}-1}\big|_{[-\sigma]} \end{matrix} \right\| \right\|$$

$$= c_2 \left\| \begin{matrix} \left(d_{q_n}^{k_n-\sigma}, d_{q_n}^{k_n+1}|_{[-\sigma]}, \dots\right) \\ \left(d_{q_n}^{k_n+1-\sigma}, d_{q_n}^{k_n+2}|_{[-\sigma]}, \dots\right) \\ \vdots \\ \left(d_{q_n}^{k_n}, d_{q_n}^{k_n+\sigma+1}|_{[-\sigma]}, \dots\right) \end{matrix} \right\|$$

and as in the case $n = 0$ it can be concluded (8).

Since, by Assumption 4,

$$\left\|d_{q_n}^{k_{n+1}-i}\right\| \le \left\|d_{q_n}^{k_{n+1}-1}\right\| \quad \forall\, i \in \{1, \dots, \sigma+1\}$$

inequality (8) yields for all $n \in \mathbb{N}$

$$\begin{aligned} \left\|y_2^0, y_2^1, \dots, y_2^n\right\| &\le c_2 \|\underbrace{1, 1, \dots, 1}_{\sigma+1}\| \|\|d_{q_n}^{k_{n+1}-1}\| \\ &\overset{(5)}{\le} c_2\|1, 1, \dots, 1\|\|\|d_{p^*}^{k_{n+1}-1}\| \\ &\overset{\text{Ass. 2}}{\le} \underbrace{c_2\|1, 1, \dots, 1\|c_1}_{=:\gamma_1} \|u_0, y_0\|. \end{aligned}$$

Hence Step 1 is finished.

STEP 2: It will be shown that

$$\exists \gamma_2 > 0 : \ \|y_2(k_0), y_2(k_1), y_2(k_2), \dots\| \le \gamma_2 \|u_0, y_0\|.$$

For $\hat{q} \in \mathcal{P}$ consider the subset $Q^{\hat{q}} \subseteq Q$ of all times where the switching strategy has switched from controller $C_{\hat{q}}$ to another one, i.e., $k \in Q^{\hat{q}} \Leftrightarrow q(k-1) = \hat{q}$. Writing $Q^{\hat{q}} = \{k_1^{\hat{q}}, k_2^{\hat{q}}, \dots\}$ it will be shown that for all $\hat{q} \in \mathcal{P}$ there exists $\gamma_2^{\hat{q}} > 0$ such that

$$\left\|y_2(k_1^{\hat{q}}), y_2(k_2^{\hat{q}}), \dots\right\| \le \gamma_2^{\hat{q}} \|u_0, y_0\|. \tag{9}$$

For

$$\gamma_2 := \left\|\gamma_2^{\hat{q}_1}, \gamma_2^{\hat{q}_2}, \dots, \gamma_2^{\hat{q}_N}\right\|$$

Step 2 would then be shown.

Let $n \in \mathbb{N}$ such that $k_n^{\hat{q}} \in Q^{\hat{q}}$, Assumption 3 then yields for $i \in \{1, 2, \dots, n\}$

$$\left|y_2(k_i^{\hat{q}})\right| \le c_2 \left\|d_{\hat{q}}^{k_i^{\hat{q}}}(k_i^{\hat{q}} - \sigma), \dots, d_{\hat{q}}^{k_i^{\hat{q}}}(k_i^{\hat{q}})\right\|.$$

Using successively Assumption 4 similar as in Step 1 one arrives at

$$\left\|y_2(k_1^{\hat{q}}), y_2(k_2^{\hat{q}}), \dots, y_2(k_n^{\hat{q}})\right\| \le c_2 \|\underbrace{1, 1, \dots, 1}_{\sigma+1}\| \left\|d_{\hat{q}}^{k_n^{\hat{q}}}\right\|.$$

If $k_n^{\hat{q}}$ is not the last element in the ordered set $Q^{\hat{q}}$ then there must exists $k > k_n^{\hat{q}}$ such that the switching strategy is switching again to the controller $C_{\hat{q}}$ at time k (otherwise it could not switch away from $\hat{q}$ later). In particular

$$\left\|d_{\hat{q}}^{k_n^{\hat{q}}}\right\| \le \left\|d_{\hat{q}}^k\right\| \le \left\|d_{p^*}^k\right\| \le c_1 \|u_0, y_0\|$$

and hence, with $\gamma_1 > 0$ as in Step 1,

$$\left\|y_2(k_1^{\hat{q}}), y_2(k_2^{\hat{q}}), \dots, y_2(k_n^{\hat{q}})\right\| \le \gamma_1 \|u_0, y_0\|. \tag{10}$$

Therefore if $Q^{\hat{q}}$ is infinite the existence of $\gamma_2^{\hat{q}} > 0$ such that (9) holds has been shown.

It remains to consider the cases where $Q^{\hat{q}}$ is finite. Define

$$F := \left\{ \ \hat{q} \in \mathcal{P} \ \middle| \ Q^{\hat{q}} \text{ is finite } \right\}$$

and let

$$K_F := \left\{ \ k \in \mathbb{N} \ \middle| \ k = \max Q^{\hat{q}}, \ \hat{q} \in F \ \right\}$$

be the set of all times at which the switching strategy switches away from a controller $C_{\hat{q}}$ the last time. Writing $K_F = \{k_1^F, k_2^F, \dots, k_{|F|}^F\}$ it will for $i \in \{1, \dots, |F|\}$ be shown that there exits $\alpha > 0$ and $\tilde{\gamma}_1 > 0$ such that

$$|y_2(k_i^F)| < \tilde{\gamma}_1 \alpha^i \|u_0, y_0\| \tag{11}$$

The proof is inductive and starts with $i = 1$. By Step 1 and (10) it is already known that

$$|y_2(k)| \le \gamma_1 \|u_0, y_0\| \quad \forall k < k_1^F.$$

With $c_0 > 0$ from (7) it follows, for all $k < k_1^F$,

$$\begin{aligned} |u_2(k)| &\le c_0 \|y_2|_{[k-\sigma+1,k]}\| \\ &\le \overbrace{c_0 \|\underbrace{1, 1, \dots, 1}_{\sigma}\| \gamma_1}^{\tilde{\gamma}_1} \|u_0, y_0\|. \end{aligned}$$

It can be assumed that $\tilde{\gamma}_1 \ge \gamma_1$. Now (1) yields

$$\begin{aligned} |y_2(k_1^F)| &\le \sum_{i=1}^{\sigma} \left|a_{p^*}^i\right| |y_2(k-i)| + |b_{p^*}| \, |u_2(k-1)| \\ &\quad + \left| y_0(k) - \sum_{i=1}^{\sigma} a_{p^*}^i y_0(k-i) - b_{p^*} u_0(k-1) \right| \\ &\le \gamma_1 \sum_{i=1}^{\sigma} |a_{p^*}^i| \|u_0, y_0\| + \tilde{\gamma}_1 |b_{p^*}| \|u_0, y_0\| \\ &\quad + \left(1 + \sum_{i=1}^{\sigma} \left|a_{p^*}^i\right| + |b_{p^*}|\right) \|u_0, y_0\| \\ &\le \tilde{\gamma}_1 \alpha \, \|u_0, y_0\|, \end{aligned}$$

where

$$\alpha := \sum_{i=1}^{\sigma} \left|a_{p^*}^i\right| + |b_{p^*}| + \frac{1}{\tilde{\gamma}_1} \left(1 + \sum_{i=1}^{\sigma} \left|a_{p^*}^i\right| + \left|b_{p^*}^i\right|\right).$$

It can be assumed that $\alpha \ge 1$. For the case $i > 1$ it is then inductively assumed that, for all $k < k_i^F$,

$$|y_2(k)| \le \tilde{\gamma}_1 \alpha^{i-1} \|u_0, y_0\|$$

and

$$|u_2(k)| \leq \tilde{\gamma}_1 \alpha^{i-1} \|u_0, y_0\|.$$

The same calculation as in the case $i = 0$ yields then

$$|y_2(k_i^F)| \leq \tilde{\gamma}_1 \alpha \ \alpha^{i-1} \|u_0, y_0\| = \tilde{\gamma}_1 \alpha^i \|u_0, y_0\|.$$

For $\hat{q} \in \mathcal{P}$ let $i_{\hat{q}}$ such that $\{k^F_{i_{\hat{q}}}\} = K_F \cap Q^{\hat{q}}$ if $Q^{\hat{q}}$ is finite and $i_{\hat{q}} = -\infty$ otherwise. Then (10) and (11) together yields (9) with

$$\gamma_2^{\hat{q}} = \|\gamma_1, \tilde{\gamma}_1 \alpha^{i_{\hat{q}}}\|.$$

qed

Remark 3: The proof of the theorem is constructive, it is possible to obtain an explicit expression for an upper bound for the gain γ. Here we highlight some salient features. Consider for example $V = l_\infty$ and either estimator A or B given in Section III. Then, for $(u_0, y_0) \in V \times V$,

$$\|y_2\|_\infty \leq \beta_\infty \ \alpha_{p^*}^{|\mathcal{P}|-1} \ \|u_0, y_0\|_\infty,$$
$$\|u_2\|_\infty \leq \beta_\infty \|y_2\|_\infty,$$

where $\beta_\infty > 0$ depends only on the distribution of the parameters and not explicitly on the number of plants and $\alpha_{p^*} > 1$ only depends on the parameters of the real plant P_{p^*}.

For the case $V = l_2$ we obtain:

$$\|y_2\|_2 \leq \sqrt{|\mathcal{P}|} \ \beta_2 \ \alpha_{p^*}^{|\mathcal{P}|-1} \ \|u_0, y_0\|_2,$$
$$\|u_2\|_2 \leq \beta_2 \|y_2\|_2,$$

where again $\beta_2 > 0$ only depends on the parameter distribution.

Observe that within the proof of Theorem 2 the worst case bounds arise when the controller switches sequentially through all the candidate plants. This gives rise to the factor $\alpha_{p^*}^{|\mathcal{P}|-1}$ in the above bounds.

Example 4: Consider the two plants P_1 and P_2 described by

$$y_1(k) = a y_1(k-1) \pm u_1(k-1),$$

i.e., in the notation of this paper, $a_1 = a_2 = a > 0$, $b_1 = 1$ and $b_2 = -1$. In [6] this setup with the additional condition $y_0 \equiv 0$ was considered, and it was shown that $\|y_2\|_2 \leq \gamma \|u_0\|_2$ for $\gamma > a + \sqrt{1 + a^2} \approx 2a$. A slightly weaker bound is obtained by following similar steps to the proof of Theorem 2 with $y_0 \equiv 0$, namely that $\gamma > \sqrt{1 + (2a+1)^2} \approx 2a$.

However, in the general case, where $y_0 \not\equiv 0$, the proof of Theorem 2 can be used directly to calculate, for example for estimator B, that for $a > 1$ we have $\|y_2\|_2 \leq \tilde{\gamma} \|u_0, y_0\|_2$, if

$$\tilde{\gamma} = \sqrt{16a^6 + 64a^5 + 64a^4 + 32a^2} = O(a^3).$$

Example 5: Consider the three plants P_{-1}, P_0 and P_1 described by

$$P_p \ : \ y_1(k) = pay_1(k-1) + u_1(k-1), \quad \text{for } a > 0.$$

The proof of Theorem 2 can be used directly to calculate, for example for estimator B, that we have $\|y_2\|_2 \leq \tilde{\gamma}_{p^*} \|u_0, y_0\|_2$ with

$$\begin{aligned} \tilde{\gamma}_{\pm 1} &= \sqrt{16a^8 + 128a^7 + 400a^6 + 576a^5 + 320a^4 + 32a^2} \\ &= O(a^4) \\ \tilde{\gamma}_0 &= \sqrt{117a^4 + 32a^2} = O(a^2). \end{aligned}$$

Observe that the gain depends crucially on the underlying real plant.

V. Conclusions

We have shown that a switched controller achieves a finite l^p gain for a certain class of systems. In addition, the proof of the main result is constructive; an upper bound for the gain can be calculated. The results obtained for some specific examples compare favourable to already known results.

Nevertheless, in the given examples the distribution of parameters is well matched to the inequalities used within the proof. There are certainly other examples of plant parameter distributions in which the inequalities utilized within our proof are conservative and the technique would need to be considerably refined to produce tight bounds.

We consider the treatment of these issues, together with the construction of the tightest achievable bounds, to be a serious topic for further study.

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