Open problems in adaptive control

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1 Introduction

The purpose of these notes is to state a series of simply stated questions in adaptive control.

2 Background

We consider causal mappings $P: U_e \rightarrow Y_e$ and $C: Y_e \rightarrow U_e$, where $P$ and $C$ represent a plant and a controller, respectively, and $U$ and $Y$ are normed vector spaces such as $L^2(\mathbb{R}_+, \mathbb{R}^n)$ and $U_e, Y_e$ are the analogous extended spaces, for example $L^2_e(\mathbb{R}_+, \mathbb{R}^m)$. Our central concern is with the system of equations:

$$\begin{bmatrix} P, C \end{bmatrix} : \begin{cases} y_1 &= Pu_1, \\ y_0 &= y_1 + y_2 \\ u_2 &= Cy_2, \\ u_0 &= u_1 + u_2, \end{cases}$$

(2.1)

where $u_0, u_1, u_2 \in U, y_0, y_1, y_2 \in Y$ and which correspond to the classical feedback configuration of a plant and controller as depicted in Figure 1. We will state our problems in terms of the operator:

$$\Pi_{P//C}: W \rightarrow W : w_0 = \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \mapsto \begin{pmatrix} u_1 \\ y_1 \end{pmatrix} = w_1.$$

We are interested in the following fundamental quantity: given a disturbance level $d \geq 0$, a nominal plant $P$ and a controller $C$,

$$B_{P,C}(d) = \sup\{ r \geq 0 \mid \delta(P, P_1) \leq r \implies \| \Pi_{P//C} w_0 \| < \infty \text{ for all } \| w_0 \| \leq d \}. \quad (2.2)$$

Note that $B_{P,C}$ is undefined if $\Pi_{P//C}$ is not BIBO stable.

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3 Global Limitations of Performance and Robustness

1. Answers to the following questions are of interest in any sensible signal space setting, e.g. \( \mathcal{U}, \mathcal{Y} = L^2(\mathbb{R}_+) \) or \( L^\infty(\mathbb{R}_+) \). Suppose \( P(\theta) : \text{dom}(P) \to \mathcal{Y} \) is defined by:

\[
P(\theta)(u_1) = y_1 \quad \text{where} \quad \dot{y}_1 = \theta y_1 + u_1, \quad y_1(0) = 0.
\]

Then we know there exists a causal controller \( C \) and \( a_\theta \geq 0, \gamma_\theta \in \mathcal{K} \) such that:

\[
\|\Pi_{P(\theta)/C} w_0\| \leq a_\theta + \gamma_\theta(\|w_0\|) \quad \forall \theta \in \mathbb{R}.
\]

(a) What is the minimal achievable rate of growth of \( \gamma_\theta \)? Concretely, find:

\[
p = \inf\{ q \geq 0 : \exists \text{ causal } C \text{ s.t. } \gamma_\theta(r) = O(r^q), \forall \theta \in \mathbb{R}. \}
\]

(b) Is the above infimum attained?

(c) Is there a control design which achieves a robustness margin independent of the disturbance level? That is, find or prove the non-existence of, a controller \( C \) and a constant \( b_{P(\theta),C} > 0 \) such that:

\[
B_{P(\theta),C}(d) = b_{P(\theta),C} > 0 \quad \forall \theta \in \mathbb{R}, \forall d \geq 0.
\]

Observe that if there exists a causal \( C \) s.t. \( \gamma_\theta(r) = O(r) \forall \theta \in \mathbb{R} \) then (c) follows, i.e. \( p = 1 \) in (a) and (b) implies (c), or \( p < 1 \) in (a) implies (c).

2. We ask the same questions in a discrete time setting, e.g. with \( \mathcal{U}, \mathcal{Y} = l^2(\mathbb{Z}_+) \) or \( l^\infty(\mathbb{Z}_+) \), and

\[
P(\theta)(u_1) = y_1 \quad \text{where} \quad y_1(k + 1) = \theta y_1(k) + u_1(k), \quad y_1(0) = 0.
\]

3. As an alternative generalisation of the concept of (e.g. \( L^2 \)) gain stability, we ask the same questions (a),(b) when (3.4) is replaced by (3.8):

\[
\|\Phi_{P(\theta)/C} w_0\| \leq a_\theta + \|\Phi_{\gamma_\theta} \circ (|w_0(\cdot)|)\| \quad \forall \tau > 0, \forall \theta \in \mathbb{R},
\]

and where \( \Phi_{\gamma_\theta}(v) := \left\{ \begin{array}{ll} v(t), & t \in [0, \tau] \\ 0, & t \in (\tau, \infty). \end{array} \right. \)

In the particular case when \( \mathcal{U} = \mathcal{Y} = L^2(\mathbb{R}_+) \), this corresponds to the existence of \( 0 \geq M > -\infty \) such that the following inequality

\[
\int_0^\tau (\gamma_\theta(|w_0(t)|))^2 - (\Pi_{P(\theta)/C} w_0(t))^2 \, dt \geq M > -\infty
\]

holds for all \( \tau > 0 \) (in the case \( \mathcal{U} = \mathcal{Y} = L^\infty(\mathbb{R}_+) \), (3.4) and (3.8) are equivalent).

4. A weaker version of questions (a),(b) is to allow \( C \) to be dependent of \( \theta_{\text{max}} \), and to replace (3.4) with the requirement that the parameterised controller set \{\( C(\theta_{\text{max}}) \)\}_{\theta_{\text{max}} \geq 0} has the property that there exists \( a_\theta > 0, \gamma_\theta \in \mathcal{K} \) such that for all \( \theta_{\text{max}} \geq 0 \),

\[
\|\Pi_{P(\theta)/C(\theta_{\text{max}})} w_0\| \leq a_\theta + \gamma_\theta(\|w_0\|) \quad \forall \theta \leq \theta_{\text{max}}.
\]

A similar replacement can also be made for (3.8). Note that it is critical that \( a_\theta, \gamma_\theta \) are independent of \( \theta_{\text{max}} \). The corresponding weak version of question (c) becomes to find or to prove the non-existence of, a controller \( C \) and a constant \( b_{P(\theta)} > 0 \) such that:

\[
B_{P(\theta),C}(d) = b_{P(\theta),C(\theta_{\text{max}})} > b_{P(\theta)} > 0 \quad \forall \theta \in \mathbb{R}, \forall d \geq 0.
\]

\(^1\mathcal{K} \) denote the class of functions \( \gamma : \mathbb{R}_+ \to \mathbb{R}_+ \) with the properties: \( \gamma(0) = 0, \gamma \) is monotonically increasing.
We now make some remarks on the classes of uncertain nominal plants considered. The two plant classes considered are deliberately simple; nevertheless the questions asked are not trivial. From a control perspective, the continuous-time class (3.3) is high gain stabilizable, hence any rationale for adaptive control has to that of (suitably formulated) superior performance. The discrete plant (3.7) represents a simple class which captures many of the difficulties of traditional adaptive control, including non-minimum phase behaviour; here the existence of stabilizing adaptive controllers already beats linear controllers.

A more general problem is to ascertain which of the above properties are achievable in the more general setting of appropriate classes of (LTI) plants $P(\theta)$ suitably parameterised by $\theta \in \mathbb{R}^n$.

4 Adaptive Control as H-Infinity Optimization

Let $\mathcal{P} = \{P_\theta\}_{\theta \in \Theta}$ be a family of finite order LTI plants (causal, discrete time), parameterized by parameter $\theta$ ranging over set $\Theta$. Each $P_\theta$ is assumed to have two vector inputs (“disturbance” $w = w_\theta$ and “control” $u = u_\theta$) and two vector outputs (“cost” $z = z_\theta$ and “sensor” $y = y_\theta$). The dimension of $y_\theta$ is assumed to be independent of $\theta$, and the same assumption is made about the dimension of $u_\theta$. In addition, assume that two functions $\gamma^+: \Theta \mapsto (0, \infty)$ and $\gamma^-: \Theta \mapsto (0, \infty)$ are given.

A general question of interest can be formulated as follows: for which families $\{P_\theta\}$ and functions $\gamma^\pm$ does there exist an efficient algorithm for either finding a single (in general, nonlinear) strictly causal feedback law $u(\cdot) = K(y(\cdot))$ which makes the closed loop gain from $w_\theta$ to $z_\theta$ less than $\gamma^+(\theta)$ for all $\theta \in \Theta$, or certifying that no single strictly causal feedback law $u(\cdot) = K(y(\cdot))$ is capable of making the closed loop gain from $w_\theta$ to $z_\theta$ less than $\gamma^-(\theta)$ for all $\theta \in \Theta$.

In the case when $\Theta$ contains a single element, we get the standard suboptimal H-Infinity optimization problem, which has an elegant and efficient solution. In the case when $\Theta$ is finite, one can inquire about existence of a polynomial time algorithm which solves this problem when $\gamma^+(\theta) = \rho \gamma^-(\theta)$ for all $\theta$, where $\rho > 1$ is a given constant.