Open problems in adaptive control

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1 Introduction

The purpose of these notes is to state a series of simply stated questions in adaptive control.

2 Background

We consider causal mappings $P\colon \mathcal{U}_e \to \mathcal{Y}_e$ and $C\colon \mathcal{Y}_e \to \mathcal{U}_e$, where P and C represent a plant and a controller, respectively, and $\mathcal{U}$ and $\mathcal{Y}$ are normed vector spaces such as $L^2(\mathbb{R}_+, \mathbb{R}^m)$ and $\mathcal{U}_e$, $\mathcal{Y}_e$ are the analogous extended spaces, for example $L^{2,e}(\mathbb{R}_+, \mathbb{R}^m)$. Our central concern is with the system of equations:

$$[P, C] \quad : \quad \left.\begin{array}{ll} y_1 = Pu_1, & y_0 = y_1 + y_2 \\ u_2 = Cy_2, & u_0 = u_1 + u_2, \end{array}\right\} \tag{2.1}$$

where $u_0, u_1, u_2 \in \mathcal{U}$, $y_0, y_1, y_2 \in \mathcal{Y}$ and which correspond to the classical feedback configuration of a plant and controller as depicted in Figure 1. We will state our problems in terms of the

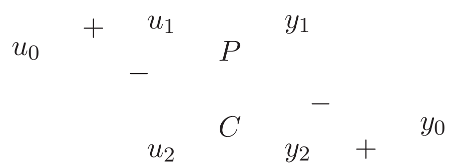

Figure 1: The closed-loop.

operator:

$$\Pi_{P//C}\colon \mathcal{W} \to \mathcal{W} \quad : \quad w_0 = \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \mapsto \begin{pmatrix} u_1 \\ y_1 \end{pmatrix} = w_1.$$

We are interested in the following fundamental quantity: given a disturbance level $d \geq 0$, a nominal plant P and a controller C,

$$B_{P,C}(d) = \sup\{r \geq 0 \mid \delta(P, P_1) \leq r \implies \|\Pi_{P_1//C} w_0\| < \infty \text{ for all } \|w_0\| \leq d\}. \tag{2.2}$$

Note that $B_{P,C}$ is undefined if $\Pi_{P//C}$ is not BIBO stable.

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3 Global Limitations of Performance and Robustness

1. Answers to the following questions are of interest in any sensible signal space setting, e.g. $\mathcal{U}, \mathcal{Y} = L^2(\mathbb{R}_+)$ or $L^\infty(\mathbb{R}_+)$. Suppose $P(\theta)\colon \mathrm{dom}(P) \to \mathcal{Y}$ is defined by:

$$P(\theta)(u_1) = y_1 \text{ where } \dot{y}_1 = \theta y_1 + u_1,\ y_1(0) = 0. \tag{3.3}$$

Then we know there exists a causal controller C and $a_\theta \geq 0$, $\gamma_\theta \in \mathcal{K}$ [1] such that:

$$\|\Pi_{P(\theta)//C} w_0\| \leq a_\theta + \gamma_\theta(\|w_0\|) \quad \forall \theta \in \mathbb{R}. \tag{3.4}$$

   (a) What is the minimal achievable rate of growth of γ_θ? Concretely, find:

$$p = \inf\{q \geq 0 \ :\ \exists \text{ causal } C \text{ s.t. } \gamma_\theta(r) = O(r^q),\ \forall \theta \in \mathbb{R}\}. \tag{3.5}$$

   (b) Is the above infimum attained?

   (c) Is there a control design which achieves a robustness margin independent of the disturbance level? That is, find or prove the non-existence of, a controller C and a constant $b_{P(\theta),C} > 0$ such that:

$$B_{P(\theta),C}(d) = b_{P(\theta),C} > 0 \quad \forall \theta \in \mathbb{R},\ \forall d \geq 0. \tag{3.6}$$

Observe that if there exists a causal C s.t. $\gamma_\theta(r) = O(r)$ $\forall \theta \in \mathbb{R}$ then (c) follows, i.e. $p = 1$ in (a) and (b) implies (c), or $p < 1$ in (a) implies (c).

2. We ask the same questions in a discrete time setting, e.g. with $\mathcal{U}, \mathcal{Y} = l^2(\mathbb{Z}_+)$ or $l^\infty(\mathbb{Z}_+)$, and

$$P(\theta)(u_1) = y_1 \text{ where } y_1(k+1) = \theta y_1(k) + u_1(k),\ y_1(0) = 0. \tag{3.7}$$

3. As an alternative generalisation of the concept of (e.g. L^2) gain stability, we ask the same questions (a),(b) when (3.4) is replaced by (3.8):

$$\|T_\tau \Pi_{P(\theta)//C} w_0\| \leq a_\theta + \|T_\tau \gamma_\theta \circ (|w_0(\cdot)|)\| \quad \forall \tau > 0,\ \forall \theta \in \mathbb{R}, \tag{3.8}$$

and where T_τ denotes the truncation operator:

$$T_\tau(v) := \begin{cases} v(t), & t \in [0, \tau] \\ 0, & t \in (\tau, \infty)\,. \end{cases}$$

In the particular case when $\mathcal{U} = \mathcal{Y} = L^2(\mathbb{R}_+)$, this corresponds to the existence of $0 \geq M > -\infty$ such that the following inequality

$$\int_0^\tau (\gamma_\theta(|w_0(t)|))^2 - \left(\Pi_{P(\theta)//C} w_0(t)\right)^2 \, dt \geq M > -\infty \tag{3.9}$$

holds for all $\tau > 0$ (in the case $\mathcal{U} = \mathcal{Y} = L^\infty(\mathbb{R}_+)$, (3.4) and (3.8) are equivalent).

4. A weaker version of questions (a),(b) is to allow C to be dependent of $\theta_{\max}$, and to replace (3.4) with the requirement that the parameterised controller set $\{C(\theta_{\max})\}_{\theta_{\max} \geq 0}$ has the property that there exists $a_\theta > 0$, $\gamma_\theta \in \mathcal{K}$ such that for all $\theta_{\max} \geq 0$,

$$\|\Pi_{P(\theta)//C(\theta_{\max})} w_0\| \leq a_\theta + \gamma_\theta(\|w_0\|) \quad \forall |\theta| \leq \theta_{\max}. \tag{3.10}$$

A similar replacement can also be made for (3.8). Note that it is critical that a_θ, γ_θ are independent of $\theta_{\max}$. The corresponding weak version of question (c) becomes to find or to prove the non-existence of, a controller C and a constant $b_{P(\theta)} > 0$ such that:

$$B_{P(\theta),C}(d) = b_{P(\theta),C(\theta_{\max})} > b_{P(\theta)} > 0 \quad \forall \theta \in \mathbb{R},\ \forall d \geq 0. \tag{3.11}$$

[1] $\mathcal{K}$ denote the class of functions $\gamma\colon \mathbb{R}_+ \to \mathbb{R}_+$ with the properties: $\gamma(0) = 0$, γ is monotonically increasing.

We now make some remarks on the classes of uncertain nominal plants considered. The two plant classes considered are deliberately simple; nevertheless the questions asked are not trivial. From a control perspective, the continuous-time class (3.3) is high gain stabilizable, hence any rationale for adaptive control has to that of (suitably formulated) superior performance. The discrete plant (3.7) represents a simple class which captures many of the difficulties of traditional adaptive control, including non-minimum phase behaviour; here the existence of stabilizing adaptive controllers already beats linear controllers.

A more general problem is to ascertain which of the above properties are achievable in the more general setting of appropriate classes of (LTI) plants $P(\theta)$ suitably parameterised by $\theta \in \mathbb{R}^n$.

4 Adaptive Control as H-Infinity Optimization

Let $\mathcal{P} = \{P_\theta\}_{\theta\in\Theta}$ be a family of finite order LTI plants (causal, discrete time), parameterized by parameter θ ranging over set Θ. Each P_θ is assumed to have two vector inputs ("disturbance" $w = w_\theta$ and "control" $u = u_\theta$) and two vector outputs ("cost" $z = z_\theta$ and "sensor" $y = y_\theta$). The dimension of y_θ is assumed to be independent of θ, and the same assumption is made about the dimension of u_θ. In addition, assume that two functions $\gamma^+ : \Theta \mapsto (0, \infty)$ and $\gamma^- : \Theta \mapsto (0, \infty)$ are given.

A general question of interest can be formulated as follows: for which families $\{P_\theta$ and functions $\gamma^\pm$ does there exist an efficient algorithm for either finding a single (in general, nonlinear) strictly causal feedback law $u(\cdot) = K(y(\cdot))$ which makes the closed loop gain from w_θ to z_θ less than $\gamma^+(\theta)$ for all $\theta \in \Theta$, or certifying that no single strictly causal feedback law $u(\cdot) = K(y(\cdot))$ is capable of making the closed loop gain from w_θ to z_θ less than $\gamma^-(\theta)$ for all $\theta \in \Theta$.

In the case when Θ contains a single element, we get the standard suboptimal H-Infinity optimization problem, which has an elegant and efficient solution. In the case when Θ is finite, one can inquire about existence of a polynomial time algorithm which solves this problem when $\gamma^+(\theta) = \rho\gamma^-(\theta)$ for all θ, where $\rho > 1$ is a given constant.