Smooth adaptive controllers have discontinuous closed loop operators

Mark French
School of Electronics and Computer Science,
University of Southampton, UK
mcf@ecs.soton.ac.uk,

MTNS 2004

1 Introduction

It is well known that the stability of the parallel projection operator mapping external disturbances to internal closed loop signals plays a critical role in both the linear and nonlinear theory of robust stabilization. Indeed the key results of [3] state that if the gain of this operator is finite, then the stability is maintained if the plant is perturbed by a distance (measured by the gap metric) less than the reciprocal of the gain.

However, this only provides a sufficient condition for stability. Recent work [1] in adaptive control has constructed controllers with robust stability properties which nevertheless violate the above sufficiency condition. The purpose of this paper is to show that this is inevitiable, namely that the specification of the adaptive control problem itself forces any 'smooth' controller to violate the finite gain condition. Furthermore, the construction also shows that it is not possible to achieve gain function stability with a class K gain function.

The problem is first motivated by considering the discontinuous local behaviour of a classical adaptive controller. We then develop a generalisation of the concept of a Fréchet derivative to enable us to relate the linearisation of an *unstable* system to an operator derivative. This in turn is utilized to show that the local behaviour of a wide class of 'smooth' adaptive controllers forces the closed loop operator to be discontinuous.

There are a number of implications of the discontinuity of the closed loop operator; most importantly it shows that the classical quantity $b_{P,C}$ is not useful in this context. In the final part of the paper we briefly argue that that a biased notion of stability overcomes these problems, and allows a systematic development of a robust stability theory for adaptive control.

2 Operator Stability and the Gap metric

We consider causal mappings $P\colon \mathcal{U}_e \to \mathcal{Y}_e$ and $C\colon \mathcal{Y}_e \to \mathcal{U}_e$, where P and C represent a plant and a controller, respectively, and $\mathcal{U}$ and $\mathcal{Y}$ are normed vector spaces such as $L^2(\mathbb{R}_+, \mathbb{R}^m)$ and $\mathcal{U}_e$, $\mathcal{Y}_e$ are the analogous extended spaces, for example $L^{2,e}(\mathbb{R}_+, \mathbb{R}^m)$. Our central concern is with the system of equations:

$$[P, C] \quad : \quad \left.\begin{array}{ll} y_1 = Pu_1, & y_0 = y_1 + y_2 \\ u_2 = Cy_2, & u_0 = u_1 + u_2, \end{array}\right\} \tag{2.1}$$

where $u_0, u_1, u_2 \in \mathcal{U}$, $y_0, y_1, y_2 \in \mathcal{Y}$ and which correspond to the classical feedback configuration of a plant and controller as depicted in Figure 1.

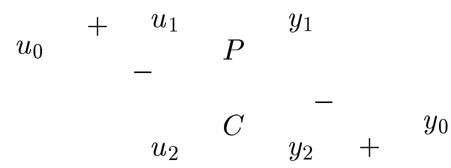

Figure 1: The closed-loop system $[P, C]$.

Let $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$. The system $[P, C]$ is said to be *well posed* if, and only if,

$$H_{P,C} \colon \mathcal{W} \to \mathcal{W}_e \times \mathcal{W}_e \,, \quad \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \mapsto \left(\begin{pmatrix} u_1 \\ y_1 \end{pmatrix}, \begin{pmatrix} u_2 \\ y_2 \end{pmatrix} \right) \quad \text{such that (2.1) holds} \tag{2.2}$$

is a causal operator.
A causal operator $F \colon \mathcal{X}_1 \to \mathcal{X}_2$ between normed spaces $\mathcal{X}_1, \mathcal{X}_2$, is said to be *gain-function stable* (or *gf-stable*) if, and only if, there exists a nonlinear function (a so called *gain-function*)

$$\gamma[F] \colon \mathbb{R} \to \mathbb{R}_+, \quad r \mapsto \gamma[F](r) = \sup_{\|x\| \le r} \|Fx\| \,. \tag{2.3}$$

A closed-loop $[P, C]$ is said to be *gf-stable* if, and only if, $H_{P,C}$ is gf-stable. Corresponding to the plant operator P is a subset of $\mathcal{W}$, called the *graph* of the plant $\mathcal{G}_P$, which is defined as

$$\mathcal{M} = \mathcal{G}_P = \left\{ \begin{pmatrix} u \\ Pu \end{pmatrix} \ : \ u \in \mathcal{U}, \ Pu \in \mathcal{Y} \right\} \subset \mathcal{W}. \tag{2.4}$$

Finally we define the following closed loop parallel projection operator:

$$\Pi_{P//C} \colon \mathcal{W} \to \mathcal{W} \quad : \quad \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \mapsto \begin{pmatrix} u_1 \\ y_1 \end{pmatrix} ,$$

3 Robust Stability Margins and Sufficient Conditions

Given normed i/o spaces $\mathcal{U}, \mathcal{Y}$, we are interested in the following fundamental quantity: given a disturbance level $d \ge 0$, a nominal plant P and a controller C,

$$B_{P,C}(d) = \sup\{ r \ge 0 \mid \delta(P, P_1) \le r \implies \|H_{P_1,C}(u_0, y_0)\| < \infty \text{ for all } \|(u_0, y_0)\| \le d \}, \tag{3.5}$$

where for the purposes of this paper we restrict our attention to minimal linear finite dimensional plants, but make no such restriction on the controllers. In this setting we interpret δ as the standard gap metric.
Note that $B_{P,C}$ is undefined if $H_{P,C}$ is not BIBO stable. For LTI systems, $(P, C \in \mathcal{RL}_\infty)$, it is well known that $B_{P,C}$ can be computed exactly in an L^2 setting, and has a constant value independent of the disturbance level $d \ge 0$:

$$B_{P,C}(d) = b_{P,C} = \|\Pi_{P//C}\|^{-1} \tag{3.6}$$

For nonlinear systems, the robust stability margin is in general dependent on the disturbance level, and the parallel projection gain only provides a lower bound:

$$B_{P,C}(d) \ge \left(\sup_{\|u_0, y_0\| \le r(d)} \frac{\|\Pi_{P//C}(u_0, y_0)\|}{\|(u_0, y_0)\|} \right)^{-1} , \tag{3.7}$$

for some appropriate choice of r. Furthermore, the reverse inequality does not necessarily hold.

4 A prototypical example

The potential lack of tightness of the lower bound (3.7) is not pathological; many adaptive controllers have the property

$$B_{P,C}(d) > 0 \quad \text{for all } d \geq 0 \tag{4.8}$$

whilst

$$\sup_{\|u_0,y_0\| \leq r} \left(\frac{\|\Pi_{P//C}(u_0,y_0)\|}{\|(u_0,y_0)\|} \right) = \infty \quad \text{for all } r > 0. \tag{4.9}$$

This arises due to a problem with small signal behaviour, where whilst $\Pi_{P//C}(0) = 0$, the operator $\Pi_{P//C}$ is not continuous at 0 – which precludes the existence of a 'local finite gain'.

An explicit example of this (in an L^2 setting) is given by the plant

$$P(\theta)(u_1) = y_1 \text{ where } \dot{y}_1 = \theta y_1 + u_1 \; y_1(0) = 0, \tag{4.10}$$

with $\theta > 0$ and the controller:

$$\begin{aligned} C(y_2)(t) &= u_2(t) \\ u_2(t) &= -k^{\frac{1}{4}} y_2(t) \\ \dot{k}(t) &= y_2^2. \end{aligned} \tag{4.11}$$

It has been shown that this closed loop is BIBO stable, see [1]. Clearly $u_0 = y_0 = 0$ implies $u_1, y_1 = 0$, ie. $\Pi_{P//C}(0) = 0$, but for any disturbance (arbitrarily small) which moves $y_1 \neq 0$, the system is unstable unless there exists a time at which $k(t) \geq \theta^4$, ie. $\|y_2\|_{L^2[0,t]} \geq \theta^2$. Hence for all $\epsilon > 0$, $\exists u_0, y_0$, $\|u_0, y_0\| \leq \epsilon$

$$\|\Pi_{P//C}\| \geq \frac{\|\Pi_{P//C}(u_0,y_0)^T\|}{\|(u_0,y_0)^T\|} = \frac{\|(u_2,y_2)^T\|}{\|(u_0,y_0)^T\|} \geq \frac{\theta^2}{\epsilon} \to \infty \quad \text{as } \epsilon \to 0. \tag{4.12}$$

Hence $\|\Pi_{P//C}\| = \infty$, and this is caused by a lack of continuity at 0. The remainder of this paper establishes a result which shows that this behaviour is inherent in all smooth adaptive controllers.

However, let us first note that this discontinuity is addressed in [1] by appending θ onto the input space, for then an inequality of the form:

$$\|u_1, y_1, \theta\|_{\mathcal{U}\times\mathcal{Y}\times\mathbb{R}} \leq g(\|u_0, y_0\|_{\mathcal{U}\times\mathcal{Y}}, |\theta|), \tag{4.13}$$

was constructed, and from this it was shown in [1] that $B_{P(\theta),C}(r) > 0$, ie. we have a non zero but disturbance dependent robustness margin.

5 Differentiation and Linearisation

Let $N(\mathcal{X}_1, \mathcal{X}_2)$ denote the set of nonlinear operators with domain $\mathcal{X}_1$ and co-range $\mathcal{X}_2$ and let $L(\mathcal{X}_1, \mathcal{X}_2)$ denote the set of *bounded* linear operators with domain $\mathcal{X}_1$ and co-range $\mathcal{X}_2$. An operator $N\colon \mathrm{dom}(N) \to \mathcal{X}_2$ where $\mathrm{dom}(N) \subset \mathcal{X}_2$ and $\mathcal{X}_1$, $\mathcal{X}_2$ are normed spaces, is said to be Fréchet differentiable at an interior point $x \in \mathrm{dom}(N)$ if there exists a bounded linear operator $A\colon \mathcal{X}_1 \to \mathcal{X}_2$ such that

$$\lim_{y \to x} \frac{\|N(y) - N(x) - A(y-x)\|}{\|y - x\|} = 0. \tag{5.14}$$

The Fréchet derivative of N at x_0 is denoted by $D_{x_0}N$, and we write $DN = D_0 N$.

Note that the Fréchet derivative is required to be a bounded linear operator, which essentially restricts the definition to operators N which are locally bounded. No sensible meaning can be given to the derivative if N is unbounded, (eg. dropping the criteria that A is bounded does not yield a sensible limiting process). However, we will need to generalize the derivative to this unbounded setting. In order to do this we now recall the definition and basic facts concerning nonlinear co-prime factorisations.

Definition 5.1 *An operator $P\colon \mathcal{U}_e \to \mathcal{Y}_e$ has a right coprime factoristion if and only if there exist causal operators $N\colon \mathcal{U} \to \mathcal{Y}$ and $D\colon \mathcal{Y} \to \mathcal{U}$ s.t. 1. D is causally invertible with $\mathrm{dom}(D^{-1}) = P^{-1}(\mathcal{Y})$, 2. $P = ND^{-1}$, and 3. there exists a stable operator $L\colon \mathcal{U} \times \mathcal{Y} \to \mathcal{U}$ s.t. $L\begin{pmatrix} D \\ N \end{pmatrix} = I$. We let $\mathbf{RCF}(\mathcal{U}_a, \mathcal{Y}_a)$ denote the set of all operators $P\colon \mathcal{U}_e \to \mathcal{Y}_e$ s.t. P admits has a right coprime factorisation $P = ND^{-1}$, and let $\mathrm{rcf}(P)$ denote the set of all coprime factors (N, D) of P.*

We now extend the definition of the Fréchet derivative as follows:

Definition 5.2 *Suppose $P \in \mathbf{RCF}(\mathcal{U}_a, \mathcal{Y}_a)$, and let $(N, D) \in \mathrm{rcf}(P)$. Then we define the extended Fréchet differential as:*

$$\tilde{D}_{x_0} N = D_{x_0} N (D_{x_0} D)^{-1}. \tag{5.15}$$

It can be easily verified that $D_{x_0} N$ is well defined, ie. that if $(N, D), (N_1, D_1) \in \mathrm{rcf}(P)$, then $D_{x_0} N (D_{x_0} D)^{-1} = D_{x_0} N_1 (D_{x_0} D_1)^{-1}$. Furthermore if $D_{x_0} N$ is defined then $D_{x_0} N = \tilde{D}_{x_0} N$, therefore henceforth we will call the 'extended Fréchet derivative' simply the 'Fréchet derivative', and write D_{x_0} for $\tilde{D}_{x_0}$. As with the standard Fréchet derivative, the extended Fréchet derivative of a smoothly stabilizable system can be interpreted as a linearisation, hence the above class of controllers includes all finite dimensional C^1 systems, $\dot{x} = f(x, y)$, $u = h(x)$, which are smoothly stabilizable.

6 The Main Results

The controller is considered to belong to a wide class of (nonlinear) operators, denoted by $\mathcal{N}(\mathcal{X}_1, \mathcal{X}_2)$, where $C \in \mathcal{N}(\mathcal{X}_1, \mathcal{X}_2)$ if and only if 1. $C = ND^{-1}$ is a co-prime factorisation, 2. N, D are Fréchet differentiable at 0, and 3. $D_0 D$, $D_0 N$ have causal, minimal finite dimensional realisations.

We can now state our two main results.

Theorem 6.1 *Suppose $P(\theta)\colon \mathcal{U}_a \to \mathcal{Y}_a$ is defined by:*

$$P(\theta)(u_1) = y_1 \text{ where } \dot{y}_1 = \theta y_1 + u_1 \; y_1(0) = 0, \tag{6.16}$$

and suppose $C \in \mathcal{N}(\mathcal{U}_a, \mathcal{Y}_a)$. Then there exists $\theta \in \mathbb{R}$ such that $H_{P(\theta),C}$ is not continuous at 0.

Proof. (Sketch). For a contradiction, we suppose $H_{P(\theta),C}$ is continuous at 0 for $\theta \in \mathbb{R}$. Since $H_{P(\theta),C}(0) = 0$, it follows that $[P(\theta), C]$ is locally γ-stable for all $\theta \in \mathbb{R}$. From this it can be shown that $D\Pi_{P(\theta)//C}$ is stable for all $\theta \in \mathbb{R}$. Furthermore, it can be shown that $D\Pi_{P(\theta)//C} = \Pi_{DP(\theta)//DC} = \Pi_{P(\theta)//DC}$, and hence that $H_{P(\theta)//DC}$ is stable for $\theta \in \mathbb{R}$. But since $C \in \mathcal{N}(\mathcal{U}_a, \mathcal{Y}_a)$, DC has a causal, minimal finite dimensional realisation, ie. DC is a universal controller for $P(\cdot)$. But it is easily shown that no such universal controller (in $\mathcal{RL}_\infty$) exists. $\square$

Theorem 6.2 *Suppose $P(\theta)\colon \mathcal{U}_a \to \mathcal{Y}_a$ is defined by:*

$$P(\theta)(u_1) = y_1 \text{ where } \dot{y}_1 = \theta y_1 + u_1 \; y_1(0) = 0, \tag{6.17}$$

and suppose $C(\theta_m) \in \mathcal{N}(\mathcal{U}_a, \mathcal{Y}_a)$ is such that $[P(\theta), C(\theta_m)]$ is locally γ_{θ,θ_m}-stable for all $\theta \in [-\theta_m, \theta_m]$. Then

$$\gamma_{0,\theta_m} \to \infty \quad \text{as } \theta_m \to \infty. \tag{6.18}$$

Proof. (Sketch). The proof is similar to the proof of Theorem 6.1, where the final contradiction is obtained by observing that within any gap neighbourhood of $\frac{1}{s}$ there exists plants of the form $P_1(s) = \frac{M-s}{M+s} \cdot \frac{1}{s+\epsilon}$, and for any such plant it is possible to find a θ s.t. no $C(s)$ can simultaneously stabilize $P_1(s)$ and $P(s) = \frac{1}{s-\theta}$. □

7 Conclusions

The results in this paper demonstrate that, for a wide class of 'smooth' universal controllers, the closed loop operator $H_{P,C}$ is not continuous, and hence it is not possible to achieve (even local) L^2 gain stability. Furthermore, if the universality requirement is dropped, then necessarily the L^2 gain performance degrades at fixed values of the parameter as the parametric uncertainty set increases in size. The construction is based on analysing small signal behaviour via appropriate linearisations. As such the proof generalises in a straightforward manner to a variety of signal space settings, both in the continuous and discrete time domains, eg. L^1, L^∞, l^1, l^2, l^∞.

It is important to observe that gain-stability is not possible even in a local sense, hence demonstrating that a relaxation of the underlying stability requirement to allow different signal gains at different signal levels (ie. the existence of a class K gain function) does not suffice. We have shown that an obstruction to good behaviour occurs at the small signal level, hence the stability condition must be relaxed even at a small signal level to obtain an appropriate robust stability and performance theory.

A construction of an appropriate robust stability theory for adaptive control based on a biased notion of stability[1] which is appropriate to the analysis of the above discontinuous operators is the subject of current work.

References

[1] M. French. Adaptive control and robustness in the gap metric. 2002. Submitted for publication.

[2] T.T. Georgiou and M.C. Smith. Biased norms and robustness analysis for nonlinear feedback systems. In *Proc. of the 36th IEEE Conf. on Decision and Control*, pages 642–643. IEEE, 1997.

[3] T.T. Georgiou and M.C. Smith. Robustness analysis of nonlinear feedback systems. *IEEE Trans. on Automatic Control*, 42(9):1200–1229, 1997.

[4] M. Vidyasagar. *Control System Synthesis.* MIT Press, Cambridge, 1985.

[5] G. Zames and A. K. El-Sakkary. Unstable systems and feedback: The gap metric. In *Proc. of the Allerton Conf.*,, pages 380–385, 1980.

[1] A causal operator $F\colon \mathcal{X}_1 \to \mathcal{X}_2$ between normed spaces $\mathcal{X}_1, \mathcal{X}_2$, is said to be γ-stable with bias β if $\|Fx\|_{\mathcal{X}_2} \leq \gamma(\|x\|_{\mathcal{X}_1}) + \beta$, where γ can be taken to be either a class K gain function, or a scalar. $[P, C]$ is said to be stable with bias if there exists $\gamma, \beta > 0$ such that $\Pi_{\mathcal{M}//\mathcal{N}}$ is γ-stable with bias β.