Robust Stability of Interconnections

Wenming Bian[*] **Mark French**[†] **Harish Pillai**[‡]

MTNS 2006
Extended Abstract

1 Introduction

We begin by observing that the graph topology with its various metrizations plays a fundamental role in the theory of robust stability for classical LTI systems([1, 2, 6]. The contribution of this note is to develop the basic theory of robust stability involving the gap-distance directly from a behavioural perspective, observing that recent approaches to generalisations of the gap metric [2] have been purely trajectory based and hence are easily amenable to such an approach. There has been previous interest in developing behavioural notions of the gap metric, see e.g. [3] for an example.

1.1 The classical result

Our concern is with the closed loop systems of equations as shown in Figure 1:

$$[P, C] \quad : \quad \begin{array}{ll} y_1 = Pu_1, & y_0 = y_1 + y_2 \\ u_2 = Cy_2, & u_0 = u_1 + u_2, \end{array}$$

where $u_0, u_1, u_2 \in H^2$, $y_0, y_1, y_2 \in H^2$ and P, C are transfer functions. For such a BIBO

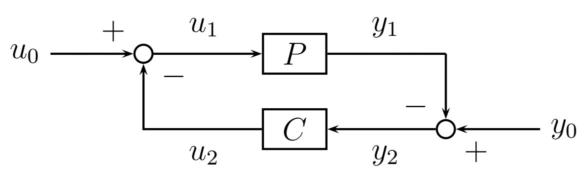

Figure 1: The closed-loop system $[P, C]$.

system, the closed loop transfer function $\Pi_{P//C}$ is of interest:

$$w_0 = \begin{pmatrix} u_0 \\ y_0 \end{pmatrix} \overset{\Pi_{P//C}}{\mapsto} \begin{pmatrix} u_1 \\ y_1 \end{pmatrix} = w_1.$$

[*]School of Electronics and Computer Science, University of Southampton, SO17 1BJ, UK, wb@ecs.soton.ac.uk
[†]School of Electronics and Computer Science, University of Southampton, SO17 1BJ, UK, mcf@ecs.soton.ac.uk,
[‡]Department of Electrical Engineering, Indian Institute of Technology, Bombay, India hp@ee.iitb.ac.in.

The classical robust stability theorem of linear control is as follows:

Theorem *If* $[P, C]$ *is BIBO stable, i.e.* $\|\Pi_{P//C}\|_{H^\infty} < \infty$, $[P_1, C]$ *is well posed, and*

$$\vec{\delta}(P, P_1)\|\Pi_{P//C}\| < 1,$$

then $[P_1, C]$ *is BIBO stable, i.e.* $\|\Pi_{P_1//C}\|_{H^\infty} < \infty$.

Here $\vec{\delta}(P_1, P_2)$ denotes the directed H^2 gap distance between P_1 and P_2. The gap measures the size of the smallest stable co-prime factor perturbation between normalised co-prime factor representations of P_1 and P_2.

1.2 A behavioural generalisation

Within a behavioural framework the above result has been generalised as follows. Consider the interconnection shown in Figure 2,
where,

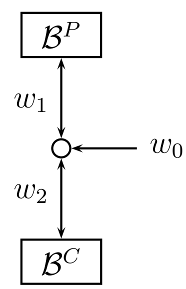

Figure 2: The interconnected behaviours: $w_i = (u_i, y_i)^T$, $i = 0, 1, 2$.

$$w_0 = \begin{pmatrix} u_0 \\ y_0 \end{pmatrix}; \quad w_1 = \begin{pmatrix} u_1 \\ y_1 \end{pmatrix}; \quad w_2 = \begin{pmatrix} u_2 \\ y_2 \end{pmatrix}.$$

$$\mathcal{B}^P = \{w_1 \in L^\infty_{\text{loc}} \mid w_1 = (u_1, y_1)^T\}; \qquad \mathcal{B}^C = \{w_2 \in L^\infty_{\text{loc}} \mid w_2 = (u_2, y_2)^T\}$$

$$\mathcal{B}^I \;=\; \{(w_0, w_1, w_2)^T \in L^\infty_{\text{loc}} \mid w_0 = w_1 + w_2\}$$

$$\mathcal{B}^{P\wedge C} = \{(w_0, w_1, w_2)^T \in \mathcal{B}^I \mid w_1 \in \mathcal{B}^P,\ w_2 \in \mathcal{B}^C\}.$$

Then we have the following result (the full version of the paper will detail all the concepts required to make this statement):

Theorem 1.1 *Suppose* $\mathcal{B}^P$, $\mathcal{B}^{P_1}$, B^C *are linear, shift invariant behaviours with finite memory. If:*

1. $\mathcal{B}^P, \mathcal{B}^C$ *are soundly stabilizable,*

2. $\mathcal{B}^{P\wedge C}$, $\mathcal{B}^{P_1\wedge C}$ *are well-posed,*

3. $\mathcal{B}^{P\wedge C}$ *is uniformly stable, and,*

4. $\vec{\delta}(\mathcal{B}^P, \mathcal{B}^{P_1})\|\Pi_{P//C}\| < 1$,

then $\mathcal{B}^{P_1\wedge C}$ *is uniformly stable.*

2 Relation to the behavioural $\mathcal{H}^\infty$ results of Trentelman and Willems

Within the context of L^2 signal spaces, classical $\mathcal{H}^\infty$ synthesis [7] provides constructions for controllers C which achieve $\|\Pi_{P//C}\| \leq 1$, i.e. solve the normalized version of the inequality required in our robustness theorems. The classical gap robustness results then provide an explicit description of plant uncertainties tolerated in the closed loop. In direct counterpart, and in the interests of a self-contained behavioural theory, it is relevant to relate the results of this paper to the behavioural approach to $\mathcal{H}^\infty$ synthesis found in [4, 5], for then our basic robust stability theorem completes a 'behavioural robust control theory' by providing an explicit robustness interpretation of the behavioural $\mathcal{H}^\infty$ synthesis results.

Therefore we explicitly describe the relationship between the problem formulation of [4, 5] and this paper. We first consider Proposition 1 of [4]. By choosing the exogenous variable d to be w_0, and the endogenous 'to be controlled' variable f to be w_1, we have

$$\mathcal{K} = \{(w_0, w_1) \in C^\infty \mid \exists w_2 \in C^\infty \text{ s.t. } (w_0, w_1, w_2) \in \mathcal{B}^{P\wedge C}\},$$

and $G_{w_0\to w_1}$ is the transfer function corresponding to $\Pi_{P//C}$. Proposition 1 asserts that if $\mathcal{B}^P$, $\mathcal{B}^C$ are smooth differential behaviours and $\mathcal{B}^{P\wedge C}$ is controllable then the following are equivalent:

1. In $\mathcal{K}$, w_0 is the input, w_1 is the output and $\|G_{w_0\to w_1}\|_{\mathcal{H}_\infty} \leq 1$;

2. $\mathcal{K}$ is Σ-dissipative on $\mathbb{R}_-$ and $m(\mathcal{K}) = \sigma_+(\Sigma)$;

3. $\|w_1\|_{\mathcal{L}^2(\mathbb{R},\mathbb{R}^f)} \leq \|w_0\|_{\mathcal{L}^2(\mathbb{R},\mathbb{R}^d)}$, w_0 is free in $\mathcal{K}$ and $(0, w_1) \in \mathcal{K}$ implies that $\lim_{t\to\infty} w_1(t) = 0$,

where $\Sigma =$diag$(I_d, -I_f)$, $\sigma_+(\Sigma)$ is the number of positive eigenvalues of Σ and $m(\mathcal{K}) = \dim(\mathcal{U} \times \mathcal{Y})$ is the number of "free" input variables. We refer to [4, 5] for the definition of Σ-dissipativity on $\mathbb{R}_-$.

We now relate the above stability concepts to the notion of uniform stability, within an L^2 context, as considered in this paper. Consider the following condition:

4. $\mathcal{B}^{P\wedge C}$ is uniformly stable, and $\|\Pi_{P//C}\|_{L^2(\mathbb{R}_+)} \leq 1$.

Then:

Proposition 2.1 *Let* $X = L^2(\mathbb{R}_+)$. *Suppose* $\mathcal{B}^p$, $\mathcal{B}^C$ *are differential behaviours and* $\mathcal{B}^{P\wedge C}$ *is controllable. Then 1,2,3 and 4 are equivalent.*

Within the context of disturbance attenuation for linear controllable differential systems in an L^2 setting, the results of [4, 5] establish conditions under which there exists a controllable differentiable behaviour $\mathcal{B}^C$ which renders Σ-dissipativity on $\mathbb{R}_-$ of the closed loop interconnection $\mathcal{B}^{P\wedge C}$. Here $\mathcal{B}^P$ is also required to be a controllable differential behaviour. Since the resulting interconnection $\mathcal{B}^{P\wedge C}$ is controllable, and by the above, it follows that this synthesis yields the uniform stability condition 4. above, and in turn, the robust stability theorem 1.1 provides an explicit description of a set of plants for which stability can be guaranteed.

It is worth noting that it is observed in [5] that the synthesis can be extended in an ad-hoc manner from the controllable case to the general case by introducing appropriate stabilizability assumptions in the analysis. The robust stability theorem 1.1 can be utilized to achieve these observations directly. Given a (soundly) stabilizable plant behaviour $\mathcal{B}^P$, follow the $\mathcal{H}^\infty$ synthesis to derive a controller $\mathcal{B}^C$ (which is controllable) for the controllable plant sub-behaviour $\mathcal{B}^P_{\text{cont}}$. Then since $\vec{\delta}(\mathcal{B}^P_{\text{cont}}, \mathcal{B}^P) = 0$, Theorem 1.1 can be applied to establish the required uniform stability for the interconnection of the derived controller behaviour $\mathcal{B}^C$ and the original plant $\mathcal{B}^P$.

3 Extensions and Examples

The final part of the paper will consider extensions to more general feedback interconnections, and within a QDF formulation will approach alternative cost structures. A variety of examples will illustrate the approaches.

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