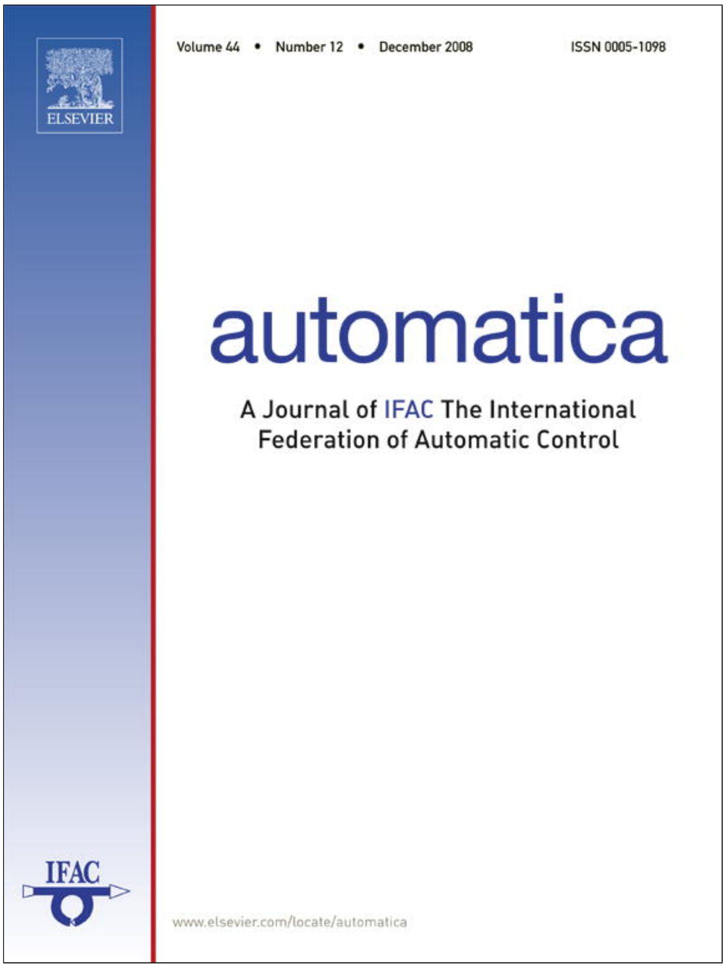
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Brief paper

A μ-based optimal finite-word-length controller design[☆]

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ABSTRACT

A novel optimal Finite Word Length (FWL) controller design is proposed in the framework of μ theory. A computationally tractable close-loop stability measure with FWL implementation considerations of the controller is derived based on the μ theory, and the optimal FWL controller realizations are obtained by solving the resulting optimal FWL realization problem using linear matrix inequality techniques.

1. Introduction

The classical digital controller design methodology often ignores the uncertainty occurring in the controller, even though in reality a control law can only be implemented with a processor of Finite Word Length (FWL). This is usually justified on the ground that the uncertainty resulting from FWL implementation of the digital controller is much smaller than the uncertainty associated with the plant model. However, researchers have realized that the FWL problem becomes critical in modern digital control practice (Franklin, Powell, & Workman, 1998; Gevers & Li, 1993) where some trends have appeared. The first trend is fast sampling which makes the closed-loop poles very close to the unit circle and hence greatly reduces the stability margin. The second one is high-order controllers produced by advanced design techniques. A small perturbation in the parameters of a high-order controller may destabilize the designed stable closed-loop system. The third trend is the growing popularity of robust controller design methods which greatly increase the robustness to plant uncertainty while may adversely decrease the robustness to controller uncertainty (Keel & Bhattacharyya, 2001). In addition, there are engineering constraints on the implementation cost in terms of chip area, operation complexity and power consumption. For mass-produced products, fixed point processors of as short a word length as possible are preferred for the implementation of digital control. The inaccuracy resulting from fixed point processors of short word length has to be considered seriously in controller design. There exist two types of FWL errors in digital controller implementation. The first one is rounding errors that occur in arithmetic operations (Liu, Skelton, & Grigoriadis, 1992; Miller, Mousa, & Michel, 1988; Miller, Michel, & Farrell, 1989) and the second one is parameter representation errors (Chen, Wu, & Li, 2002; Fialho & Georgiou, 1994, 2001; Li, 1998; Mantey, 1968; Whidborne, Wu, & Istepanian, 2000; Wu, Istepanian, & Chen, 1999; Yu & Ko, 2003), both due to finite precision. Typically, these two types of error are investigated separately for the reason of mathematical tractability. Although FWL rounding errors can lead to instability through bounded limit cycles or unbounded response (Miller et al., 1988, 1989), they do not affect the closed-loop poles. FWL parameter representation errors by contrast directly change the closed-loop poles and they are concerned with the critical issue of closed-loop stability. This paper deals with this second type of FWL error.

The question asked in this study is as follows. If a control law has been constructed by an existing controller synthesis method which may not take into account FWL effects, how can one "re-design" it to take into account controller implementation uncertainty? Because a control law can be implemented with

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different realizations and these different realizations possess different levels of robustness to FWL errors, this property can be utilized to select controller realizations that are most robust to FWL errors from all the realizations of the given control law. Since the most critical requirement for a closed-loop control system is to maintain its stability, most of the researches focus on developing various FWL stability measures to quantify FWL effects on closed-loop stability and obtaining optimal realizations by optimizing the related FWL stability measures. The FWL stability measure proposed in Fialho and Georgiou (1994), referred to as the ideal measure υ, is known to best describe the FWL stability characteristics of a controller realization. Unfortunately, how to calculate the value of this ideal measure υ and to optimize it is still an open problem. For this reason, various computationally tractable FWL stability measures are proposed and adopted in practice to replace the ideal measure υ. These tractable measures include the Frobenius-norm pole sensitivity measure υ_f (Li, 1998), the l_1-based stability measure υ_l (Whidborne et al., 2000), the 1-norm pole sensitivity measure υ_1 (Mantey, 1968; Wu et al., 1999), the complex stability radius measure υ_r (Chen et al., 2002; Fialho & Georgiou, 2001) and the pole sensitivity sum measure υ_s (Yu & Ko, 2003). Finding optimal realizations by optimizing υ_f, υ_l or υ_1 leads to some complicated nonconvex optimization problems for which systematic solutions are lacked. Consequently, numerical search algorithms have to be adopted which may possibly be trapped at local solutions. Although optimizing υ_r and υ_s can be done effectively through Linear Matrix Inequality (LMI) and another analytic technique, respectively, the measure υ_r only provides a statistical word length for guaranteeing stability with probability no less than 0.9777 while the measure υ_s often yields considerably conservative word length estimate for guaranteeing stability because it utilizes the sensitivity sum of all eigenvalues. These observations motivate us to search for some new tractable FWL stability measure. This paper studies the optimal FWL controller design problem based on the μ theory (Doyle, 1982; Fan, Tits, & Doyle, 1991; Young, 1993). Our novel contributions are as follows. We show that the ideal stability measure υ can be expressed with an inequality of μ. From this expression, a μ-based FWL stability measure is developed. This μ-based FWL stability measure can be evaluated conveniently and the corresponding optimal FWL realization problem can be solved efficiently by means of LMI techniques.

2. Notations and preliminaries

Let $\mathcal{R}$ be the field of real numbers, $\mathcal{C}$ the field of complex numbers, and $\mathcal{U}$ the closed unit disk in $\mathcal{C}$. For a matrix $\mathbf{A} = [a_{i,j}]$, $\|\mathbf{A}\|_{\mathrm{m}} \triangleq \max_{i,j} |a_{i,j}|$, $\mathbf{A}^*$ denotes the complex conjugate transpose of $\mathbf{A}$, and $\overline{\sigma}(\mathbf{A})$ the largest singular value of $\mathbf{A}$. Let $\rho(\mathbf{A})$ represent the spectral radius of square matrix $\mathbf{A}$. $\mathbf{I}_n$ denotes the $n \times n$ identity matrix, while $\mathbf{I}$ and $\mathbf{0}$ represent the identity and zero matrices of appropriate dimensions, respectively. Let $\mathbf{d}_n = [\,1 \ \cdots \ 1\,] \in \mathcal{R}^{1\times n}$whose elements are all equal to 1. The notation $\sharp$ within a matrix represents the symmetric term of the matrix. $\mathbf{A} \bigotimes \mathbf{B}$ is the Kronecker product of matrices $\mathbf{A}$ and $\mathbf{B}$.

Denote $\mathcal{F}$ the set of all the causal finite-dimensional linear time-invariant discrete-time systems. Any system in $\mathcal{F}$ can be described as

$$\begin{cases} \mathbf{x}(k+1) &= \mathbf{A}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k) \\ \mathbf{y}(k) &= \mathbf{C}\mathbf{x}(k) + \mathbf{D}\mathbf{u}(k) \end{cases} \tag{1}$$

where $\mathbf{x}(k) \in \mathcal{R}^{n_x}$, $\mathbf{u}(k) \in \mathcal{R}^{n_u}$ and $\mathbf{y}(k) \in \mathcal{R}^{n_y}$ are state, input and output, respectively, and the real constant matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ and $\mathbf{D}$ have appropriate dimensions. The transfer function matrix of the above system is

$$\hat{\mathbf{G}}(\varpi) \triangleq \varpi\mathbf{C}(\mathbf{I} - \varpi\mathbf{A})^{-1}\mathbf{B} + \mathbf{D}. \tag{2}$$

Lemma 1. *For $\hat{\mathbf{G}}(\varpi) \in \mathcal{F}$ given in (2), the following statements are true.*

(a) *$\hat{\mathbf{G}}(\varpi)$ is stable ($\mathbf{A}$ is stable) if and only if $\rho(\mathbf{A}) < 1$ or equivalently $\forall \varpi \in \mathcal{U}$, $\det(\mathbf{I} - \varpi\mathbf{A}) \neq 0$.*

(b) *If $\hat{\mathbf{G}}(\varpi)$ is stable, $\|\hat{\mathbf{G}}(\varpi)\|_\infty \triangleq \sup_{\varpi\in\mathcal{U}} \overline{\sigma}(\hat{\mathbf{G}}(\varpi)) < \infty$.*

(c) *$\hat{\mathbf{G}}(\varpi)$ is stable and $\|\hat{\mathbf{G}}(\varpi)\|_\infty < 1$ if and only if there exists a $\mathbf{E} = \mathbf{E}^{\mathrm{T}} > 0$ such that*

$$\begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} - \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{E} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} > 0. \tag{3}$$

The following results of μ theory are from Young (1993). Suppose that we have a matrix $\mathbf{M} \in \mathcal{C}^{n_a\times n_a}$ and two non-negative integers p and q with $p + q \le n_a$, which specify the numbers of uncertainty blocks of two types: repeated complex scalars and repeated real scalars. A $(p+q)$-tuple of positive integers

$$\mathbf{k}(p, q) = \left[k_1 \cdots k_p \ k_{p+1} \cdots k_{p+q}\right]^{\mathrm{T}} \tag{4}$$

specifies the dimensions of the perturbation blocks, and we require $\sum_{i=1}^{p+q} k_i = n_a$, in order that these dimensions are compatible with $\mathbf{M}$. The block structure $\mathbf{k}(p, q)$ determines the set of allowable perturbations, namely

$$\mathcal{K} \triangleq \left\{ \boldsymbol{\Upsilon} \left| \begin{array}{l} \boldsymbol{\Upsilon} = \mathrm{diag}\left(\zeta_1\mathbf{I}_{k_1}, \cdots, \zeta_{p+q}\mathbf{I}_{k_{p+q}}\right); \\ \forall i \in \{1, \ldots, p\}, \zeta_i \in \mathcal{C}; \\ \forall i \in \{p+1, \ldots, p+q\}, \zeta_i \in \mathcal{R}; \end{array} \right. \right\}. \tag{5}$$

The structured singular value μ of a matrix $\mathbf{M} \in \mathcal{C}^{n_a\times n_a}$ with respect to a perturbation set $\mathcal{K}$ is defined as

$$\mu_{\mathcal{K}}(\mathbf{M}) \triangleq \left(\inf_{\boldsymbol{\Upsilon}\in\mathcal{K}} \{\overline{\sigma}(\boldsymbol{\Upsilon}) |\det(\mathbf{I} - \boldsymbol{\Upsilon}\mathbf{M}) = 0\}\right)^{-1}. \tag{6}$$

with $\mu_{\mathcal{K}}(\mathbf{M}) = 0$ if no $\boldsymbol{\Upsilon} \in \mathcal{K}$ solves $\det(\mathbf{I} - \boldsymbol{\Upsilon}\mathbf{M}) = 0$.

Lemma 2. *Suppose that $p = 1$ and $q = 0$. Then $\mu_{\mathcal{K}}(\mathbf{M}) = \rho(\mathbf{M})$.*

Presently, except for a few special cases, how to compute $\mu_{\mathcal{K}}(\mathbf{M})$ is unknown. However, an upper bound of $\mu_{\mathcal{K}}(\mathbf{M})$ provided in the following is easy to compute and is often used to replace $\mu_{\mathcal{K}}(\mathbf{M})$ in practice. Define

$$\mathcal{E}_{\mathcal{K}} \triangleq \left\{ \mathbf{E} \left| \begin{array}{l} \mathbf{E} = \mathrm{diag}\left(\mathbf{E}_1, \ldots, \mathbf{E}_{p+q}\right); \\ \forall i \in \{1, \ldots, p+q\}, 0 < \mathbf{E}_i \in \mathcal{C}^{k_i\times k_i}; \end{array} \right. \right\}, \tag{7}$$

$$\mathcal{G}_{\mathcal{K}} \triangleq \left\{ \mathbf{G} \left| \begin{array}{l} \mathbf{G} = \mathrm{diag}\left(0\mathbf{I}_{k_1}, \ldots, 0\mathbf{I}_{k_p}, \mathbf{G}_{p+1}, \ldots, \mathbf{G}_{p+q}\right); \\ \forall i \in \{p+1, \ldots, p+q\}, \\ \mathbf{G}_i = \mathbf{G}_i^* \in \mathcal{C}^{k_i\times k_i} \end{array} \right. \right\}. \tag{8}$$

Then

$$\alpha_{\mathcal{K}}(\mathbf{M}) \triangleq \inf_{\substack{\mathbf{E}\in\mathcal{E}_{\mathcal{K}} \\ \mathbf{G}\in\mathcal{G}_{\mathcal{K}} \\ 0<\alpha\in\mathcal{R}}} \left\{ \alpha \left| \begin{array}{l} \alpha^2\mathbf{E} - \mathbf{M}^*\mathbf{E}\mathbf{M} \\ -\sqrt{-1}(\mathbf{G}\mathbf{M} - \mathbf{M}^*\mathbf{G}) > 0 \end{array} \right. \right\} \tag{9}$$

is an upper bound of $\mu_{\mathcal{K}}(\mathbf{M})$, i.e. $\mu_{\mathcal{K}}(\mathbf{M}) \le \alpha_{\mathcal{K}}(\mathbf{M})$. When the real scalars of $\boldsymbol{\Upsilon} \in \mathcal{K}$ are not repeated and $\mathbf{M}$ is a real matrix, $\alpha_{\mathcal{K}}(\mathbf{M})$ can be expressed in a simpler form and computed more easily. Define

$$\mathcal{E}_{\mathcal{RK}} \triangleq \left\{\mathbf{E} \in \mathcal{E}_{\mathcal{K}} \left| \mathbf{E} \in \mathcal{R}^{n_a\times n_a} \right.\right\}. \tag{10}$$

The following lemma is due to Theorem 5.12 in Young (1993).

Lemma 3. *Suppose that we have a real matrix* $\mathbf{M} \in \mathcal{R}^{n_a \times n_a}$ *and a perturbation set* $\mathcal{K}$ *with* $k_i = 1$ *for* $i \in \{p+1, \dots, p+q\}$ *(i.e. none of the real scalars are repeated). Then*

$$\alpha_{\mathcal{K}}(\mathbf{M}) = \inf_{\substack{\mathbf{E} \in \mathcal{E}_{\mathcal{RK}} \\ 0<\alpha \in \mathcal{R}}} \{\alpha \mid \alpha^2 \mathbf{E} - \mathbf{M}^{\mathrm{T}} \mathbf{E} \mathbf{M} > 0\}. \tag{11}$$

Corollary 1. *For* $\mathbf{M}$ *and* $\mathcal{K}$ *as in Lemma 3,* $\alpha_{\mathcal{K}}(\mathbf{M}) < 1$ *if and only if there exists* $\mathbf{E} \in \mathcal{E}_{\mathcal{RK}}$ *such that* $\mathbf{E} - \mathbf{M}^{\mathrm{T}}\mathbf{E}\mathbf{M} > 0$.

Consider a matrix $\mathbf{M} \in \mathcal{C}^{n_a \times n_a}$ partitioned as

$$\mathbf{M} = \begin{bmatrix} \mathbf{M}_{1,1} & \mathbf{M}_{1,2} \\ \mathbf{M}_{2,1} & \mathbf{M}_{2,2} \end{bmatrix}$$

where $\mathbf{M}_{1,1}$ and $\mathbf{M}_{2,2}$ are square. Suppose that we have a perturbation set $\mathcal{K}_1$ compatible with $\mathbf{M}_{1,1}$ and a perturbation set $\mathcal{K}_2$ compatible with $\mathbf{M}_{2,2}$. Then the perturbation set

$$\mathcal{K}_f \triangleq \{\boldsymbol{\Upsilon} = \mathrm{diag}\,(\boldsymbol{\Upsilon}_1, \boldsymbol{\Upsilon}_2) \mid \boldsymbol{\Upsilon}_1 \in \mathcal{K}_1, \boldsymbol{\Upsilon}_2 \in \mathcal{K}_2\} \tag{12}$$

defined by the block structure $\mathbf{k}_f = [\mathbf{k}_1^{\mathrm{T}}\ \mathbf{k}_2^{\mathrm{T}}]^{\mathrm{T}}$ is compatible with $\mathbf{M}$. From Young (1993), we have

Lemma 4. *For* $0 < \alpha \in \mathcal{R}$, $\mu_{\mathcal{K}_f}(\mathbf{M}) < \alpha$ *if and only if* $\mu_{\mathcal{K}_1}(\mathbf{M}_{1,1}) < \alpha$ *and* $\forall \boldsymbol{\Upsilon}_1 \in \mathcal{K}_1$ *with* $\overline{\sigma}(\boldsymbol{\Upsilon}_1) \le \frac{1}{\alpha}$, $\mu_{\mathcal{K}_2}(F(\mathbf{M}, \boldsymbol{\Upsilon}_1)) < \alpha$, *where* $F(\mathbf{M}, \boldsymbol{\Upsilon}_1)$ *is defined by*

$$F(\mathbf{M}, \boldsymbol{\Upsilon}_1) \triangleq \mathbf{M}_{2,2} + \mathbf{M}_{2,1}(\mathbf{I} - \boldsymbol{\Upsilon}_1 \mathbf{M}_{1,1})^{-1} \boldsymbol{\Upsilon}_1 \mathbf{M}_{1,2}. \tag{13}$$

3. A μ-Based FWL Stability Measure

Consider a discrete-time closed-loop control system consisting of a plant $\hat{\mathbf{P}}(\varpi) \in \mathcal{F}$ and a digital stabilizing controller $\hat{\mathbf{C}}(\varpi) \in \mathcal{F}$. The plant model $\hat{\mathbf{P}}(\varpi)$ is assumed to be strictly proper with a state-space description

$$\begin{cases} \mathbf{x}_P(k+1) = \mathbf{A}_P \mathbf{x}_P(k) + \mathbf{B}_P \mathbf{u}_P(k) \\ \mathbf{y}_P(k) = \mathbf{C}_P \mathbf{x}_P(k) \end{cases} \tag{14}$$

where $\mathbf{A}_P \in \mathcal{R}^{n \times n}$, $\mathbf{B}_P \in \mathcal{R}^{n \times s}$ and $\mathbf{C}_P \in \mathcal{R}^{t \times n}$. The digital controller $\hat{\mathbf{C}}(\varpi)$ is described by

$$\begin{cases} \mathbf{x}_C(k+1) = \mathbf{A}_C \mathbf{x}_C(k) + \mathbf{B}_C \mathbf{y}_P(k) \\ \mathbf{u}_P(k) = \mathbf{C}_C \mathbf{x}_C(k) + \mathbf{D}_C \mathbf{y}_P(k) \end{cases} \tag{15}$$

with $\mathbf{A}_C \in \mathcal{R}^{m \times m}$, $\mathbf{B}_C \in \mathcal{R}^{m \times t}$, $\mathbf{C}_C \in \mathcal{R}^{s \times m}$ and $\mathbf{D}_C \in \mathcal{R}^{s \times t}$. Denote the *realization* of $\hat{\mathbf{C}}(\varpi)$ as

$$\mathbf{X} \triangleq \begin{bmatrix} \mathbf{D}_C & \mathbf{C}_C \\ \mathbf{B}_C & \mathbf{A}_C \end{bmatrix} \in \mathcal{R}^{(s+m)\times(t+m)}. \tag{16}$$

The stability of the closed-loop control system depends on the spectral radius of the closed-loop transition matrix

$$\overline{\mathbf{A}}(\mathbf{X}) = \begin{bmatrix} \mathbf{A}_P & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{B}_P & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} \mathbf{X} \begin{bmatrix} \mathbf{C}_P & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix}$$
$$\triangleq \mathbf{M}_0 + \mathbf{M}_1 \mathbf{X} \mathbf{M}_2 \in \mathcal{R}^{(n+m)\times(n+m)}. \tag{17}$$

Since the digital control system has been designed to be stable, $\rho(\overline{\mathbf{A}}(\mathbf{X})) < 1$. However, when $\mathbf{X}$ is implemented in fixed-point format of FWL, it is perturbed into $\mathbf{X} + \boldsymbol{\Delta}$. $\boldsymbol{\Delta}$ belongs to the hypercube

$$\mathcal{D}_\beta \triangleq \{\boldsymbol{\Delta} \mid \boldsymbol{\Delta} \in \mathcal{R}^{(s+m)\times(t+m)}, \|\boldsymbol{\Delta}\|_{\mathrm{m}} \le \beta\} \tag{18}$$

where $0 \le \beta \in \mathcal{R}$ is the maximum representation error of the fixed-point digital processor. Thus, $\overline{\mathbf{A}}(\mathbf{X})$ is moved to

$$\overline{\mathbf{A}}(\mathbf{X} + \boldsymbol{\Delta}) = \overline{\mathbf{A}}(\mathbf{X}) + \mathbf{M}_1 \boldsymbol{\Delta} \mathbf{M}_2. \tag{19}$$

If $\rho(\overline{\mathbf{A}}(\mathbf{X} + \boldsymbol{\Delta})) \ge 1$, the closed-loop system, designed to be stable, becomes unstable with the finite-precision implemented $\mathbf{X}$. It is therefore critical to know how robust the closed-loop stability to the FWL error $\boldsymbol{\Delta}$ for a realization $\mathbf{X}$. This means that we would like to know the largest $\mathcal{D}_\beta$ within which the closed-loop system remains stable. Based on this consideration, Fialho and Georgiou (1994) proposed the following ideal FWL stability measure

$$\upsilon(\mathbf{X}) \triangleq \inf_{\boldsymbol{\Delta} \in \mathcal{R}^{(s+m)\times(t+m)}} \{\|\boldsymbol{\Delta}\|_{\mathrm{m}} \mid \overline{\mathbf{A}}(\mathbf{X} + \boldsymbol{\Delta}) \text{ is unstable}\}$$
$$= \sup_{0 \le \beta \in \mathcal{R}} \{\beta \mid \forall \boldsymbol{\Delta} \in \mathcal{D}_\beta, \overline{\mathbf{A}}(\mathbf{X} + \boldsymbol{\Delta}) \text{ is stable}\}. \tag{20}$$

Although the measure $\upsilon(\mathbf{X})$ characterizes well the FWL robustness of closed-loop stability for controller realization $\mathbf{X}$, how to compute explicitly the value of $\upsilon(\mathbf{X})$ is still an unsolved open problem. This has motivated the derivations of alternative tractable FWL stability measures (Chen et al., 2002; Fialho & Georgiou, 2001; Li, 1998; Mantey, 1968; Whidborne et al., 2000; Wu et al., 1999; Yu & Ko, 2003). Here we derive a new μ-based FWL stability measure.

Let us denote

$$N \triangleq (s+m)(t+m), \tag{21}$$

$$\mathcal{O} \triangleq \{\boldsymbol{\Lambda} \mid \boldsymbol{\Lambda} \in \mathcal{R}^{N \times N}, \boldsymbol{\Lambda} \text{ is diagonal}\}, \tag{22}$$

$$\mathcal{O}_\beta \triangleq \{\boldsymbol{\Lambda} \mid \boldsymbol{\Lambda} \in \mathcal{O}, \overline{\sigma}(\boldsymbol{\Lambda}) \le \beta\}. \tag{23}$$

Further let us revisit (19) and express $\boldsymbol{\Delta}$ as

$$\boldsymbol{\Delta} \triangleq \begin{bmatrix} \delta_{1,1} & \delta_{1,2} & \cdots & \delta_{1,t+m} \\ \delta_{2,1} & \delta_{2,2} & \cdots & \delta_{2,t+m} \\ \vdots & \vdots & \cdots & \vdots \\ \delta_{s+m,1} & \delta_{s+m,2} & \cdots & \delta_{s+m,t+m} \end{bmatrix}. \tag{24}$$

It is easy to check that

$$\overline{\mathbf{A}}(\mathbf{X} + \boldsymbol{\Delta}) = \overline{\mathbf{A}}(\mathbf{X}) + \mathbf{B}_u \boldsymbol{\Lambda} \mathbf{C}_u \tag{25}$$

where

$$\mathbf{B}_u \triangleq \mathbf{d}_{t+m} \bigotimes \mathbf{M}_1 \in \mathcal{R}^{(n+m)\times N}, \tag{26}$$

$$\mathbf{C}_u \triangleq \mathbf{M}_2 \bigotimes \mathbf{d}_{s+m}^{\mathrm{T}} \in \mathcal{R}^{N \times (n+m)}, \tag{27}$$

$$\boldsymbol{\Lambda} \triangleq \mathrm{diag}\left(\delta_{1,1}, \delta_{2,1}, \dots, \delta_{s+m,1}, \delta_{1,2}, \dots, \delta_{s+m,2}, \right.$$
$$\left. \dots, \delta_{1,t+m}, \dots, \delta_{s+m,t+m}\right) \in \mathcal{O} \tag{28}$$

with $\overline{\sigma}(\boldsymbol{\Lambda}) = \|\boldsymbol{\Delta}\|_{\mathrm{m}}$. Hence, the FWL stability measure $\upsilon(\mathbf{X})$ can also be stated by means of $\overline{\sigma}(\boldsymbol{\Lambda})$,

$$\upsilon(\mathbf{X}) \triangleq \inf_{\boldsymbol{\Lambda} \in \mathcal{O}} \{\overline{\sigma}(\boldsymbol{\Lambda}) \mid \overline{\mathbf{A}}(\mathbf{X}) + \mathbf{B}_u \boldsymbol{\Lambda} \mathbf{C}_u \text{ is unstable}\}$$
$$= \sup_{0 \le \beta \in \mathcal{R}} \left\{\beta \,\middle|\, \forall \boldsymbol{\Lambda} \in \mathcal{O}_\beta,\ \overline{\mathbf{A}}(\mathbf{X}) + \mathbf{B}_u \boldsymbol{\Lambda} \mathbf{C}_u \text{ is stable}\right\}. \tag{29}$$

We further formulate $\upsilon(\mathbf{X})$ with μ. For $0 \le \beta \in \mathcal{R}$, denote

$$\mathbf{H}(\mathbf{X}, \beta) \triangleq \begin{bmatrix} \overline{\mathbf{A}}(\mathbf{X}) & \mathbf{B}_u \\ \beta \mathbf{C}_u & 0\mathbf{I}_N \end{bmatrix} \in \mathcal{R}^{(n+m+N)\times(n+m+N)}, \tag{30}$$

$$\mathcal{K}_h \triangleq \left\{ \boldsymbol{\Upsilon}_h = \begin{bmatrix} \varpi \mathbf{I}_{n+m} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda} \end{bmatrix} \middle|\, \varpi \in \mathcal{C}, \boldsymbol{\Lambda} \in \mathcal{O} \right\}. \tag{31}$$

The perturbation set $\mathcal{K}_h$ is compatible with $\mathbf{H}(\mathbf{X}, \beta)$ and hence there exists $\mu_{\mathcal{K}_h}(\mathbf{H}(\mathbf{X}, \beta))$.

Theorem 1. *$\upsilon(\mathbf{X}) > \beta$ if and only if $\mu_{\mathcal{K}_h}(\mathbf{H}(\mathbf{X},\beta)) < 1$.*

Proof. Define the perturbation sets

$$\mathcal{K}_a \triangleq \{\varpi \mathbf{I}_{n+m} \mid \varpi \in \mathcal{C}\}, \qquad \mathcal{K}_0 \triangleq \mathcal{O}, \tag{32}$$

which are compatible with $\bar{\mathbf{A}}(\mathbf{X})$ and $0\mathbf{I}_N$, respectively. Since $\bar{\mathbf{A}}(\mathbf{X})$ is stable and $\mathcal{K}_a$ contains perturbations of one repeated complex scalar, we conclude by Lemma 2 that

$$\mu_{\mathcal{K}_a}(\bar{\mathbf{A}}(\mathbf{X})) = \rho(\bar{\mathbf{A}}(\mathbf{X})) < 1. \tag{33}$$

Thus, from Lemma 4, $\mu_{\mathcal{K}_h}(\mathbf{H}(\mathbf{X},\beta)) < 1$ if and only if

$$\mu_{\mathcal{K}_0}(F(\mathbf{H}(\mathbf{X},\beta), \varpi \mathbf{I}_{n+m})) < 1, \ \forall \varpi \in \mathcal{U}. \tag{34}$$

By the definitions (6) and (13), condition (34) is equivalent to

$$\inf_{\substack{\boldsymbol{\Lambda}\in\mathcal{O}\\ \varpi\in\mathcal{U}}} \left\{\bar{\sigma}(\boldsymbol{\Lambda}) \mid \det(\mathbf{I} - \beta\varpi\boldsymbol{\Lambda}\mathbf{C}_u(\mathbf{I} - \varpi\bar{\mathbf{A}}(\mathbf{X}))^{-1}\mathbf{B}_u) = 0\right\} > 1. \tag{35}$$

On the other hand, from Lemma 1, $\bar{\mathbf{A}}(\mathbf{X})$ is stable means that $\mathbf{I} - \varpi\bar{\mathbf{A}}(\mathbf{X})$ is invertible for any $\varpi \in \mathcal{U}$. Then

$$\upsilon(\mathbf{X}) = \inf_{\substack{\boldsymbol{\Lambda}\in\mathcal{O}\\ \varpi\in\mathcal{U}}} \{\bar{\sigma}(\boldsymbol{\Lambda}) \mid \det(\mathbf{I} - \varpi\bar{\mathbf{A}}(\mathbf{X}) - \varpi\mathbf{B}_u\boldsymbol{\Lambda}\mathbf{C}_u) = 0\}$$

$$= \inf_{\substack{\boldsymbol{\Lambda}\in\mathcal{O}\\ \varpi\in\mathcal{U}}} \{\bar{\sigma}(\boldsymbol{\Lambda}) \mid \det(\mathbf{I} - \varpi\boldsymbol{\Lambda}\mathbf{C}_u(\mathbf{I} - \varpi\bar{\mathbf{A}}(\mathbf{X}))^{-1}\mathbf{B}_u) = 0\}.$$

Combining the above equality with (35) gives Theorem 1. □

Corollary 2. $\upsilon(\mathbf{X}) = \sup_{0\le\beta\in\mathcal{R}}\{\beta \mid \mu_{\mathcal{K}_h}(\mathbf{H}(\mathbf{X},\beta)) < 1\}$.

The above corollary gives the quantitative relation between υ and a special μ. Replacing the intractable $\mu_{\mathcal{K}_h}(\mathbf{H}(\mathbf{X},\beta))$ in Corollary 2 with its upper bound $\alpha_{\mathcal{K}_h}(\mathbf{H}(\mathbf{X},\beta))$ produces the measure

$$\upsilon_\mu(\mathbf{X}) \triangleq \sup_{0\le\beta\in\mathcal{R}} \{\beta \mid \alpha_{\mathcal{K}_h}(\mathbf{H}(\mathbf{X},\beta)) < 1\}. \tag{36}$$

The following plain result states that $\upsilon_\mu(\mathbf{X})$ is a lower bound of $\upsilon(\mathbf{X})$ and it is a suitable FWL stability measure.

Theorem 2. (a) $\upsilon(\mathbf{X}) \ge \upsilon_\mu(\mathbf{X})$. (b) *If $\beta < \upsilon_\mu(\mathbf{X})$, then $\forall\boldsymbol{\Delta} \in \mathcal{D}_\beta$, $\bar{\mathbf{A}}(\mathbf{X}+\boldsymbol{\Delta})$ is stable.*

It should be noticed that $\mathbf{H}(\mathbf{X},\beta)$ is a real matrix and the real scalars in perturbation $\boldsymbol{\Upsilon}_h \in \mathcal{K}_h$ are not repeated. This is the case for which Corollary 1 is applicable. Hence, given a stabilizing controller realization $\mathbf{X}$, denote the LMI

$$\mathbf{H}^{\mathrm{T}}(\mathbf{X},\beta)\mathrm{diag}\,(\mathbf{E}_1, e_1, \ldots, e_N)\,\mathbf{H}(\mathbf{X},\beta)$$
$$< \mathrm{diag}\,(\mathbf{E}_1, e_1, \ldots, e_N)\,,$$
$$0 < \mathbf{E}_1 \in \mathcal{R}^{(n+m)\times(n+m)}, \ 0 < e_i \in \mathcal{R}, \quad 1 \le i \le N,$$

as $\mathcal{L}(\beta)$. Then the value of the FWL stability measure $\upsilon_\mu(\mathbf{X})$ is obtained by the following computational problem

$$\upsilon_\mu(\mathbf{X}) = \sup_{0\le\beta\in\mathcal{R}} \beta, \quad \text{s.t. } \mathcal{L}(\beta). \tag{37}$$

This problem can be solved conveniently by a combination of LMI technique (Boyd, El Ghaoui, Feron, & Balakrishan, 1994) and bisection search (Quarteroni, Sacco, & Saleri, 2000). The detailed computational procedure for solving (37) is as follow.

Step (1) Specify a precision $\varepsilon > 0$. Initially set a sufficiently small $\beta_{\min} \ge 0$ such that $\mathcal{L}(\beta_{\min})$ has solutions and a sufficiently large $\beta_{\max} > 0$ such that $\mathcal{L}(\beta_{\max})$ has no solution.

Step (2) Let $\beta_t = (\beta_{\min} + \beta_{\max})/2$, solve $\mathcal{L}(\beta_t)$ with the LMI toolbox of MATLAB.

Step (3) If $\mathcal{L}(\beta_t)$ has solutions, let $\beta_{\min} = \beta_t$; if $\mathcal{L}(\beta_t)$ has no solution, let $\beta_{\max} = \beta_t$.

Step (4) If $\beta_{\max} - \beta_{\min} < \varepsilon$, set $\upsilon_\mu(\mathbf{X}) = \beta_t$ and terminate the algorithm; if $\beta_{\max} - \beta_{\min} \ge \varepsilon$, go to Step (2).

4. Optimal FWL realizations of controller

It is well known that the realizations of $\hat{\mathbf{C}}(\varpi)$ are not unique. In fact, given an initial realization

$$\mathbf{X}_0 \triangleq \begin{bmatrix} \mathbf{D}_C^0 & \mathbf{C}_C^0 \\ \mathbf{B}_C^0 & \mathbf{A}_C^0 \end{bmatrix} \in \mathcal{R}^{(s+m)\times(t+m)} \tag{38}$$

of $\hat{\mathbf{C}}(\varpi)$, all the realizations of $\hat{\mathbf{C}}(\varpi)$ form a set

$$\mathcal{X} \triangleq \left\{\mathbf{X} \middle| \mathbf{X} = \begin{bmatrix} \mathbf{I}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \mathbf{X}_0 \begin{bmatrix} \mathbf{I}_t & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^{-1} \end{bmatrix}, \mathbf{T} \in \mathcal{R}^{m\times m}, \det\mathbf{T} \ne 0 \right\}. \tag{39}$$

Note that all the different realizations $\mathbf{X} \in \mathcal{X}$ have the same level of robustness to the plant parameter perturbations $\boldsymbol{\Delta}_A$, $\boldsymbol{\Delta}_B$ and $\boldsymbol{\Delta}_C$. In fact

$$\rho\left(\begin{bmatrix} \widetilde{\mathbf{A}}_P & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \widetilde{\mathbf{B}}_P & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} \mathbf{X} \begin{bmatrix} \widetilde{\mathbf{C}}_P & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix}\right)$$
$$= \rho\left(\begin{bmatrix} \widetilde{\mathbf{A}}_P & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \widetilde{\mathbf{B}}_P & \mathbf{0} \\ \mathbf{0} & \mathbf{T} \end{bmatrix} \mathbf{X}_0 \begin{bmatrix} \widetilde{\mathbf{C}}_P & \mathbf{0} \\ \mathbf{0} & \mathbf{T}^{-1} \end{bmatrix}\right)$$
$$= \rho\left(\begin{bmatrix} \widetilde{\mathbf{A}}_P & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} + \begin{bmatrix} \widetilde{\mathbf{B}}_P & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix} \mathbf{X}_0 \begin{bmatrix} \widetilde{\mathbf{C}}_P & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix}\right) \tag{40}$$

with $\widetilde{\mathbf{A}}_P \triangleq \mathbf{A}_P + \boldsymbol{\Delta}_A$, $\widetilde{\mathbf{B}}_P \triangleq \mathbf{B}_P + \boldsymbol{\Delta}_B$, $\widetilde{\mathbf{C}}_P \triangleq \mathbf{C}_P + \boldsymbol{\Delta}_C$. However, different realizations have different levels of robustness to the FWL errors in controller parameters. Certain realizations in $\mathcal{X}$ possess the largest value of υ_μ, and they have superior FWL closed-loop stability robustness over rest of the realizations in $\mathcal{X}$. It is highly desired to find an optimal realization in $\mathcal{X}$ that maximizes the measure υ_μ.

Given an initial realization $\mathbf{X}_0$, $\mathcal{X}$ is specified and, moreover, $\mathbf{X} \in \mathcal{X}$ depends on the non-singular transformation $\mathbf{T}$, i.e. $\mathbf{X} = \mathbf{X}(\mathbf{T})$. Thus, this optimal FWL realization problem is formally defined as

$$\gamma = \sup_{\substack{\mathbf{T}\in\mathcal{R}^{m\times m}\\ \det\mathbf{T}\ne 0}} \upsilon_\mu(\mathbf{X}(\mathbf{T})). \tag{41}$$

Equivalently, the optimal FWL realization problem (41) is stated as

$$\gamma = \sup_{0\le\beta\in\mathcal{R}} \beta, \tag{42}$$
$$\text{s.t. } \alpha_{\mathcal{K}_h}(\mathbf{H}(\mathbf{X}(\mathbf{T}),\beta)) < 1, \mathbf{T} \in \mathcal{R}^{m\times m}, \det\mathbf{T} \ne 0.$$

In the matrix inequality form, the necessary and sufficient condition of $\alpha_{\mathcal{K}_h}(\mathbf{H}(\mathbf{X}(\mathbf{T}),\beta)) < 1$ is stated in the following theorem.

Theorem 3. *Given a $0 \le \beta \in \mathcal{R}$, if and only if there exist*

$$0 < \mathbf{E}_1 \in \mathcal{R}^{(n+m)\times(n+m)}, \tag{43}$$
$$0 < e_i \in \mathcal{R}, \quad 1 \le i \le N, \tag{44}$$
$$\mathbf{T} \in \mathcal{R}^{m\times m}, \quad \det\mathbf{T} \ne 0, \tag{45}$$

such that

$$\begin{bmatrix} \mathbf{E}_1 - \bar{\mathbf{A}}^{\mathrm{T}}(\mathbf{X}_0)\mathbf{E}_1\bar{\mathbf{A}}(\mathbf{X}_0) & \sharp \\ -\mathbf{B}_u^{\mathrm{T}}\mathbf{E}_1\bar{\mathbf{A}}(\mathbf{X}_0) & \tilde{\mathbf{U}} \end{bmatrix}$$
$$> \mathrm{diag}\left(\beta^2\mathbf{C}_P^{\mathrm{T}}\mathbf{E}_{\mathbf{C}}\mathbf{C}_P, \beta^2\mathbf{T}^{\mathrm{T}}\mathbf{E}_{\mathbf{T}}\mathbf{T}, \mathbf{B}_u^{\mathrm{T}}\mathbf{E}_1\mathbf{B}_u\right) \tag{46}$$

where

$$\tilde{\mathbf{U}} \triangleq \mathrm{diag}\left(\mathbf{J}_1, \mathbf{T}^{\mathrm{T}}\mathbf{L}_1\mathbf{T}, \ldots, \mathbf{J}_{t+m}, \mathbf{T}^{\mathrm{T}}\mathbf{L}_{t+m}\mathbf{T}\right), \tag{47}$$

$$\mathbf{E}_{\mathbf{C}} \triangleq \mathrm{diag}\left(\sum_{j=1}^{s+m} e_j, \sum_{j=1}^{s+m} e_{(s+m)+j}, \ldots, \sum_{j=1}^{s+m} e_{(t-1)(s+m)+j}\right), \tag{48}$$

$$\mathbf{E_T} \triangleq \text{diag}\left(\sum_{j=1}^{s+m} e_{t(s+m)+j}, \sum_{j=1}^{s+m} e_{(t+1)(s+m)+j}, \ldots, \sum_{j=1}^{s+m} e_{(t+m-1)(s+m)+j}\right), \quad (49)$$

$$\mathbf{J}_i \triangleq \text{diag}\left(e_{(i-1)(s+m)+1}, e_{(i-1)(s+m)+2}, \ldots, e_{(i-1)(s+m)+s}\right), \quad (50)$$

$$\mathbf{L}_i \triangleq \text{diag}\left(e_{(i-1)(s+m)+s+1}, e_{(i-1)(s+m)+s+2}, \ldots, e_{i(s+m)}\right), \quad i \in \{1, \ldots, t+m\}, \quad (51)$$

then $\alpha_{\mathcal{K}_h}(\mathbf{H}(\mathbf{X}(\mathbf{T}), \beta)) < 1$.

Proof. From Corollary 1, we know that $\exists \mathbf{T} \in \mathcal{R}^{m\times m}$, $\det \mathbf{T} \neq 0$ satisfying $\alpha_{\mathcal{K}_h}(\mathbf{H}(\mathbf{X}(\mathbf{T}), \beta)) < 1$ if and only if there exist

$$0 < \mathbf{P}_1 \in \mathcal{R}^{(n+m)\times(n+m)}, \quad 0 < e_i \in \mathcal{R}, 1 \le i \le N, \quad (52)$$

such that

$$\begin{bmatrix} \mathbf{P}_1 & \\ & \mathbf{P}_2 \end{bmatrix} > \mathbf{H}^{\mathrm{T}}(\mathbf{X}(\mathbf{T}), \beta) \begin{bmatrix} \mathbf{P}_1 & \\ & \mathbf{P}_2 \end{bmatrix} \mathbf{H}(\mathbf{X}(\mathbf{T}), \beta)$$

or equivalently

$$\begin{bmatrix} \mathbf{P}_1 - \overline{\mathbf{A}}^{\mathrm{T}}(\mathbf{X}(\mathbf{T}))\mathbf{P}_1\overline{\mathbf{A}}(\mathbf{X}(\mathbf{T})) & \sharp \\ -\mathbf{B}_u^{\mathrm{T}}\mathbf{P}_1\overline{\mathbf{A}}(\mathbf{X}(\mathbf{T})) & \mathbf{P}_2 \end{bmatrix} > \text{diag}\left(\beta^2\mathbf{C}_u^{\mathrm{T}}\mathbf{P}_2\mathbf{C}_u, \mathbf{B}_u^{\mathrm{T}}\mathbf{P}_1\mathbf{B}_u\right), \quad (53)$$

where

$$\mathbf{P}_2 = \text{diag}(e_1, e_2, \ldots, e_N). \quad (54)$$

From (17) and (39), we have $\overline{\mathbf{A}}(\mathbf{X}(\mathbf{T})) = \mathbf{T}_n\overline{\mathbf{A}}(\mathbf{X}_0)\mathbf{T}_n^{-1}$ with $\mathbf{T}_n = \text{diag}(\mathbf{I}_n, \mathbf{T})$. Hence, the inequality (53) becomes

$$\begin{bmatrix} \mathbf{P}_1 - \mathbf{T}_n^{-\mathrm{T}}\overline{\mathbf{A}}^{\mathrm{T}}(\mathbf{X}_0)\mathbf{E}_1\overline{\mathbf{A}}(\mathbf{X}_0)\mathbf{T}_n^{-1} & \sharp \\ -\mathbf{B}_u^{\mathrm{T}}\mathbf{P}_1\mathbf{T}_n\overline{\mathbf{A}}(\mathbf{X}_0)\mathbf{T}_n^{-1} & \mathbf{P}_2 \end{bmatrix} > \text{diag}\left(\beta^2\mathbf{C}_u^{\mathrm{T}}\mathbf{P}_2\mathbf{C}_u, \mathbf{B}_u^{\mathrm{T}}\mathbf{P}_1\mathbf{B}_u\right) \quad (55)$$

where

$$\mathbf{E}_1 = \mathbf{T}_n^{\mathrm{T}}\mathbf{P}_1\mathbf{T}_n > 0. \quad (56)$$

Furthermore, the inequality (55) can be written as

$$\begin{bmatrix} \mathbf{E}_1 - \overline{\mathbf{A}}^{\mathrm{T}}(\mathbf{X}_0)\mathbf{E}_1\overline{\mathbf{A}}(\mathbf{X}_0) & \sharp \\ -\mathbf{E}_u\mathbf{B}_u^{\mathrm{T}}\mathbf{T}_n^{-\mathrm{T}}\mathbf{E}_1\overline{\mathbf{A}}(\mathbf{X}_0) & \mathbf{E}_u\mathbf{P}_2\mathbf{E}_u^{\mathrm{T}} \end{bmatrix} > \text{diag}\left(\beta^2\mathbf{T}_n^{\mathrm{T}}\mathbf{C}_u^{\mathrm{T}}\mathbf{P}_2\mathbf{C}_u\mathbf{T}_n, \mathbf{E}_u\mathbf{B}_u^{\mathrm{T}}\mathbf{P}_1\mathbf{B}_u\mathbf{E}_u^{\mathrm{T}}\right) \quad (57)$$

with

$$\mathbf{E}_u \triangleq \mathbf{I}_{t+m} \bigotimes \text{diag}\left(\mathbf{I}_s, \mathbf{T}^{\mathrm{T}}\right) \in \mathcal{R}^{N\times N}. \quad (58)$$

It can be deduced from (17), (26), (27), (54), (56) and (58) that

$$\mathbf{E}_u\mathbf{B}_u^{\mathrm{T}}\mathbf{T}_n^{-\mathrm{T}} = \mathbf{B}_u^{\mathrm{T}}, \quad (59)$$

$$\mathbf{E}_u\mathbf{B}_u^{\mathrm{T}}\mathbf{P}_1\mathbf{B}_u\mathbf{E}_u^{\mathrm{T}} = \mathbf{B}_u^{\mathrm{T}}\mathbf{E}_1\mathbf{B}_u, \quad (60)$$

$$\mathbf{T}_n^{\mathrm{T}}\mathbf{C}_u^{\mathrm{T}}\mathbf{P}_2\mathbf{C}_u\mathbf{T}_n = \text{diag}\left(\mathbf{C}_P^{\mathrm{T}}\mathbf{E_C}\mathbf{C}_P, \mathbf{T}^{\mathrm{T}}\mathbf{E_T}\mathbf{T}\right), \quad (61)$$

$$\mathbf{E}_u\mathbf{P}_2\mathbf{E}_u^{\mathrm{T}} = \tilde{\mathbf{U}}. \quad (62)$$

Finally, substituting (59)–(62) into (57) results in (46). □

The above necessary and sufficient condition means that the optimal realization problem (42) can be expressed as

$$\gamma = \sup_{0\le\beta\in\mathcal{R}} \beta, \quad \text{s.t. (43)–(46).} \quad (63)$$

The inequality (46) is not an LMI and at present we do not have efficient means to solve it. By means of

$$\mathbf{T}^{\mathrm{T}}\mathbf{L}_i\mathbf{T} + \mathbf{L}_i^{-1} \ge \mathbf{T}^{\mathrm{T}} + \mathbf{T}, \quad (64)$$

$$\mathbf{J}_i + \mathbf{J}_i^{-1} \ge 2\mathbf{I}_s \quad (65)$$

and Schur complements (Boyd et al., 1994), it is easy to arrive at a sufficient condition in an LMI form of $\alpha_{\mathcal{K}_h}(\mathbf{H}(\mathbf{X}(\mathbf{T}), \beta)) < 1$ as follow.

Theorem 4. *Given a $0 \le \beta \in \mathcal{R}$, if there exist*

$$0 < \mathbf{E}_1 \in \mathcal{R}^{(n+m)\times(n+m)}, \quad (66)$$

$$0 < \mathbf{E}_2 \in \mathcal{R}^{n\times n}, \quad (67)$$

$$0 < \mathbf{E}_3 \in \mathcal{R}^{m\times m}, \quad (68)$$

$$0 < q_i \in \mathcal{R}, \quad 1 \le i \le N, \quad (69)$$

$$\mathbf{T} \in \mathcal{R}^{m\times m}, \quad (70)$$

such that

$$\begin{bmatrix} \mathbf{E}_1 - \overline{\mathbf{A}}^{\mathrm{T}}(\mathbf{X}_0)\mathbf{E}_1\overline{\mathbf{A}}(\mathbf{X}_0) & \sharp \\ -\mathbf{B}_u^{\mathrm{T}}\mathbf{E}_1\overline{\mathbf{A}}(\mathbf{X}_0) & \mathbf{I}_{t+m} \bigotimes \begin{bmatrix} 2\mathbf{I}_s & \\ & \mathbf{T}^{\mathrm{T}} + \mathbf{T} \end{bmatrix} \end{bmatrix} > \text{diag}\left(\beta^2\mathbf{E}_2, \beta^2\mathbf{E}_3, \mathbf{B}_u^{\mathrm{T}}\mathbf{E}_1\mathbf{B}_u + \mathbf{Q}\right), \quad (71)$$

$$\begin{bmatrix} \mathbf{E}_2 & \mathbf{C}_P^{\mathrm{T}} & \cdots & \mathbf{C}_P^{\mathrm{T}} \\ \mathbf{C}_P & \mathbf{V}_1 & & \\ \vdots & & \ddots & \\ \mathbf{C}_P & & & \mathbf{V}_{s+m} \end{bmatrix} > 0, \quad (72)$$

$$\begin{bmatrix} \mathbf{E}_3 & \mathbf{T}^{\mathrm{T}} & \cdots & \mathbf{T}^{\mathrm{T}} \\ \mathbf{T} & \mathbf{W}_1 & & \\ \vdots & & \ddots & \\ \mathbf{T} & & & \mathbf{W}_{s+m} \end{bmatrix} > 0, \quad (73)$$

with

$$\mathbf{Q} \triangleq \text{diag}(q_1, q_2, \ldots, q_N), \quad (74)$$

$$\mathbf{V}_j \triangleq \text{diag}\left(q_j, q_{(s+m)+j}, \ldots, q_{(t-1)(s+m)+j}\right), \quad (75)$$

$$\mathbf{W}_j \triangleq \text{diag}\left(q_{t(s+m)+j}, q_{(t+1)(s+m)+j}, \ldots, q_{(t+m-1)(s+m)+j}\right), \quad j \in \{1, \ldots, s+m\}, \quad (76)$$

then $\alpha_{\mathcal{K}_h}(\mathbf{H}(\mathbf{X}(\mathbf{T}), \beta)) < 1$.

In Theorem 4, the nonsingularity of $\mathbf{T}$ is guaranteed by (71) which includes $\mathbf{T}^{\mathrm{T}} + \mathbf{T} > 0$. This constraint does not allow any singular $\mathbf{T}$: If $\det \mathbf{T} = 0$, there exists a nonzero $\mathbf{x} \in \mathcal{R}^m$ such that $\mathbf{Tx} = 0$ and hence $\mathbf{x}^{\mathrm{T}}\mathbf{T}^{\mathrm{T}} = 0$. Thus $\mathbf{x}^{\mathrm{T}}(\mathbf{T}^{\mathrm{T}} + \mathbf{T})\mathbf{x} = 0$ which contradicts $\mathbf{T}^{\mathrm{T}} + \mathbf{T} > 0$.

Theorem 4 obviously gives a lower bound of γ as

$$\gamma_1 = \sup_{0\le\beta\in\mathcal{R}} \beta, \quad \text{s.t. (66)– (73),} \quad (77)$$

which can be solved by a combination of LMI technique and bisection search. Through solving the problem (77), an optimal transformation $\mathbf{T}_{\text{opt}}$ is obtained. This leads to the corresponding optimal FWL realization $\mathbf{X}_{\text{opt}} = \mathbf{X}(\mathbf{T}_{\text{opt}})$.

5. A numerical design example

The plant was given by

Table 1
Measures and estimated word length for different realizations.

	υ_μ	υ_l	υ_1 $\times 10^{-3}$	υ_f	υ_r	υ_s	Bit length estimate
$\mathbf{X}_0$	**4.32**	2.10	1.95	1.08	3.10	0.211	$9(b_\mu(\mathbf{X}_0))$
$\mathbf{X}_\mu$	**13.1**	7.55	5.47	4.88	12.0	0.505	$8(b_\mu(\mathbf{X}_\mu))$
$\mathbf{X}_l$	10.6	**8.16**	6.71	4.75	11.3	0.516	$8(b_l(\mathbf{X}_l))$
$\mathbf{X}_1$	10.7	5.36	**8.93**	4.90	10.2	0.632	$8(b_1(\mathbf{X}_1))$
$\mathbf{X}_f$	12.5	7.49	5.28	**4.90**	12.0	0.546	$9(b_f(\mathbf{X}_f))$
$\mathbf{X}_r$	10.5	6.27	5.88	4.03	**12.0**	0.555	$9(b_r(\mathbf{X}_r))$
$\mathbf{X}_s$	9.62	4.85	4.48	3.38	8.40	**0.650**	$12(b_s(\mathbf{X}_s))$

$$\mathbf{A}_P = \begin{bmatrix} 0.99513 & -9.7260 & 4.8724\times10^{-3} \\ 9.9614\times10^{-4} & 0.98843 & -9.9614\times10^{-4} \\ 6.6995\times10^{-3} & 13.373 & 0.99330 \end{bmatrix},$$

$$\mathbf{B}_P = \begin{bmatrix} 0.24863 \\ 1.2427\times10^{-4} \\ 5.5656\times10^{-4} \end{bmatrix}, \qquad \mathbf{C}_P = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

and an initial realization of the controller was given by

$$\mathbf{X}_0 = \left[\begin{array}{c|cc} 1.3512 & 1.4260\times10^{-2} & 1.1956 \\ \hline -1 & 1 & 0 \\ -1 & 0 & 0.33330 \end{array}\right]$$

The value of the μ-based FWL stability measure for this initial controller realization, computed by the algorithm given in Section 3, was $\upsilon_\mu(\mathbf{X}_0) = 4.32 \times 10^{-3}$. The design approach reported in Section 4 was used to find an optimal FWL realization of the controller, yielding

$$\mathbf{X}_\mu = \left[\begin{array}{c|cc} 1.3512 & -0.11460 & 1.0970 \\ \hline -1.3490\times10^{-3} & 0.98538 & 6.5647\times10^{-2} \\ -1.1030 & 0.14523 & 0.34792 \end{array}\right]$$

with the μ-based FWL stability measure $\upsilon_\mu(\mathbf{X}_\mu) = 1.31 \times 10^{-2}$, which is more than three times larger than that of $\mathbf{X}_0$.

When a fixed point processor of b-bit length is used to implement a controller realization $\mathbf{X}$, the b bits are assigned as follows: 1 bit for the sign, b_{int} bits for the integer part and b_{fra} bits for the fraction part with $b = 1 + b_{\text{int}} + b_{\text{fra}}$. In order to supply a sufficient dynamic range for $\mathbf{X}$, b_{int} should at least equal to $\lceil \log_2 \|\mathbf{X}\|_{\text{m}} \rceil$, where $\lceil x \rceil$ denotes the closest integer greater than or equal to $x \in \mathcal{R}$. With a fraction bit length b_{fra}, the absolute values of the FWL errors are bounded by $2^{-(b_{\text{fra}}+1)}$. Comparing this bound with the stability measure $\upsilon_\mu(\mathbf{X})$ within which the closed-loop system remains stability, we know that b_{fra} should at least be $\lceil \log_2 \upsilon_\mu(\mathbf{X}) \rceil - 1$. Hence, when implementing $\mathbf{X}$ with fixed point processor, $b_\mu(\mathbf{X}) = \lceil \log_2 \|\mathbf{X}\|_{\text{m}} \rceil + \lceil \log_2 \upsilon_\mu(\mathbf{X}) \rceil$ can be viewed as the minimal word length guaranteeing closed-loop stability, estimated based on $\upsilon_\mu(\mathbf{X})$. In our example, $b_\mu(\mathbf{X}_0) = 9$ and $b_\mu(\mathbf{X}_\mu) = 8$.

The existing stability measures, $\upsilon_l, \upsilon_1, \upsilon_f, \upsilon_r$ and υ_s, were also maximized to obtain the associated optimal realizations, $\mathbf{X}_l, \mathbf{X}_1, \mathbf{X}_f, \mathbf{X}_r$ and $\mathbf{X}_s$, respectively. Table 1 lists the values of various FWL stability measures for all the 7 different controller realizations. Similar to b_μ, the minimum stable word lengths, b_l, b_1, b_f, b_r and b_s, can be estimated based on the values of $\upsilon_l, \upsilon_1, \upsilon_f, \upsilon_r$ and υ_s, respectively. For each optimal realization obtained by optimizing the corresponding stability measure, Table 1 also lists its minimum stable word length estimated based on the value of the related stability measure. For this example, by employing υ_μ and the other tractable measures, we have found several good controller realizations with the respective estimated minimum stable word lengths, which are helpful to practical engineers. The results of Table 1 also confirm the fact that the pole sensitivity sum measure υ_s gives conservative word length estimate. Specifically, it suggests the minimum stable bit length of 12 bits for $\mathbf{X}_s$, while a bit length of 10 bits is sufficient for $\mathbf{X}_s$ if the other stability measures are used to estimate the minimum stable bit length of $\mathbf{X}_s$.

6. Conclusions

A new μ-based FWL stability measure has been derived. This FWL stability measure can be evaluated conveniently and the corresponding optimal FWL realization problem can be solved efficiently by means of LMI techniques. Thus our proposed design is also computationally more attractive than many existing optimal FWL realization problems where non-convex optimization problems must be solved. As future research work, it is worth rearranging the full-structured uncertainty problem (24) of $(s+m)(t+m)$ independent uncertainties with a reduced size standard μ problem (of $2m + s + t - 1$ independent uncertainties (Young, 1997)).

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