Modeling and Control of Bi-directional Discrete Linear Repetitive Processes

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Abstract—Repetitive processes are characterized by a series of sweeps or passes through a set of dynamics defined over a finite duration where the output produced on any pass acts as a forcing function on, and hence contributes to, the dynamics of the next pass. The resulting control problem is that the output sequence of pass profiles can contain oscillations that increase in amplitude in the pass-to-pass direction. This paper considers bi-directional operation, i.e. a pass is completed and at the end the next one begins but in the opposite direction. In particular, a model for such a process in the case of discrete dynamics is first proposed and new results on stability and control law design for stabilization and performance developed.

Index Terms—uni-directional and bi-directional repetitive process, linear matrix inequality (LMI), required pass profile

I. INTRODUCTION

The unique characteristic of a repetitive, or multipass, process is a series of sweeps, termed passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass an output, termed the pass profile, is produced which acts as a forcing function on, and hence contributes to, the dynamics of the next pass profile. This, in turn, leads to the unique control problem in that the output sequence of pass profiles generated can contain oscillations that increase in amplitude in the pass-to-pass direction.

Physical examples include long-wall coal cutting and metal rolling operations [1] (and the references given in this article to other examples). In long-wall coal cutting, the cutting machine is hauled along the coal face resting on a so-called semi-flexible conveyor and the objective during this operation is to steer the cutting head within the undulating confines of the coal seam to cut (or extract) the maximum amount of coal without penetrating the stone/coal interface. The pass profile here is the height of the floor of the coal seam above some datum. At the end of each pass, the machine is hauled back to the start at (relatively) high speed and then hydraulic rams are used to push the complete installation (machine but the supporting infrastructure) over so that the conveyor and machine now rest on the floor profile cut during the previous pass.

Simple geometric considerations now confirm that this is indeed a repetitive process and, in particular, the pass profile produced on any pass clearly has a major influence on that produced on the next (and damage to this by the machines weight during the pusher phase is a major contributing factor). Moreover, if the floor profile becomes too undulating (or rough) then productive work must be halted in order for the floor profile to be levelled out. This is the reason for the stop-start operating pattern in coal cutting.

The analysis and control law design methods currently available for repetitive processes all apply to uni-directional operation. In particular, the process completes a pass and then is reset to the starting position before the start of the next pass and hence dynamics are only produced in one direction, e.g. from left to right across the coal face in the mining application. Moreover, in some applications the time between completing one pass and starting the next can, if required, be used to pre-compute at least part of the control input for the next pass or to do pre-conditioning on the previous pass profile. In some applications, however, it is bi-directional operation which takes place. For example, in metal rolling uni-directional operation means that the workpiece is passed through the rolling devices and then it is returned to the starting side for the next pass and so on. This can be very wasteful and an alternative is to simply reverse the operation at the end of each pass, i.e. pass the workpiece through the rolling devices in the opposite direction.

This paper first proposes a model for a bi-directional linear repetitive processes with discrete dynamics. Then we show how this can be written in a form which enables stability to be examined using existing theory and give new results on control law design for both stability and suitably specified performance objectives.

Throughout this paper, the null matrix and the identity matrix are denoted by 0 and I respectively. Moreover, $M > 0$ ($M < 0$) denotes a real symmetric positive (negative) definite matrix and diag $(\cdot)$ a block-diagonal matrix.

II. MODELLING AND STABILITY ANALYSIS/CONTROL LAW DESIGN

The state-space model of a discrete linear repetitive process is of the form

\begin{align}
    x_{k+1}(p+1) &= Ax_{k+1}(p) + Bu_{k+1}(p) + B_0y_k(p) + D_1w_k(p) \\
    y_{k+1}(p) &= Cx_{k+1}(p) + Du_{k+1}(p) + D_0y_k(p) + D_1w_k(p) \\
\end{align}

over $k = 0, 1, \ldots$ and $p = 0, 1, \ldots, (\alpha - 1)$ where $x_{k+1}(p) \in \mathbb{R}^n$ is the state vector, $u_{k+1}(p) \in \mathbb{R}^m$ is the input (control) vector, $y_k(p) \in \mathbb{R}^m$ is the output or pass profile vector, $w_k(p) \in \mathbb{R}^p$ is a disturbance signal (noting the unique feature of these processes, a current pass disturbance signal is nothing new from an analysis point of view) from the previous pass,
$k = 0, 1, \ldots$ is the pass number, $p$ is a point along the pass, and $\alpha$ is the finite pass length. To complete the process description, it is necessary to specify the boundary conditions, i.e. the initial state vector on each pass and the initial pass profile. Without loss of generality here we can assume that these are zero or have known constant entries.

This model represents uni-directional dynamics, i.e. a pass is completed, the process is reset and the next pass begins. To model a bi-directional process, i.e. successive passes are completed, the process is reset and the next pass begins. Such dynamics do not arise in standard linear systems and hence a standard lifting approach cannot be applied to the bi-directional case. Instead, we use an approach where part of the current pass state vector indexes backwards resulting in the following state-space model to which the existing stability theory can be applied (where the disturbance terms have been omitted as they play no role in stability analysis)

\[
X_{k+1}(p) = \hat{A}X_{k+1}(p) + \hat{B}U_{k+1}(p) + \hat{B}_0Y_k(p) \\
Y_{k+1}(p) = \hat{C}X_{k+1}(p) + \hat{D}U_{k+1}(p) + \hat{D}_0Y_k(p)
\]

over $k = 0, 1, \ldots$ and $p = 0, 1, \ldots, \alpha$, where

\[
\hat{A} = \begin{bmatrix} A_1 & 0 \\ B_0C_1 & A_2 \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B_1 \\ B_0D_1 \\ B_2 \end{bmatrix} \\
\hat{B}_0 = \begin{bmatrix} B_{01} \\ B_{02}D_{01} \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} C_0 \\ D_0C_1 \\ C_2 \end{bmatrix} \\
\hat{D} = \begin{bmatrix} D_0D_1 \\ D_2 \end{bmatrix}, \quad \hat{D}_0 = D_0D_{01}
\]

with the rule that the components out of range, i.e. $x_{2k+1}(\alpha+1)$ and $x_{2k+2}(-1)$, are discarded (this will be done without specific reference where required in what follows).

The key feature in (4) is the modified forward/backward ($\pm$) shift which is, however, completely recurrent. This allows us to treat it as a standard shift and, in effect, apply the stability theory for the uni-directional case.

It is also possible to develop a 2D transfer-function matrix of the process dynamics, where there are a number of possible input/output couplings that could be of interest. It turns out in opposite directions. Such updating is not present in other classes of 2D discrete linear systems and note also that second equation in (2) is not anti-causal since the process is recurrent.

Consider the question of how to determine the stability properties of a bi-directional repetitive process of the form considered here. Then the existing stability theory is for a uni-directional process is based on an abstract model in a Banach space setting [1] and consists of two concepts termed asymptotic stability and stability along the pass respectively. Recalling the control problem for these processes, the stability theory is formulated as the requirement that bounded inputs produce bounded outputs, i.e. sequences of pass profiles. Asymptotic stability requires this property over the, finite by definition, pass length and stability along the pass is stronger in that it demands this property for all possible values the pass length. The stability theory for discrete linear repetitive processes of the form (1) is well developed and many sets of conditions are known [1]. Also for discrete processes a lifting approach can be used to obtain easily computed conditions for asymptotic stability (and partially for stability along the pass since asymptotic stability is a necessary condition for this property) and this extends to enable the development of algorithms for control law design to ensure this property and/or meet specified control objectives.

The process state dynamics considered here consists of discrete updating with two equations with, critically, contributions from the previous pass dynamics where these evolve along the pass number.
that the 2D transfer-function matrix for all of these cases can be developed in the same way. Here as an exemplar, we detail the case when the output of interest is the even numbered pass profiles, i.e. \( y_{2k+2}(\alpha - p) \) in (3) where no loss of generality arises from assuming zero boundary conditions and disturbances.

Introduce the shift operators applied, e.g. to \( y_k(p) \) and \( x_k(p) \) respectively, as

\[
z_1 y_{2k}(p) = y_{2k+1}(p), \quad z_2 x_k(p) = x_k(p+1)
\]

and, for example, let \( Z\{x_k(p)\} \) denote the result of applying them to \( x_k(p) \). Then in this case

\[
Z \{ y_{2k+2}(\alpha - p) \} = G(z_1, z_2) \left[ \begin{array}{c} u_{2k+1}(p) \\ u_{2k+2}(\alpha - p) \end{array} \right]
\]

where the 2D transfer-function matrix \( G(z_1, z_2) \) is given by

\[
G(z_1, z_2) = \left( I_n - \left( \bar{C} \left( z_2^{\pm 1} I_{2n} - \bar{A} \right)^{-1} \bar{B}_0 + \bar{D}_0 \right) z_1^{-2} \right)^{-1} \left( \bar{C} \left( z_2^{\pm 1} I_{2n} - \bar{A} \right)^{-1} \bar{B} + \bar{D} \right)
\]

where

\[
z_2^{\pm 1} I_{2n} = \begin{bmatrix} z_2 I_n & 0 \\ 0 & z_2^{-1} I_n \end{bmatrix}
\]

Theorem 1 below characterizes stability along the pass for the bi-directional processes considered here. Its proof relies on the following result [2].

**Lemma 1:** [2] Let \( W, L \) and \( V \) be given matrices of appropriate dimensions with \( W = W^T \) and \( V > 0 \). Then the matrix inequality

\[
W + L^T VL < 0
\]

holds if, and only if,

\[
\begin{bmatrix} W & L^T G \\ G^T L & V - G - G^T \end{bmatrix} < 0
\]

where \( G \) is an arbitrary matrix of the same dimensions as \( V \).

**Theorem 1:** A bi-directional discrete repetitive process described by (2) and (3) and written in the form (4) is stable along the pass if there exists a matrix \( X > 0 \), and non-singular matrices \( \hat{V} \) and \( \hat{Z} \) such that

\[
\begin{bmatrix} -X & \Psi \hat{Z} & \Phi \hat{V} \\ \hat{Z}^T \Psi^T & -\hat{Z} - \hat{Z}^T & \Theta \hat{V}^T \\ \hat{V}^T \Phi^T & \hat{V}^T \Theta^T & X - \hat{V} - \hat{V}^T \end{bmatrix} < 0
\]

where \( X = \text{diag} \left( X_1, X_2 \right) \) and

\[
\Phi = \begin{bmatrix} A_1 & 0 & B_{01} \\ 0 & A_2 & B_{02} \\ 0 & C_2 & B_{02} \end{bmatrix}, \quad \Psi = \begin{bmatrix} 0 & 0 & 0 \\ B_{02} & 0 & B_{02} \\ D_{02} & 0 & D_{02} \end{bmatrix}, \quad \Theta = \begin{bmatrix} C_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_{01} \end{bmatrix}
\]

**Proof:** First introduce the so-called augmented plant matrix

\[
\hat{\Phi} = \begin{bmatrix} \bar{A} & \bar{B}_0 \\ \bar{C} & \bar{D}_0 \end{bmatrix} = \Phi + \Psi \Theta
\]

and from [1] we have that stability along the pass in this case holds if the following condition holds (i.e. the so-called 2D Lyapunov equation condition which results from a Lyapunov function based approach to stability)

\[
X > 0, \quad \hat{\Phi}^T X \hat{\Phi} - X < 0
\]

where \( X = X_1 \oplus X_2 \) and \( \oplus \) denotes the direct sum, i.e. \( X = \text{diag} \left( X_1, X_2 \right) \). These last matrix inequalities can be rewritten as

\[
X > 0 \quad \text{and} \quad -X + (\Phi + \Psi \Theta) X (\Phi + \Psi \Theta)^T
\]

\[
= \begin{bmatrix} I & \Psi \\ Z^T \Psi^T + \Theta \Psi \Phi^T & \Theta \Psi \Theta^T - Z - Z^T \end{bmatrix} < 0
\]

where \( Z \) is an additional non-singular matrix of compatible dimensions. Hence if

\[
X > 0 \quad \text{and} \quad \begin{bmatrix} -X + \Phi \Psi \Phi^T & \Psi \hat{Z} + \Phi \Psi \Theta^T \\ Z^T \Psi^T + \Theta \Psi \Phi^T & \Theta \Psi \Theta^T - Z - Z^T \end{bmatrix}
\]

\[
= \begin{bmatrix} I & \Psi \hat{Z} \Psi^T - Z - Z^T \end{bmatrix} + \begin{bmatrix} \Phi & \Psi \Theta \end{bmatrix} X \begin{bmatrix} \Phi^T & \Theta^T \end{bmatrix} < 0
\]

holds then (10) also holds. Applying Lemma 1 to this last inequality yields (8). ■

Suppose now that the following control law is to be used to ensure stability along the pass

\[
u_{2k+1}(p) = K_a x_{2k+1}(p) + K_b y_{2k}(p)
\]

\[
u_{2k+2}(\alpha - p) = K_c x_{2k+2}(\alpha - p) + K_d y_{2k+2}(\alpha - p)
\]

over \( k = 0, 1, \ldots \) and \( p = 0, 1, \ldots, \alpha \) or, in lifted form,

\[
u_{k+1}(p) = \hat{K}_1 x_{k+1}(p) + \hat{K}_2 y_k(p)
\]

where

\[
\hat{K}_1 = \begin{bmatrix} K_a \\ K_d (C_1 + D_1 K_a) \\ K_c \end{bmatrix}, \quad \hat{K}_2 = \begin{bmatrix} K_b \\ K_d (D_0 + D_1 K_b) \end{bmatrix}
\]

Also introduce

\[
\hat{B} = \begin{bmatrix} B_1 & 0 & B_3 \\ 0 & B_2 & 0 \\ 0 & D_2 & 0 \end{bmatrix}, \quad \hat{B}_a = \begin{bmatrix} 0 & 0 & 0 \\ B_2 & 0 & B_2 \\ D_2 & 0 & D_2 \end{bmatrix}
\]

\[
\hat{B}_b = \begin{bmatrix} D_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & D_1 \end{bmatrix}
\]

Then we have the following result.

**Theorem 2:** Suppose that a control law of the form (12) is applied to a bi-directional discrete repetitive process described by (2) and (3) and written in the form (4). Then the resulting process is stable along the pass if there exists a matrix \( X > 0 \),
where $X = \text{diag} (X_1, X_2)$, $\bar{V} = \text{diag} (V_1, V_2, V_3)$, $\bar{Z} = \text{diag} (Z_1, Z_2, Z_3)$, $\bar{L} = \text{diag} (L_1, L_2, L_3)$, $\bar{N} = \text{diag} (N_1, N_2, N_3)$. If this condition holds, stabilizing control law matrices are given by

$$\begin{align*}
\bar{K}_1 & = \bar{L} \bar{V}^{-1} = \text{diag} (L_1 V_1^{-1}, L_2 V_2^{-1}, L_3 V_3^{-1}) \\
\bar{K}_2 & = \bar{N} \bar{Z}^{-1} = \text{diag} (N_1 Z_1^{-1}, N_2 Z_2^{-1}, N_3 Z_3^{-1})
\end{align*}$$

(15)

Proof: This result is proved by interpreting the result of Theorem 1 in terms of the controlled process dynamics and hence the details are omitted here.

III. CONTROL FOR PERFORMANCE AND DISTURBANCE REJECTION

Suppose that stability along the pass holds. Then for the processes considered here this guarantees the existence of the so-called limit profile, i.e. the sequence of pass profiles generated by $\{y_k\}$ converges in $k$ to the so-called limit profile which is denoted by $y_{\infty}$ which is also described by a 1D discrete linear systems state-space model. Hence one overall control objective is to achieve a specified limit profile with appropriate (1D) transient response characteristics.

As a first attempt at control law design, we consider the case when the disturbance terms are assumed to be the same for each pair (i.e. forward and backward respectively in the terminology associated with (2) and (3)) of passes i.e. the disturbances are periodic with period equal to $2\alpha$.

Suppose that $y_{ref}(p)$ denotes the required limit profile and that as $k \to \infty$, $x_{2k+1}(p) \to x_{1\infty}(p)$, $x_{2k+2}(\alpha - p) \to x_{2\infty}(\alpha - p)$, $u_{2k+1}(p) \to u_{1\infty}(p)$, $u_{2k+2}(\alpha - p) \to u_{2\infty}(\alpha - p)$, and define the total tracking error for the forward and backward passes (at point $p$ on the pass number $2k+1$) and backward pass (at point $\alpha - p$ on the pass number $2k+2$) as

$$\begin{align*}
\chi_{2k+1}(p) & = \sum_{i=0}^{2k+1} (y_{ref}(p) - y_i(p)) \\
\chi_{2k+2}(\alpha - p) & = \sum_{i=0}^{2k+2} (y_{ref}(\alpha - p) - y_i(\alpha - p))
\end{align*}$$

(16)

and

$$\begin{align*}
\chi_{2k+1}(p) & = \sum_{i=0}^{2k+1} (y_{ref}(p) - y_i(p)) \\
\chi_{2k+2}(\alpha - p) & = \sum_{i=0}^{2k+2} (y_{ref}(\alpha - p) - y_i(\alpha - p))
\end{align*}$$

(17)

respectively over $k = 0, 1, \ldots$ and $p = 0, 1, \ldots, \alpha$, or

$$\begin{align*}
\chi_{2k+1}(p) & = \chi_{2k}(p) + y_{ref}(p) - y_{2k+1}(p) \\
\chi_{2k+2}(\alpha - p) & = \chi_{2k+1}(\alpha - p) + y_{ref}(\alpha - p) - y_{2k+2}(\alpha - p)
\end{align*}$$

(18)

over $k = 0, 1, \ldots$ and $p = 0, 1, \ldots, \alpha$. Define also the following incremental vectors

$$\begin{align*}
\bar{x}_{2k+1}(p) & = x_{2k+1}(p) - x_{1\infty}(p) \\
\bar{x}_{2k+2}(\alpha - p) & = x_{2k+2}(\alpha - p) - x_{2\infty}(\alpha - p)
\end{align*}$$

$$\begin{align*}
\bar{u}_{2k+1}(p) & = u_{2k+1}(p) - u_{1\infty}(p) \\
\bar{u}_{2k+2}(\alpha - p) & = u_{2k+2}(\alpha - p) - u_{2\infty}(\alpha - p)
\end{align*}$$

\begin{align*}
\bar{y}_{2k+1}(p) & = y_{2k+1}(p) - y_{ref}(p) \\
\bar{y}_{2k+2}(\alpha - p) & = y_{2k+2}(\alpha - p) - y_{ref}(\alpha - p)
\end{align*}$$

and

$$\begin{align*}
\bar{x}_{2k+1}(p) & = x_{2k+1}(p) - x_{1\infty}(p) \\
\bar{x}_{2k+2}(\alpha - p) & = x_{2k+2}(\alpha - p) - x_{2\infty}(\alpha - p)
\end{align*}$$

over $k = 0, 1, \ldots$ and $p = 0, 1, \ldots, \alpha$. Then, we can construct the so-called augmented incremental model as

$$\begin{align*}
\bar{x}_{2k+1}(p+1) & = A_1 \bar{x}_{2k+1}(p) + B_1 \bar{u}_{2k+1}(p) \\
& + [B_{01} 0] \bar{y}_{2k+1}(p) \\
\bar{x}_{2k+2}(\alpha - p - 1) & = A_2 \bar{x}_{2k+2}(\alpha - p) + B_2 \bar{u}_{2k+2}(\alpha - p) \\
& + [B_{02} 0] \bar{y}_{2k+2}(\alpha - p) - \bar{y}_{ref}(\alpha - p) \\
\bar{x}_{2k+1}(p) & = \chi_{2k+1}(p) - \chi_{1\infty}(p) \\
\bar{x}_{2k+2}(\alpha - p) & = \chi_{2k+2}(\alpha - p) - \chi_{2\infty}(\alpha - p)
\end{align*}$$

(19)

over $k = 0, 1, \ldots$ and $p = 0, 1, \ldots, (\alpha - 1)$, and

$$\begin{align*}
\bar{y}_{2k+1}(p) & = C_1 \bar{x}_{2k+1}(p) - C_1 \bar{x}_{2k+1}(p) \\
& + [D_1 0] \bar{u}_{2k+1}(p) + [D_{01} 0] \bar{y}_{2k+1}(p) \\
\bar{y}_{2k+2}(\alpha - p) & = C_2 \bar{x}_{2k+2}(\alpha - p) - C_2 \bar{x}_{2k+2}(\alpha - p) \\
& + [D_2 0] \bar{u}_{2k+2}(\alpha - p) + [D_{02} 0] \bar{y}_{2k+2}(\alpha - p) - \bar{y}_{ref}(\alpha - p) \\
\bar{y}_{2k+1}(p) & = C_1 \bar{x}_{2k+1}(p) - C_1 \bar{x}_{2k+1}(p) \\
& + [D_1 0] \bar{u}_{2k+1}(p) + [D_{01} 0] \bar{y}_{2k+1}(p)
\end{align*}$$

(20)

over $k = 0, 1, \ldots$ and $p = 0, 1, \ldots, (\alpha - 1)$. This model has the same structure as (2) and (3) but the influence of the disturbances has been completely decoupled. Moreover, it has a periodic disturbance along the pass property and if this holds then

$$\begin{align*}
\lim_{k \to \infty} \bar{x}_{2k+1}(p) & = 0, \lim_{k \to \infty} \bar{x}_{2k+2}(\alpha - p) = 0 \\
\lim_{k \to \infty} \bar{y}_{2k+1}(p) & = 0, \lim_{k \to \infty} \bar{y}_{2k+2}(\alpha - p) = 0
\end{align*}$$

Hence, $\lim_{k \to \infty} \bar{y}_{2k+2}(\alpha - p) = y_{ref}(\alpha - p)$ as $k \to \infty$ and we achieve our objectives.
over \( k = 0, 1, \ldots \) and \( p = 0, 1, \ldots, \alpha \). If we assume that
\[
\begin{align*}
\alpha_{2k+1}(p) &= K_e x_{2k+1}(p) \\
&\quad + [K_{c1} K_{c2}] \begin{bmatrix} y_2(p) \\ \chi_{2k}(p) \end{bmatrix} \\
\alpha_{2k+2}(\alpha-p) &= K_f x_{2k+2}(\alpha-p) \\
&\quad + [K_{f1} K_{f2}] \begin{bmatrix} y_{2k+1}(\alpha-p) \\ \chi_{2k+1}(\alpha-p) \end{bmatrix}
\end{align*}
\]
(22)
over \( k = 0, 1, \ldots \) and \( p = 0, 1, \ldots, \alpha \) then it follows immediately from (21) that
\[
\begin{align*}
u_{1\infty}(p) &= K_e x_{1\infty}(p) - K_{c1} y_{ref}(p) - K_{c2} \chi_{2\infty}(p) = 0 \\
u_{2\infty}(\alpha-p) &= K_f x_{2\infty}(\alpha-p) - K_{f1} y_{ref}(\alpha-p) - K_{f2} \chi_{2\infty}(\alpha-p) = 0
\end{align*}
\]
over \( k = 0, 1, \ldots \) and \( p = 0, 1, \ldots, \alpha \) and hence we can apply finally to the process the direct, non-incremental control law of (22) which is composed of two terms. The former is static (the proportional action controller), the latter is dynamic (the integral action controller). Note here that we do not need to know the disturbances exactly, it is enough to know that they are periodic with the period equal to the pass length \( \alpha \).

Finally, the control law matrices are obtained by interpreting Theorem 2 in terms of the incremental model with the control law applied.

IV. NUMERICAL EXAMPLE

Consider the case when \( \alpha = 100 \) and
\[
\begin{bmatrix}
A_1 & B_{01} & B_1 \\
C_1 & D_{01} & D_1 \\
A_2 & B_{02} & B_2 \\
C_2 & D_{02} & D_2
\end{bmatrix}
= 10^{-3} \begin{bmatrix}
963.8554 & 96.3855 & 13.5542 & -0.3614 \\
-361.4458 & 963.8554 & 135.5422 & -3.6145 \\
963.8554 & 96.3855 & 638.5542 & -0.3614 \\
981.5951 & 98.1595 & 6.9018 & -0.1840 \\
-184.0491 & 981.5951 & 69.0184 & -1.8405 \\
981.5951 & 98.1595 & 631.9018 & -0.1840
\end{bmatrix}
\]

Fig. 2 shows the disturbance signals. Finally, the reference signal is shown in Fig. 3 (upper plot). The linear matrix inequality of Theorem 2 is feasible in this case and
\[
\begin{bmatrix}
K_e & K_{c1} & K_{c2} \end{bmatrix} = \begin{bmatrix}
1557.2200 & 266.6667 \\
206.6831 & -77.4039 \\
3154.5404 & 533.3333 \\
\end{bmatrix}
\]
\[
\begin{bmatrix}
K_f & K_{f1} & K_{f2} \end{bmatrix} = \begin{bmatrix}
369.7512 & -152.0100 \\
0.10 & 0.20 \\
-0.20 & -0.20
\end{bmatrix}
\]

The lower plot in Fig. 3 confirms that the overall design task is achieved and the next stage would be to attempt to tune the design.
V. CONCLUSIONS

The major contributions in this short paper are i) the application of lifting techniques to transform the bi-directional dynamics into those of an equivalent uni-directional repetitive process model and hence the availability of a stability theory and control law design to achieve this basic property, and ii) the first results on stability plus performance in the case when there are disturbances present which are assumed to be periodic over twice the pass length. Also there is clearly much work to do before these results can be evaluated on physical examples. This includes a wide range of algorithms for control law design, robustness analysis, and allowing for more general disturbance terms. Also there could well be alternative lifting type approaches, such as that proposed in [3] for iterative learning control, which may be advantageous for bi-directional repetitive processes. Finally, extending the control design analysis to allow for different reference signals in each direction needs to be addressed.

REFERENCES