Robust $\mathcal{H}_\infty$ Control for Model-Based Networked Control Systems with Uncertainties and Packet Dropouts

Dongxiao Wu$^1$, Jun Wu$^1$, Sheng Chen$^2$

$^1$Institute of Cyber-Systems and Control
Zhejiang University

$^2$School of Electronics and Computer Science
University of Southampton

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An NCS is a control system in which the control loop is closed via a shared communication network.

The advantages:
- Low installation cost.
- Reducing system wiring.
- Easy maintenance.

The inherited problems:
- Packet dropout.
- Packet delay.
- Bandwidth constraint.
Robust $\mathcal{H}_\infty$ control has been investigated for the NCS with delays [8,13–15].

Most of existing works use fixed controller.

In Model-based NCS, the network is only located between sensor and controller [16,17].
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- Study robust $\mathcal{H}_\infty$ control for NCS with packet dropouts.
- Consider the generic MB-NCS
  - the plant has time-varying norm-bounded parameter uncertainties;
  - packet dropouts occur in both the S/C and C/A channels.
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Model Based NCS

Figure: Networked control system $\hat{P}_K$. 
Plant Description

The plant $\hat{P}$:

\[
\begin{cases}
    x(k + 1) = [A + \Delta A(k)]x(k) \\
    + [B + \Delta B(k)]u(k) + B_w w(k), & \forall k \in \mathbb{N}.
\end{cases}
\]

The states $z(k) = Cx(k) + Du(k)$, $\forall k \in \mathbb{N}$.

The time-varying parameter uncertainties satisfy:

\[
[\Delta A(k) \Delta B(k)] = M F(k) [N_a \ N_b].
\]

with $F^T(k)F(k) \leq I$. 

\[\text{Motivations}\] \[\text{Problem Formulation}\] \[\text{Main Results}\] \[\text{Example}\] \[\text{Conclusions}\]
Network Assumptions

- Packet dropouts indicators:
  \[
  \begin{align*}
  \theta_{k+1}^s & \in \{0, 1\}, \quad \text{in S/C channel;} \\
  \theta_k^a & \in \{0, 1\}, \quad \text{in C/A channel.}
  \end{align*}
  \]

  Then the system index \( r_k = f(\theta_{k+1}^s, \theta_k^a) \in \mathcal{N} \triangleq \{1, 2, 3, 4\} \).

- \( r_k \) is driven by Markov chain.

- TCP-like protocol.
Controller is running as:

\[
\hat{x}(k+1) = \theta^s_{k+1} x(k+1) + (1 - \theta^s_{k+1})(A\hat{x}(k) + Bu(k))
\]

\[
= \begin{cases} 
  x(k+1), & \theta^s_{k+1} = 1, \\
  A\hat{x}(k) + Bu(k), & \theta^s_{k+1} = 0.
\end{cases}
\]

where \( u(k) = \theta^a_k \hat{u}(k) \) with \( \hat{u}(k) = K_{r_k} \hat{x}(k), \ r_k \in \mathcal{N} \).

State feedback gain matrices \( K_i, \ i \in \mathcal{N} \), but only \( K_3 \) and \( K_4 \) are needed, as \( \theta^a_k = 0 \) for \( i = 1, 2 \).
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NCS Dynamics

The NCS $\hat{P}_K$ in the form of Markovian jump linear system:

$$
\begin{bmatrix}
\bar{x}(k + 1) \\
z(k)
\end{bmatrix} =
\begin{bmatrix}
\bar{A}_{r_k}(k) & \bar{B}_{r_k} \\
\bar{C}_{r_k} & 0
\end{bmatrix}
\begin{bmatrix}
\bar{x}(k) \\
w(k)
\end{bmatrix},
\quad r_k \in \mathcal{N}
$$

where $e(k) = x(k) - \hat{x}(k), \quad \bar{x}(k) \triangleq [x^T(k) \ e^T(k)]^T$. 
NCS Dynamics

\[ \overline{A}_i(k) = \Phi_i + \overline{M} \overline{F}(k)\Gamma_i, \quad i \in \mathcal{N}, \text{ where} \]

\[
\Phi_i = \begin{bmatrix}
A + \theta^a_k B K_i & -\theta^a_k B K_i \\
0 & (1 - \theta^s_{k+1}) A
\end{bmatrix},
\]

\[
\Gamma_i = \begin{bmatrix}
N_a + \theta^a_k N_b K_i & -\theta^a_k N_b K_i \\
(1 - \theta^s_{k+1})(N_a + \theta^a_k N_b K_i) & -(1 - \theta^s_{k+1})\theta^a_k N_b K_i
\end{bmatrix}.
\]

\[ \overline{M} = \text{diag}\{M, M\}, \quad \overline{F}(k) = \text{diag}\{F(k), F(k)\}. \]

Only \( K_3 \) and \( K_4 \) are needed, as \( \theta^a_k = 0 \) for \( i = 1, 2 \).
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Robust stability

**Theorem 1**: The NCS $\hat{P}_K$ with $w(k) \equiv 0$ and driven by the Markov chain $r_k \in \mathcal{N}$ is robustly stochastically stable if there exist scalars $\epsilon_i > 0$ and matrices $X_i > 0$ for $i \in \mathcal{N}$ such that $\forall i \in \mathcal{N}$

$$
\begin{bmatrix}
-X_i & X_i \Phi_i^T W_i & X_i \Gamma_i^T \\
* & \epsilon_i W_i - X & 0 \\
* & * & -\epsilon_i I
\end{bmatrix} < 0,
$$

where $\overline{M}$, $\Phi_i$ and $\Gamma_i$ are given by the NCS dynamics, while

$$W_i = \begin{bmatrix}
\sqrt{\rho_{i1}} I & \sqrt{\rho_{i2}} I & \sqrt{\rho_{i3}} I & \sqrt{\rho_{i4}} I
\end{bmatrix},$$

$$\overline{W}_i = W_i^T \overline{M} \overline{M}^T W_i,$$

$$X = \text{diag}\{X_1, X_2, X_3, X_4\}.$$
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Robust Stabilisation Control

**Theorem 2:** The NCS $\hat{P}_K$ with $w(k) \equiv 0$ and driven by the Markov chain $r_k \in \mathcal{N}$ is robustly stochastically stable if there exist $\epsilon_i > 0$, $Q_i > 0$ and $Y_i$ for $i \in \mathcal{N}$ such that $\forall i \in \mathcal{N}$

$$
\begin{bmatrix}
-\tilde{Q}_i & \tilde{\Phi}^T_i W_i & \tilde{\Gamma}^T_i \\
* & \epsilon_i W_i - \tilde{Q} & 0 \\
* & * & -\epsilon_i \mathbf{I}
\end{bmatrix} \triangleq \Theta_i < 0,
$$

where $\tilde{Q}_i = \text{diag}\{Q_i, Q_i\}$, $\tilde{Q} = \text{diag}\{\tilde{Q}_1, \tilde{Q}_2, \tilde{Q}_3, \tilde{Q}_4\}$,

$$
\tilde{\Phi}_i = \begin{bmatrix}
A Q_i + \theta^a_k B Y_i & -\theta^a_k B Y_i \\
0 & (1 - \theta^s_{k+1})A Q_i
\end{bmatrix},
$$

$$
\tilde{\Gamma}_i = \begin{bmatrix}
N_a Q_i + \theta^a_k N_b Y_i & -\theta^a_k N_b Y_i \\
(1 - \theta^s_{k+1})(N_a Q_i + \theta^a_k N_b Y_i) & -(1 - \theta^s_{k+1})\theta^a_k N_b Y_i
\end{bmatrix}.
$$

In this case, state feedback gain matrices $K_i = Y_i Q_i^{-1}$, $i = 3, 4$. 
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**Theorem 3:** Given a scalar $\gamma > 0$, the NCS $\hat{P}_K$ driven by the Markov chain is robustly stochastically stable with disturbance attenuation level $\gamma$, if there exist scalars $\epsilon_i > 0$, matrices $Q_i > 0$ and $Y_i$ for $i \in \mathcal{N}$ such that $\forall i \in \mathcal{N}$

$$
\begin{bmatrix}
-\tilde{Q}_i & 0 & \tilde{\Phi}_i^T W_i & \tilde{\Gamma}_i^T & \tilde{C}_i^T \\
* & -\gamma^2 I & B_i^T W_i & 0 & 0 \\
* & * & \epsilon_i W_i - \tilde{Q} & 0 & 0 \\
* & * & * & -\epsilon_i I & 0 \\
* & * & * & * & -I
\end{bmatrix} < 0,
$$

where $\tilde{C}_i = \begin{bmatrix} CQ_i + \theta_k^a D Y_i & -\theta_k^a D Y_i \end{bmatrix}$, $W_i$, $\overline{W}_i$, $\tilde{Q}_i$, $\tilde{Q}$, $\tilde{\Phi}_i$ and $\tilde{\Gamma}_i$ are given before.

In this case, state feedback gain matrices $K_i = Y_i Q_i^{-1}$, $i = 3, 4$. 
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Plant and Network

- Unstable uncertain NCS of $x(t) \in \mathbb{R}^3$, $u(t) \in \mathbb{R}^2$, $z(t) \in \mathbb{R}$ and $w(t) \in \mathbb{R}$, with

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0.4 & 0.6 & 0.2 \\ 1 & 0.2 & -1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 & 1 \\ 0.5 & 0.2 \\ 1 & 0.4 \end{bmatrix}, \quad B_w = \begin{bmatrix} 0.1 \\ 0.1 \\ -0.2 \end{bmatrix},$$

$$M = \begin{bmatrix} 0.1 \\ -0.1 \\ 0.2 \end{bmatrix}, \quad C = \begin{bmatrix} 0.2 & 0.3 & 0.3 \end{bmatrix}, \quad D = \begin{bmatrix} 0.7 & 0.9 \end{bmatrix},$$

$$N_a = \begin{bmatrix} 0.5 & 0.2 & 0.3 \end{bmatrix}, \quad N_b = \begin{bmatrix} 0.1 & 0.2 \end{bmatrix}.$$
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Give disturbance attenuation level $\gamma = 0.45$.

According to Theorem 3, we can obtain $\epsilon_i$ and $Q_i$, $1 \leq i \leq 4$, as well as $Y_3$ and $Y_4$.

Thus, derive state feedback gain matrices

$$K_3 = \begin{bmatrix} 0.0004 & 0.0170 & 0.0331 \\ -0.0707 & -0.0964 & -0.1062 \end{bmatrix},$$

$$K_4 = \begin{bmatrix} -0.5959 & -0.1417 & 0.4485 \\ 0.3396 & -0.3326 & -0.5625 \end{bmatrix},$$

as the solution of robust $H_\infty$ control problem.
We have studied a generic class of model-based NCSs, where

- the plant has time-varying norm-bounded uncertainties;
- both the sensor-to-controller and controller-to-actuator channels experience random packet dropouts.

We have derived sufficient conditions, in the form of LMIs, for

- guaranteeing the robust stochastic stability;
- synthesising the stochastic stabilisation controller;
- designing the $H_\infty$ controller.