Complexity reduction of Nonlinear Systems

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***Abstract*— A common problem in nonlinear control is the need to consider systems of high complexity. Here we consider systems, which although may be low order, have high complexity due to a complex right hand side of a differential equation (e.g. a right hand side which has many terms – such systems arise from coordinate transformations in constructive nonlinear control designs). This contribution develops a systematic method for the reduction of this complexity, complete with error bounds. In the case when the underling nonlinear system input/output operator is stable and differentiable, the operator Taylor expansion, truncated after a finite number of terms, is taken as the approximation. If the nonlinear system i/o operator is not stable, but admits a coprime factorizations, the Taylor approximation is made to both coprime factors. By bounding the gap between the polynomial system and the original nominal plant, and applying gap robust stability approaches, it is proved that local stability of approximation implies the local stability of the underlining nonlinear systems, and explicit robust stability margins and performance bounds obtained. For systems specified by a finite dimensional first order differential equation, the first order approximant is the system linearisation and the higher order approximants have greater state dimension but with polynomial right hand sides.**

1. INTRODUCTION

In linear control, model order reduction plays an important role in reducing the complexity (e.g. order) of control designs. In nonlinear control the need for complexity reduction arises from both model order and the complexity of the nonlinear terms of the equations themselves. This task is concerned with the latter need. As motivation, observe that most constructive control designs (e.g. backstepping [14], feedback linearisation [13]) involve repeated partial differentiation of the right hand side of the system model to construct a normal form, and consequently the number of terms in the closed form expression for the control action (in the original system coordinates) grows exponentially with the order of the system. For systems of moderate order, it is necessary to use computer algebra packages to calculate the closed forms, and the resulting expressions can become unusable for systems of quite low order, simply due to the number of terms involved.

In this paper, approximations based on Taylor expansions of the i/o plant operator (based on Fréchet differentiation of the i/o operator), are related to polynomial approximants to the right-hand side of an ordinary differential system describing the nominal plant. The first order approximation of this type is well-known since linearisations of ordinary differential systems correspond to the Fréchet operator differential of the associated i/o operator of the plant([12]), which states that if the linearisation is stable, then the nonlinear plant is locally stable.

The approach we use is the grounded in the nonlinear gap metric theory [1], [3], [7], [8]: this is utilised to provide the expliciter error bounds. Furthermore, Fréchet differentials of smooth co-prime factors provide good local approximants in the gap metric. Bounding the gap between the polynomial system and the original nominal plant, sacrificing global stability for semi-global stability and applying gap robust stability approaches should yield the desired results, together with estimations of the basin of attraction. It is anticipated this will provide a systematic and theoretically grounded approach to complexity reduction.

It is noted that Weierstrass theorem based polynomial approximations to general nonlinear operators have been studied in [9], [10], [11] and references therein, for which the domain needs to be compact and the approximation is in a sense that system gain is impossible to calculate.

In Section 2, we present with some preliminaries on signal spaces and control systems. We also recall the definition of Fréchet differentiability and some of its properties that will be used in this paper. In section 3, approximants and robust stability margin for stable and differentiable i/o operators will be discussed. When the operators are given by a differential equation, the approximant is calculated explicitly in Section 4, where the result shows that the approximant is a system recursive linear control system, for which each step is a linear system with the same state matrix and different inputs consisting of polynomial combinations of the control input and state variables obtained in previous steps. Examples are given, showing the advantage of high order approximation over the traditional linearisation. In Section 6, we study the case when the operators are unstable (even not differentiable) but have coprime factorisations.

2. PRELIMINARIES

We first recall some notions on signal spaces and gap metric theory, all can be found in [1], [3], [7]. Let $\mathcal{X}$ be a nonempty set. For $0<\omega\leq\infty$, let $\mathcal{S}_\omega$ denote the set of all locally integrable functions from $[0,\omega)$ to $\mathcal{X}$. For $\sigma\in(0,\omega)$ and $x\in\mathcal{S}_\omega$, let $x|_{[0,\sigma)}$ be the restriction of x on $[0,\sigma)$ and define a truncation operator T_σ as follows:

$$T_\sigma:\mathcal{S}_\omega\to\mathcal{S}_\infty,(T_\sigma x)(t)=\begin{cases} x(t), & \text{for } t\in[0,\sigma);\\ 0, & \text{otherwise.}\end{cases}$$

Let $\mathcal{V}\subset\mathcal{S}_\infty$ be a normed vector space. and the norm $\|\cdot\|=\|\cdot\|_\mathcal{V}$ be defined for signals of the form $T_\sigma|_{[0,\sigma)}, v\in\mathcal{V}$. We can define a norm $\|\cdot\|_\sigma$ on $\mathcal{S}_\sigma$ by $\|v\|_\sigma=\|T_\sigma v\|$. We

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associate spaces as follows:

$$\begin{aligned}
&\mathcal{V}[0,\sigma) = \left\{ v \in \mathcal{S}_\sigma : v = w|_{[0,\sigma)}, w \in \mathcal{V}, \|v\|_\sigma < \infty \right\};\\
&\mathcal{V}_e = \left\{ v \in \mathcal{S}_\infty : \forall \sigma > 0, v|_{[0,\sigma)} \in \mathcal{V}[0,\sigma) \right\};\\
&\mathcal{V}_\omega = \left\{ v \in \mathcal{S}_\omega : \forall \sigma \in (0,\omega), v|_{[0,\sigma)} \in \mathcal{V}[0,\sigma) \right\};\\
&\mathcal{V}_a = \cup_{\omega \in (0,\infty]} \mathcal{V}_\omega,
\end{aligned}$$

where $\sigma, \omega \geq 0$. Let $\mathcal{U}, \mathcal{Y}$ be two normed signal spaces (such as $L^{\mathbf{p}}(\mathbb{R}_+, \mathbb{R}^n), 1 \leq \mathbf{p} \leq \infty$) with norm $\|\cdot\|_{\mathcal{U}}, \|\cdot\|_{\mathcal{Y}}$ respectively. If there is no ambiguous occur on the notation, we will drop the subscripts. A mapping $Q : \mathcal{U}_a \to \mathcal{Y}_a$ is said to be causal if for any $x, y \in \mathcal{U}_a$ and any $\sigma \in \mathrm{dom}(x) \cap \mathrm{dom}(Qx)$, we have

$$x|_{[0,\sigma)} = y|_{[0,\sigma)} \quad \text{implies } (Qx)|_{[0,\sigma)} = (Qy)|_{[0,\sigma)}.$$

Let $P : \mathcal{U}_a \to \mathcal{Y}_a$ and $K : \mathcal{Y}_a \to \mathcal{U}_a$ be two causal mappings representing the plant and the controller, respectively. We consider the system of equations

$$[P, C] \;:\; \begin{array}{l} y_1 = Pu_1, \quad u_2 = Cy_2, \\ u_0 = u_1 + u_2, \quad y_0 = y_1 + y_2, \end{array} \tag{2.1}$$

corresponding to the closed-loop feedback configuration in Figure 1,

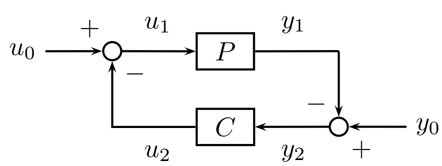

Fig. 1. The closed-loop $[P, C]$.

Let $\mathcal{W} = \mathcal{U} \times \mathcal{Y}$ with the product norm $\|(u,y)\|_{\mathcal{W}} = \max\{\|u\|_{\mathcal{U}}, \|y\|_{\mathcal{Y}}\}$. Let

$$\mathrm{graph}(P) = \left\{ \begin{pmatrix} u \\ Pu \end{pmatrix} : u \in \mathcal{U}, Pu \in \mathcal{Y} \right\},$$

$$\mathrm{graph}(K) = \left\{ \begin{pmatrix} Ky \\ y \end{pmatrix} : Cy \in \mathcal{U}, y \in \mathcal{Y} \right\}.$$

be the graph of P and the graph of K, respectively. Suppose both P and C are stabilizable, i.e. for all $w = (u, y) \in \mathcal{W}_a$ satisfying $Pu = y$ (resp. $Ky = u$) and for all $\sigma > 0$, there exists $w' \in \mathrm{graph}(P)$ (resp. $\mathrm{graph}(K)$) such that $w|_{[0,\sigma)} = w'|_{[0,\sigma)}$.

For $w_0 = (u_0, y_0) \in \mathcal{W}$, a pair $w_1 = (u_1, y_1), w_2 = (u_2, y_2) \in \mathcal{W}_a$ is said to be a solution of the system if (2.1) holds on $\mathrm{dom}(w_1) \cap \mathrm{dom}(w_2)$ which is an interval $[0, \omega)$ or $[0, \omega]$ with $\omega > 0$. Let $\mathcal{Z}_{w_0}$ be the set of all solutions to the system corresponding to the given w_0, which could be empty. Assume that $[P, K]$ has both existence property, i.e. $\mathcal{Z}_{w_0} \neq$ for each $w_0 \in \mathcal{W}$, and uniqueness property, i.e.

$$\begin{aligned}
&(w_1, w_2), (\tilde{w}_1, \tilde{w}_2) \in \mathcal{Z}_{w_0} \text{ implies}\\
&(w_1, w_2) = (\tilde{w}_1, \tilde{w}_2) \text{ on } \mathrm{dom}(w_1, w_2) \cap \mathrm{dom}(\tilde{w}_1, \tilde{w}_2).
\end{aligned}$$

Then, for each $w_0 \in \mathcal{W}$, we define a number ω_{w_0} by

$$(0, \omega_{w_0}) = \cup_{(\tilde{w}_1, \tilde{w}_2) \in \mathcal{Z}_{w_0}} \mathrm{dom}(\tilde{w}_1) \cap \mathrm{dom}(\tilde{w}_2),$$

and define a pair $(w_1, w_2) \in \mathcal{W}_a \times \mathcal{W}_a$, with domain $\mathrm{dom}(w_1, w_2) = [0, \omega_{w_0})$, by the property $(w_1, w_2)|_{[0,t)} \in \mathcal{Z}_{w_0}$ for all $t < \omega_{w_0}$. This induces the system operator:

$$H_{P,K} : \mathcal{W} \to \mathcal{W}_a \times \mathcal{W}_a, \quad \Pi_{P,K} w_0 = (w_1, w_2).$$

Let $\Pi_i : \mathcal{W}_a \times \mathcal{W}_a \to \mathcal{W}_a$ be the projection onto the i-th component of $\mathcal{W}_a \times \mathcal{W}_a$ for $i = 1, 2$. We define

$$\Pi_{P//K} = \Pi_1 H_{P,K}, \quad \text{and} \quad \Pi_{K//P} = \Pi_2 H_{P,K}.$$

Definition 2.1: Let $\Omega \subset \mathcal{W}$. The closed-loop (2.1) is said to be:

- *locally well posed on* Ω if it has the existence and uniqueness properties and the operator $H_{P,K}|_\Omega : \Omega \to \mathcal{W}_a \times \mathcal{W}_a$ is causal;
- *globally well posed on* Ω if it is locally well posed and $H_{P,K}(\Omega) \subset \mathcal{W}_e \times \mathcal{W}_e$.

Definition 2.2: Let $\Omega \subset \mathcal{W}$. The closed-loop $[P, K]$ given by (2.1) is said to be:

- *stable* on Ω if for any $w \in \Omega$, $\|H_{P,K} w\| < \infty$;
- *gain stable* on Ω if it is globally well posed on Ω and

$$\|H_{P,K}\| := \sup \left\{ \frac{\|T_\sigma H_{P,K} w\|}{\|T_\sigma w\|}, \sigma > 0, w \in \Omega\} < \infty \right\}.$$

When Ω is a (small) neighbourhood of 0, we also say the system is locally (gain) stable.

It is noticed that stability of $[P, K]$ is equivalent to the same stability of either $\Pi_{P//K}$ or $\Pi_{K//P}$ since $H_{P,K} = (\Pi_{P//K}, \Pi_{K//P})$ and $\Pi_{P//K} + \Pi_{K//P} = I$. So the stability of control system $[P, K]$ depends on the the calculation of the induced norm of operator $\Pi_{P//K}$. For robustness, given P the nominal plant and P_1 the perturbed plant, we aim to bound $\|\Pi_{P_1//C}\|$ in terms of $\|\Pi_{P//K}\|$. Gap metric provides a practical way of doing so. A general gap metric is presented in [7] by Georgiou and Smith using surjective mappings between graphs of the plant and controllers:

Definition 2.3: The gap metric distance between causal operators $P, P_1 : \mathcal{U}_a \to \mathcal{Y}_a$ is defined to be

$$\vec{\delta}(P, P_1) = \begin{cases} \inf_{\Phi \in \Theta} \|(I - \Phi)|_{\mathrm{graph}(P)}\| & \text{if } \Theta \neq \emptyset, \\ \infty & \text{if } \Theta = \emptyset \end{cases}$$

with

$$\Theta = \left\{ \Phi : \begin{array}{l} \Phi : (\Phi) \subset \mathrm{graph}(P) \to \mathrm{graph}(P_1) \text{ is a} \\ \text{causal, gain stable and surjective mapping} \end{array} \right\}.$$

Lemma 2.4: Let $[P, K]$ be globally well-posed, $[P_1, K]$ be locally well-posed, $H_{P,K}$ be bounded on $\mathcal{S}_r$ and $\|\Pi_{P//K}|_{\mathcal{S}_r}\| \leq p$. Suppose that there exists a mapping $\Phi : E_{pr} := \mathrm{graph}(K) \cap \mathcal{S}_{pr} \to \mathrm{graph}(K_1)$ such that $\|(\Phi - I)|_{\mathcal{S}_{pr}}\| = q < 1/p$ and $T_\sigma(\Phi - I)\Pi_{P//K}|_{\mathcal{S}_r}$ is continuous, compact. Then $H_{P_1,K}$ is globally well-posed and is bounded on $\mathcal{S}_{(1-pq)r}$ and

$$\|\Pi_{P_1//K}|_{\mathcal{S}_{(1-pq)r}}\| \leq \frac{p(1+q)}{1 - pq}.$$

At the end of this section, we recall the notion of Fréchet differentiability.

Let $G : \mathcal{U} \to \mathcal{Y}$ be an operator. For any $u_0 \in \mathcal{U}$, G is said Fréchet differentiable at u_0 if there exists a bounded linear operator $G'(u_0) : U \to Y$ such that

$$\lim_{\|v\|\to 0} \frac{\|G(u_0+v) - G(u_0) - G'(u_0)v\|}{\|v\|} = 0.$$

The linear operator $G'(u_0)$ is called the derivative of G at u_0 and also denoted by $\frac{dG}{du}|_{u=u_0}$. In the case where G is a multi-variable operator, its partial derivative at u_0 is denoted by $\frac{\partial G}{\partial u}|_{u=u_0}$.

If $G'(u)$ exists on a neighbourhood of u_0 and the operator $u \mapsto G'(u)$ is differentiable at u_0, we say G is second order differentiable at u_0 and denote by $G''(u_0)$ the second order derivative. Similarly, the derivative of nth order can be defined, denoted by $G^{(n)}(u_0)$. In general, $G^{(n)}(u_0)$ can be identified as a bounded n-linear operator from $\mathcal{U}^n$ to $\mathcal{Y}$, that is for each $i = 1, \ldots, n$, the operator $v_i \to G^{(n)}(u_0)(v_1, \cdots, v_i, \cdots, v_n)$ is bounded and linear, and there exists a constant $M = M_n(u_0) > 0$ such that $\|G^{(n)}(u_0)(v_1, \cdots, v_n)\| \le M\|v_1\| \cdots \|v_n\|$ for all $(v_1, \cdots, v_n) \in \mathcal{U}^n$.

The Fréchet derivative has the following properties ([5], [6]):

Lemma 2.5: i) If G is n times Fréchet differentiable at u_0, then

$$G^{(n)}(u_0)(v_1, \cdots, v_n) = \frac{dG^{(n-1)}(u)(v_1, \cdots, v_{n-1})}{du} v_n|_{u=u_0};$$

$$\frac{\partial G^{(n)}(u_0)(v_1, \cdots, v_i, \cdots, v_n)}{\partial v_i} w = G^{(n)}(u_0)(v_1, \cdots, v_{i-1}, w, v_{i+1}, \cdots, v_n);$$

$$\frac{\partial G^{(n)}(u_0)v^n}{\partial v} w = \sum_{i=0}^{n-1} G^{(n)}(u_0)v^i w v^{n-i-1}.$$

ii) If G is n-times differentiable at u_0 and the derivatives up to the order $n-1$ are continuous, then

$$\left\| G(u_0+u) - G(u_0) - \sum_{j=1}^{n-1} \frac{1}{j!} G^{(j)}(u_0)u^j \right\| \le \frac{\|u\|^n}{n!} \sup_{z\in[u_0,u_0+u]} \|G^{(n)}(z)\|.$$

If, in addition, $u \mapsto G^{(n)}(u)$ is continuous, then for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left\| G(u_0+u) - \sum_{j=0}^{n} \frac{1}{j!} G^{(j)}(u_0)(u, \cdots, u) \right\| \le \varepsilon \|u\|^n$$

for all $u \in U$ and $\|u\| \le \delta$.

3. Approximation of stable operators

In this section, we consider the approximation of BIBO stable operators, that is operators for which finite inputs yield finite outputs. Since the resulting notion of robust stability will be local, the operators it will suffice to consider operators which are only locally stable.

Let $P : \mathcal{U}_a \to \mathcal{Y}_a$ be the input to output operator of a control system with $\mathcal{U}, \mathcal{Y}$ two given normed signal spaces, and let $K : \mathcal{Y}_a \to \mathcal{U}_a$ be a controller. Suppose P has continuous Fréchet derivatives up to order n at 0. Let P_n denote the Taylor expansion of order n at 0, that is

$$P_n u = P(0) + P'(0)u + \frac{1}{2}P''(0)u^2 + \cdots + \frac{1}{n!}P^{(n)}(0)u^n.$$

Here we identify u^n with the vector $(u, \cdots, u)$ of n components.

For any $u \in U$, define two graph mappings, as below:

$$\Phi \begin{pmatrix} u \\ Pu \end{pmatrix} = \begin{pmatrix} u \\ P_n u \end{pmatrix} \tag{3.1}$$

and

$$\Psi \begin{pmatrix} u \\ P_n u \end{pmatrix} = \begin{pmatrix} u \\ Pu \end{pmatrix}. \tag{3.2}$$

Since each $P^{(k)}$ is a bounded operator from U^k to Y, $\mathrm{dom}(P_n) = U$ and P_n is global stable. As assumed, P is also stable. Hence we have

Lemma 3.1: Under the assumptions above, both Φ : $\mathrm{graph}(P) \mapsto \mathrm{graph}(P_n)$ and $\Psi : \mathrm{graph}(P_n) \mapsto \mathrm{graph}(P)$ are surjective.

In the case when P is only locally stable, i.e., $\|Pu\| < \infty$ for $\|u\| \le r$ with $r > 0$, then Ψ maps $\mathrm{graph}(P_n) \cap \mathcal{S}_r$ into $\mathrm{graph}(P_n)$. This is will be enough for local robustness.

We next use these two graph mappings to evaluate the gap distance between P and P_n, and then to establish robust stability. Since the well-established gap metric theory requires $P0 = 0$ though there are publications ([4], [8]) considering the biased case, the assumption $P(0) = 0$ is imposed in the rest of this section. However this it is not necessary to assume that $P(0) = 0$ in the calculation of P_n as shown in later sections.

Theorem 3.2: Suppose $n > 1, r > 0, k_n := k_n(r) > 0$, P is continuously differentiable up to n times on an open set containing the disc centered at 0 with radius r and

$$\|P^{(n+1)}(u)\| \le k_n \quad \text{for all } u \in X, \|u\| \le r. \tag{3.3}$$

Let $[P, K]$ be globally well-posed, $[P_n, K]$ be locally well-posed, and $\|\Pi_{P//K}|_{\mathcal{S}_r}\| \le p$ with $p \ge 0$. Let $(\Phi - I)\Pi_{P//K}|_{\mathcal{S}_r}$ be continuous, compact. Then, for any $r_1 \in (0, r)$ such that

$$q := \frac{p^{n+1} r_1^n}{(n+1)!} k_n < 1, \; pr_1 < r, \tag{3.4}$$

$H_{P_n,K}$ is globally well-posed and bounded on $\mathcal{S}_{(1-pq)r_1}$,

$$\|\Pi_{P_n//K}|_{\mathcal{S}_{(1-pq)r_1}}\| \le \frac{p(1+q)}{1-pq}, \tag{3.5}$$

where

$$q = \frac{p^n r_1^n}{(n+1)!} k_n.$$

Proof: According to Lemma 2.5 (iii) and our assumptions

$$\|Pu - P_n u\| \le \frac{k_n}{(n+1)!} \|u\|^{n+1}$$

for all $u \in X$ with $\|u\| \le r$. Let $r_1 > 0$ satisfy (3.4). Then for any $(u, Pu) \in \text{graph}(P)$ with $\|(u, Pu)\| \le pr_1$, we have $\|T_\sigma u\| \le \|u\| \le pr_1 < r$. So

$$\begin{aligned} \|PT_\sigma u - P_n T_\sigma u\| &\le \frac{k_n}{(n+1)!}\|T_\sigma u\|^{n+1} \\ &\le \frac{p^n r_1^n}{(n+1)!} k_n \|T_\sigma u\|, \end{aligned}$$

for all $\sigma > 0$. Considering the mapping Φ given in (3.1), which maps $\text{graph}(P)$ into $\text{graph}(P_n)$. Since

$$\begin{aligned} \left\| T_\sigma(\Phi - I)\begin{pmatrix} u \\ Pu \end{pmatrix} \right\| &= \|T_\sigma(Pu - P_n u)\| \\ &= \|T_\sigma(PT_\sigma u - P_n T_\sigma u)\|, \end{aligned}$$

we see

$$\|(\Phi - I)|_{\mathcal{S}_{pr_1}}\| \le \frac{p^n r_1^n}{(n+1)!} k_n.$$

Since

$$\frac{p^n r_1^n}{(n+1)!} k_n p = \frac{p^{n+1} r_1^n}{(n+1)!} k_n < 1,$$

by Lemma 2.4, $[P_n, K]$ is globally well posed and bounded on $\mathcal{S}_{(1-pq)r_1}$ with $q = \frac{p^n r_1^n}{(n+1)!} k_n$. ■

In the above proof, the order of P and P_n plays no role since only the input u and the differentiability of P are crucial. So replacing Φ by Ψ and using the same proof, we have the following theorem.

Theorem 3.3: Suppose $k_n, r > 0$, P is continuously differentiable up to n times on an open set containing the disc centered at 0 with radius r and suppose condition (3.3) holds. Let $[P_n, K]$ be globally well-posed, $[P, K]$ be locally well-posed, P is local stable and $\|\Pi_{P_n//K}|_{\mathcal{S}_r}\| \le p$ with $p \ge 0$. Let $(\Psi - I)\Pi_{P_n//K}|_{\mathcal{S}_r}$ be continuous, compact. Then, for any $r_1 \in (0, r)$ satisfying (3.4), $H_{P,K}$ is globally well-posed and is bounded on $\mathcal{S}_{(1-pq)r_1}$, where q is the same as in Theorem 3.2, and

$$\|\Pi_{P//K}|_{\mathcal{S}_{(1-pq)r_1}}\| \le \frac{p(1+q)}{1-pq}, \qquad (3.6)$$

The robust margin corresponds to the largest tolerable gap between nominal and perturbed systems. In the above two theorems, the gap is bounded by $\frac{(pr)^n}{(n+1)!}k_n$. If k_n has certain power growth with respect to n such that $\frac{(pr)^n}{(n+1)!}k_n$ tends to zero as $n \to \infty$, then, the higher the order of the approximation is, the larger the robust margin will be, which explains the necessity of considering high order approximation. Later on, we will use concrete examples to illustrate this. Secondly, it also shows that, in the case when p or $\|\Pi_{P//K}|_{\mathcal{S}_r}\|$ (resp $\|\Pi_{P_n//K}|_{\mathcal{S}_r}\|$) is large, we need larger n so that assumption (3.4) holds.

In Theorems 3.2 and 3.3, condition (3.3) is necessary. But as the expression for the derivative becomes more complicated as the order goes large, it is not easy to find the upper bound K, particularly for operators with memory. In this situation, we have the following alternative.

Theorem 3.4: Suppose P is continuously differentiable up to n times at 0. Let $[P, K]$ be globally well-posed, $[P_n, K]$ be locally well-posed, $H_{P,K}$ be bounded on $\mathcal{S}_r$ and $\|\Pi_{P//K}|_{\mathcal{S}_r}\| \le p$. Let $(\Phi - I)\Pi_{P//K}|_{\mathcal{S}_r}$ be continuous, compact. Then, for any $\varepsilon > 0$ there exists $r_1 \in (0, r)$ such that $H_{P_n,K}$ is globally well-posed and is bounded on $\mathcal{S}_{(1-pq)r_1}$, where

$$q = \varepsilon p^n r_1^{n-1}.$$

The same conclusion holds if P_n is the nominal plant, P is the perturbed plant and local stable, and Φ is replaced by Ψ.

Proof: Let $\varepsilon > 0$ be given. According to Lemma 2.5 (iii), there exists $\delta > 0$ such that

$$\|Pu - P_n u\| \le \varepsilon \|u\|^n$$

for all $u \in U$ with $\|u\| \le \delta$. Let $r_1 \in (0, r)$ be such that

$$pr_1 < \delta \quad \text{and} \quad \varepsilon p^n r_1^{n-1} < 1.$$

Then for any $(u, Pu) \in \text{graph}(P)$ with $\|(u, Pu)\| \le pr_1$, we have $\|u\| \le pr_1 < \delta$. So, for all $\sigma > 0$,

$$\|PT_\sigma u - P_n T_\sigma u\| \le \varepsilon \|T_\sigma u\|^n \le \varepsilon p^{n-1} r_1^{n-1} \|T_\sigma u\|.$$

This shows

$$\|(\Phi - I)|_{\mathcal{S}_{pr_1}}\| \le \varepsilon p^{n-1} r_1^{n-1}.$$

Since

$$\varepsilon p^{n-1} r_1^{n-1} p = \varepsilon p^n r_1^{n-1} < 1,$$

by Lemma 2.4, $[P_n, K]$ is globally well posed and bounded on $\mathcal{S}_{(1-pq)r_1}$.

The second part of the theorem can be proved similarly. ■

4. Systems governed by differential equations

In this section, we investigate the approximation operator P_n, given that P is the input-to-output operator of control system:

$$x'(t) = F(x(t), u(t)), x(0) = x_0, \qquad (4.1)$$

$$y(t) = Cx(t) + C_1 u(t) \qquad (4.2)$$

in the $L^\infty(\mathbb{R}^+)$ setting, where $F : \mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \to \mathbb{R}^{m_1}$ is in general a nonlinear function, $C : \mathbb{R}^{m_1} \to \mathbb{R}^{m_3}$ and $C_1 : \mathbb{R}^{m_2} \to \mathbb{R}^{m_3}$ are linear or nonlinear functions, representing the output matrix and feed-through matrix respectively.

Let $\mathcal{F} : u \mapsto x$ and $\mathcal{C} : x \mapsto y$ be the operators determined by

$$\mathcal{F} : u \mapsto x, \quad x'(t) = F(x(t), u(t)), \quad x(0) = x_0, \quad (4.3)$$

$$\mathcal{C}_1 : u \mapsto x_1, \quad x_1(t) = C_1(u(t)),$$

and

$$\mathcal{C} : x \mapsto y, \quad y(t) = C(x(t)), \qquad (4.4)$$

respectively. Then

$$Pu = \mathcal{C}\mathcal{F}u + \mathcal{C}_1(u)$$

and, provided derivatives of each order exist

$$\begin{aligned}
P'(u_0)v &= \mathcal{C}'(\mathcal{F}(u_0))\circ\mathcal{F}'(u_0)v+\mathcal{C}_1'(u_0)v,\\
P''(u_0)v^2 &= \mathcal{C}''(\mathcal{F}(u_0))\circ(\mathcal{F}'(u_0)v)^2\\
&\quad ++\mathcal{C}'(\mathcal{F}(u_0))\circ\mathcal{F}''(u_0)v^2+\mathcal{C}_1''(u_0)v^2,\\
P'''(u_0)v^3 &= \mathcal{C}'''(\mathcal{F}(u_0))\circ(\mathcal{F}'(u_0)v)^3\\
&\quad +3\mathcal{C}''(\mathcal{F}(u_0))\circ(\mathcal{F}'(u_0)v,(\mathcal{F}''(u_0)v)^2)\\
&\quad +\mathcal{C}'(\mathcal{F}(u_0))\circ(\mathcal{F}'''(u_0)v^3+\mathcal{C}_1'''(u_0)v^3\\
&\quad \cdots
\end{aligned}$$

So the approximation system is governed by the approximations of memory operator $\mathcal{F}$ and the memoryless operators $\mathcal{C}$ and $\mathcal{C}_1$.

Let's first consider the derivatives of memoryless operators. It is known that for a general nonlinear memoryless operator (say) $\mathcal{C}$ between functions over infinite time interval, even assuming C continuously differentiable is still not sufficient to ensure the differentiability of $\mathcal{C}$. However, memoryless operators generated by most basic functions are differentiable, such as, in the scalar case:

1) linear operators: if C is linear, then $\mathcal{C}'(u_0)=\mathcal{C}$ and $\mathcal{C}^{(k)}(u_0)=0$ for $k\geq 2$;
2) polynomial operators: if $Cu=u^n$, then $\mathcal{C}^{(k)}(u_0)=\frac{n!}{(n-k)!}u_0^{n-k}$ for $k\geq 1$;
3) delay operators: if $Cu(t)=u(t-\tau)$ for some $\tau\geq 0$, then $\mathcal{C}'(u_0)v(t)=v(t-\tau)$ and $\mathcal{C}^{(k)}(u_0)=0$ for $k\geq 2$.

In this paper, we suppose that any memoryless operators $\mathcal{C}$ and $\mathcal{C}_1$ generated by functions C and C_1 are differentiable as many times as required.

To study the approximation of $\mathcal{F}$, we let function $F(x,u)$ be Fréchet differentiable up to n times with respect to each variable and let the partial derivatives be denoted by $F_1'(x,u),\ F_2'(x,u),\ F_{11}''(x,u),\ F_{12}''(x,u),\cdots$ respectively. For any $u\in\mathcal{U}$, we let $x(t,u)$ be the solution to Equation (4.1). Then we have

Theorem 4.1: For any $n\geq 1$, $\mathcal{F}^{(n)}(u_0)v^n$ is the solution to the system of equations

$$\begin{aligned}
z_n'(t) &= F_1'(x(t,u_0),u_0(t))z_n(t)\\
&\quad +\left[\sum_{i=2}^{n-1}\frac{\partial G_{n-1}}{\partial z_{i-1}}z_i(t)+\frac{\partial G_{n-1}}{\partial x}z_1(t)\right]_{u=u_0}\\
&\quad +\frac{\partial G_{n-1}}{\partial u}v(t)\Big|_{u=u_0} \qquad (4.5)\\
z_n(0) &= 0,
\end{aligned}$$

where

$$\begin{aligned}
z_1(t) &= \frac{\partial x(t,u)}{\partial u}\Big|_{u=u_0}v(t),\\
G_1 &= G_1(z_1,x,u,v)=F_1'(x,u)z_1+F_2'(x,u)v
\end{aligned}$$

and for $j=2,\cdots,n-1$,

$$\begin{aligned}
z_j(t) &= \frac{\partial z_{j-1}(t)}{\partial u}\Big|_{u=u_0}v(t),\\
G_j &= G_j(z_j,\cdots,z_1,x,u,v)=F_1'(x,u)z_j\\
&\quad +\sum_{i=2}^{j-1}\frac{\partial G_{j-1}}{\partial z_{i-1}}z_i+\frac{\partial G_{j-1}}{\partial x}z_1+\frac{\partial G_{j-1}}{\partial u}v, \qquad (4.6)
\end{aligned}$$

each derivative of G_j is in the function level, e.g. $\frac{\partial G_1}{\partial z_1}=\frac{\partial G_1(z_1(t),x(t),u(t),v(t))}{\partial z_1(t)}$.

Proof: The subscription $|_{u=u_0}$ will be omitted in the proof.

As supposed

$$x(t,u)=x_0+\int_0^t F(x(s,u),u(s))ds,$$

so

$$\begin{aligned}
\frac{\partial}{\partial u}x(t,u)v &= \int_0^t\frac{\mathrm{d}}{\mathrm{d}u}F(x(s,u),u(s))v(s)ds\\
&= \int_0^t\left[F_1'(x(s,u),u(s))\frac{\partial}{\partial u}x(s,u)\right]v(s)ds\\
&\quad +\int_0^t\left[F_2'(x(s,u),u(s))\right]v(s)ds
\end{aligned}$$

and

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial}{\partial u}x(t,u)v(t)\right) &= F_2'(x(t,u),u(t))\frac{\partial}{\partial u}x(t,u)v\\
&\quad +F_2'(x(t,u),u(t))v(t).
\end{aligned}$$

This shows that $z_1(t)=\frac{\partial}{\partial u}x(t,u)v$ is the solution to the equation:

$$z_1'(t)=F_1'(x(t,u),u)z_1(t)+F_2'(x(t,u),u)v,\quad z_1(0)=0. \qquad (4.7)$$

Similarly, since $G_1(z_1,x,u)=F_1'(x,u)z_1+F_2'(x,u)v$, omitting the time variables, we have (note that z_1 is also dependent of u):

$$\begin{aligned}
z_2(t) &= \frac{\partial}{\partial u}z_1(t)v=\int_0^t\frac{\mathrm{d}}{\mathrm{d}u}G_1(z_1,x,u)vds\\
&= \int_0^t F_1'(x,u)z_2ds\\
&\quad +\int_0^t\left[\frac{\partial}{\partial x}G_1(z_1,x,u)z_1+\frac{\partial}{\partial u}G_1(z_1,x,u)v\right]ds.
\end{aligned}$$

Hence, our claim holds for $n=1,2$.

Suppose our claim holds for $n=k$, i.e. $z_k(t)=$

$\int_0^t G_k(z_k, \cdots, z_1, x, u)ds$ with G_k as given by (4.6). Then

$$
\begin{aligned}
z_{k+1}(t) =& \frac{\partial z_k(t)}{\partial u} v = \int_0^t \frac{\mathrm{d}}{\mathrm{d}u} G_k(z_k, \cdots, z_1, x, u) v ds \\
=& \int_0^t \Big[\frac{\partial G_k}{\partial z_k} \frac{\partial z_k}{\partial u} v + \sum_{j=1}^{k-1} \frac{\partial G_k}{\partial z_j} \frac{\partial z_j}{\partial u} v \Big] ds \\
&+ \int_0^t \Big[\frac{\partial G_k}{\partial x} \frac{\partial x}{\partial u} v + \frac{\partial G_k}{\partial u} v \Big] ds \\
=& \int_0^t F_1'(x,u) z_{k+1} ds \\
&+ \int_0^t \Big[\sum_{j=1}^{k-1} \frac{\partial G_k}{\partial z_j} z_{j+1} + \frac{\partial G_k}{\partial x} z_1 + \frac{\partial G_k}{\partial u} v \Big] ds,
\end{aligned}
$$

which shows the claim holds for $n = k+1$.

By induction, the proof is completed.

■

It is noted that Formula (4.5) is recursive, each z_n is given by a linear system with multi-input v and $z_1, \cdots, z_{n-1}$ obtained in previous steps. In each step, the partial derivative F_1' is the state matrix and if it is (gain) stable, the approximation system will be (gain) stable globally. For example, $\mathcal{F}''(u_0)(v,v)$ is the solution to the equation

$$
\begin{aligned}
z_2'(t) =& F_1'(x(t,u_0), u_0(t)) z_2(t) + F_{11}''(x(t,u_0), u_0(t)) z_1^2(t) \\
&+ F_{12}''(x(t,u_0), u_0(t)) z_1(t) v(t) \\
&+ F_{21}''(x(t,u_0), u_0(t)) v(t) z_1(t) \\
&+ F_{22}''(x(t,u_0), u_0(t)) v^2(t), \qquad (4.8) \\
z_2(0) =& 0
\end{aligned}
$$

with z_1 as obtained in (4.7), and $\mathcal{F}^{(3)}(u_0)(v,v,v)$ is the solution to the equation (we omit the variables $(x(t,u_0), u_0(t))$ for each derivative of F)

$$
\begin{aligned}
z_3'(t) =& F_1' z_3(t) + F_{11}'' z_1(t) z_2(t) + 2F_{11}'' z_2(t) z_1(t) \\
&+ F_{12}'' z_2(t) v(t) + F_{21}'' v(t) z_2(t) + F_{21}'' z_2(t) v(t) \\
&+ F_{111}''' z_1^3(t) + F_{112}''' z_1^2(t) v(t) \\
&+ F_{121}''' z_1(t) v(t) z_1(t) + F_{122}''' z_1(t) v^2(t) \\
&+ F_{211}''' v(t) z_1^2(t) + F_{212}''' v(t) z_1(t) v(t) \\
&+ F_{221}''' v^2(t) z_1(t) + F_{222}''' v^3(t), \\
z_3(0) =& 0
\end{aligned}
$$

where z_1, z_2 are as obtained in (4.7),(4.8) respectively.

Example 4.2: Suppose that $P : u \mapsto y$ is the input-output operator of the one dimensional system

$$
\begin{aligned}
x'(t) &= f(x) + g(x)u, \quad x(0) = 0, \\
y(t) &= Cx(t)
\end{aligned}
$$

with f, g both continuously differentiable to any order as required, $f(0) = 0$ and C a bounded linear operator, then $P'(0), P''(0), P'''(0)$ are, respectively:

$$
\begin{aligned}
P'(0) :& \ v \mapsto Cz_1 : \\
& z_1' = f'(0) z_1 + g(0) v, \quad z(0) = 0, \qquad (4.9) \\
P''(0) :& \ v \mapsto Cz_2 : \\
& z_2' = f'(0) z_2 + f''(0) z_1^2 + 2g'(0) z_1 v, \qquad (4.10) \\
& z(0) = 0, \\
P'''(0) :& \ v \mapsto Cz_3 : \\
& z_3' = f'(0) z_3 + f'''(0) z_1^3 + 3g''(0) z_1^2 v \\
& \quad + 3f''(0) z_1 z_2 + 3g'(0) z_2 v, \qquad (4.11) \\
& z(0) = 0.
\end{aligned}
$$

Therefore the approximation operator of order 3

$$
y = P_3 u = P'(0)u + \frac{1}{2} P''(0) u^2 + \frac{1}{6} P'''(0) u^3
$$

is given by the system of equations (4.9),(4.10),(4.11) and

$$
z = z_1 + \frac{1}{2} z_2 + \frac{1}{6} z_3, \qquad (4.12)
$$

$$
y = Cz \qquad (4.13)
$$

We now look at a concrete case when $f(x) = -x - x^3, g(x) = 1$ and $C = I$, i.e. when P is given as

$$
P : u \mapsto x, \ x' = -x - x^3 + u, x(0) = 0.
$$

For any u such that $\|u\| \leq r \leq 1$, we let $x_u = Pu$. It can be shown $\|x_u\| + \|x_u\|^3 \leq \|u\|$ which gives

$$
\|x_u\| \leq 0.7\|u\| \leq 0.7, \|x_u\|^3 \leq 0.35\|u\|. \qquad (4.14)
$$

The derivatives of $P(u)$ are, respectively

$$
\begin{aligned}
P'(u) :& \ v \mapsto z_1 : z_1' = -(1 + 3x_u^2) z_1 + v, \quad z_1(0) = 0 \\
P''(u) :& \ v \mapsto z_2 : z_2' = -(1 + 3x_u^2) z_2 - 6x_u z_1^2, \\
& \qquad z_2(0) = 0 \\
P'''(u) :& \ v \mapsto z_3 : z_3' = -(1 + 3x_u^2) z_3 - 18x_u z_1 z_2 - 6z_1^3, \\
& \qquad z_3(0) = 0 \\
P^{(4)}(u) :& \ v \mapsto z_4 : z_4' = -(1 + 3x_u^2) z_4 - 24x_u z_1 z_3 \\
& \qquad - 36z_1^2 z_2 - 18x_u z_2^2, \ z_4(0) = 0 \\
& \cdots\cdots
\end{aligned}
$$

It can be proved that

$$
\|z_1\| \leq \|v\|, \|z_2\| \leq 6\|x_u\|\|v\|^2,
$$

$$
\|z_3\| \leq 6(3\|x_u\| + 1)\|v\|^3, \ \|z_4\| \leq 6(7\|x_u\| + 6)\|v\|^4
$$

and therefore

$$
\|P'(u)\| \leq 1, \ \|P''(u)\| \leq 6\|x_u\| \leq 4.2,
$$

$$
\|P'''(u)\| \leq 18.6, \ \|P^{(4)}(u)\| \leq 66.
$$

According to Theorem, an upper bound for each of the gaps $\vec{\delta}(P, P_1)$ and $\vec{\delta}(P, P_3)$ is

$$
\vec{\delta}(P, P_1) \leq \frac{pr_1}{2} \times 4.2 = 2.1pr_1,
$$

$$
\vec{\delta}(P, P_3) \leq \frac{(pr_1)3}{4!} \times 66 \leq 2.6(pr_1)^2 pr_1.
$$

If $pr_1 \leq 0.8$, then the above estimation shows that the third order approximation P_3 will give a better robust margin than the linearisation P_1.

5. Approximation of un-stable operators

In this section, we study the complexity reduction for un-stable operators which has co-prime factorisations, the operators is even not necessarily differentiable as long as the co-prime factors are. Recall that a causal operator $P : \mathcal{U}_a \to \mathcal{Y}_a$ is said to admit a (right) coprime factorization if and only if there exist causal stable operators $D : \mathrm{dom}(D) \subset \mathcal{U} \to \mathcal{V} \subset \mathcal{U}$ and $N : \mathcal{V} \subset \mathcal{U} \to \mathcal{Y}$ such that

(i) D is causally invertible with $\mathrm{dom}(D^{-1}) = \mathrm{dom}(P)$,

(ii) $P = ND^{-1}$ and

(iii) there exists a causal stable mapping $L : \mathcal{U} \times \mathcal{Y} \to \mathcal{U}$ such that $L(D, N)^\top = I$.

In the case when D, N are coprime factorisation of P, we write $P = ND^{-1}$. Sufficient conditons for the existence of co-prime factorisations to differential systems can be found in [2].

Now, suppose $P = ND^{-1}$ with D, N stable Fréchet differentiable, D^{-1} is not necessarily stable nor differentiable. The discussion in the last sections cannot be applied directly to P, but can be applied to both N and D to obtain approximations N_n and D_n respectively. If D_n has an "inverse"$[D_n]^{-1}$, then we treat $N_n[D_n]^{-1}$ as the approximation to P. This is the idea of this section.

Let

$$N_n u = N u_0 + N'(u_0)u + \frac{1}{2}N''(u_0)u^2 + \cdots + \frac{1}{n!}N^{(n)}u^n, \tag{5.1}$$

$$D_n u = D u_0 + D'(u_0)u + \frac{1}{2}D''(u_0)u^2 + \cdots + \frac{1}{n!}D^{(n)}u^n. \tag{5.2}$$

For each v from the range of D_n, we choose an arbitrary $u \in U$ such that $D_n u = v$ and denoted by

$$[D_n]^{-1} v = u.$$

In other word, $[D_n]^{-1}$ is the inverse of $D_n|_V$ with $V \subset U$ a subset such that

$$D_n : V \to \text{Range}(D_n) \quad \text{is one-one.}$$

Note, if D_n is invertible, then $[D_n]^{-1}$ is the inverse of D_n.

We now define the approximation of P as

$$P_n = N_n[D_n]^{-1}.$$

It is known that an operator may have more than than one coprime factorisations([1]), therefore the approximation for unstable systems defined in this way is not unique.

Since (N, D) is a right coprime factorization of P, we have

$$\mathrm{graph}(P) = \left\{ \begin{pmatrix} Du \\ Nu \end{pmatrix} : u \in U \right\}.$$

(N_n, D_n) is not necessarily the coprime factorisation of P_n, we cannot have the same graph representation. However, the definitions of P_n and $[D_n]^{-1}$ shows the following inclusion

$$\mathrm{graph}(P_n) = \left\{ \begin{pmatrix} D_n v \\ N_n v \end{pmatrix} v \in V \right\} \subset \left\{ \begin{pmatrix} D_n v \\ N_n v \end{pmatrix} : v \in U \right\}$$

where $V \subset U$ is the subset such that $[D_n]^{-1} = [D_n|_V]^{-1}$. In the case when D_n is invertible, the inclusion becomes equal. This shows that the mapping Ψ defined by

$$\Psi \begin{pmatrix} D_n u \\ N_n u \end{pmatrix} = \begin{pmatrix} Du \\ Nu \end{pmatrix} \tag{5.3}$$

maps $\mathrm{graph}(P_n)$ into $\mathrm{graph}(P)$. Moreover

$$\left\| (\Psi - I) \begin{pmatrix} D_n u \\ N_n u \end{pmatrix} \right\| = \left\| \begin{pmatrix} (D - D_n)u \\ (N - N_n)u \end{pmatrix} \right\|.$$

Note, by the definitions of coprime factorisaton and differentiability, both N, D and N_n, D_n are stable.

Again, as in Section 3, in the rest of this section, we suppose $N0 = 0$ and $D0 = 0$ for the applications of gap metric theory.

Theorem 5.1: Let $[P_n, K]$ be globally well-posed, $[P, K]$ be locally well-posed, and $\|\Pi_{P_n//C}|_{\mathcal{S}_r}\| \leq p$ with $r, p \geq 0$. Suppose both N and D are continuously differentiable up to n times on an open set containing the disc centered at 0 with radius r and there exist number $h(pr) > 0$ and functions

$$k_n, c : [0, \infty] \to [0, \infty]$$

such that

$$\|N^{(n+1)}(u)\| \leq k_n(R), \quad \|D^{(n+1)}(u)\| \leq k_n(R) \tag{5.4}$$

for all $u \in U$ with $\|u\| \leq R$, and

$$\|Du\| \geq c(\|u\|) \quad \text{for all } u \in U, \tag{5.5}$$

$$c(t) - \frac{k_n(t)}{(n+1)!} t^{n+1} \leq pr \text{ implies } t \leq h(pr). \tag{5.6}$$

If $T_\tau(\Psi - I)\Pi_{P_n//C}|_{\mathcal{S}_{pr}}$ is continuous, compact for all $\tau \geq 0$ and

$$0 < \frac{pk_n(h(pr))}{b(n+1)! - k_n(h(pr))} < 1, \tag{5.7}$$

where $b = \inf\{c(t)/t^{n+1} : t \in (0, h(pr)]\}$, then $H_{P,K}$ is globally well-posed and gain stable on $\mathcal{S}_{(1-pq)r}$ and

$$\|\Pi_{P_n//C}|_{\mathcal{S}_{(1-pq)r}}\| \leq \frac{p(1+q)}{1-pq},$$

where

$$q = \frac{k_n(h(pr))}{b(n+1)! - k_n(h(pr))}.$$

Proof: Let $\sigma > 0$. Apply the same procedure in Theorem 3.2 to N and D, respectively, to obtain

$$\|(N - N_n)T_\sigma u\|, \|(D - D_n)T_\sigma u\| \leq \frac{k_n(\|u\|)}{(n+1)!}\|T_\sigma u\|^{n+1}$$

for all $u \in U$. Let $u \in U$ such that $\|(D_n u, N_n u)\| \leq pr$. Then $\|T_\sigma D T_\sigma u\| - \|T_\sigma(D T_\sigma u - D_n T_\sigma u)\| \leq \|T_\sigma D_n T_\sigma u\| \leq pr$ and therefore

$$c(\|T_\sigma u\|) - \frac{k_n(\|T_\sigma u\|)}{(n+1)!}\|T_\sigma u\|^{n+1} \leq pr.$$

By (5.6), it follows $\|T_\sigma u\| \le \|u\| \le h(pr)$. Hence

$$\begin{aligned}
&\|T_\sigma(\Psi - I)|_{\mathcal{S}_{pr}}\| \\
&= \sup_{\substack{u\in V\\ \|(D_n u, N_n u)\|\le pr}} \frac{\left\|T_\sigma(\Psi - I)\begin{pmatrix} D_n u\\ N_n u\end{pmatrix}\right\|}{\left\|T_\sigma\begin{pmatrix} D_n u\\ N_n u\end{pmatrix}\right\|} \\
&\le \sup_{\substack{u\in U\\ \|u\|\le h(pr)}} \frac{k_n(h(pr))\|T_\sigma u\|^{n+1}/(n+1)!}{c(\|T_\sigma u\|) - k_n(h(pr))\|T_\sigma u\|^{n+1}/(n+1)!} \\
&= \frac{k_n(h(pr))}{b(n+1)! - k_n(h(pr))}
\end{aligned}$$

By assumption (3.4) and Lemma 2.4, the proof is completed. ■

This is a local stability result, but conditions (5.4) and (5.5) are made globally. However, as assumed, there is no specific conditions imposed to k_n or c, so we can choose the two functions piecewisely based on their local behaviour.

For local stability, requiring (5.4)-(5.5) to hold for all $u \in U$ is a bit strict. If the coercive condition is imposed to D_n, this restriction can be weakened. The proof is similar as above.

Theorem 5.2: Let $[P_n, K]$ be globally well-posed, $[P, K]$ be locally well-posed, and $\|\Pi_{P_n//K}|_{\mathcal{S}_r}\| \le p$ with $r, p \ge 0$. Suppose both N and D are continuously differentiable up to n times on an open set containing the disc centered at 0 with radius r and there exist constants $\alpha, c > 0$ and $k_n \ge 0$ such that

$$\|D_n u\| \ge c\|u\|^\alpha \quad \text{for all } u \in U, \|u\| \le r, \tag{5.8}$$

and

$$\|N^{(n+1)}(u)\| \le k_n, \quad \|D^{(n+1)}(u)\| \le k_n \tag{5.9}$$

for all $u \in U, \|u\| \le pr/c$. If $T_\tau(\Psi - I)\Pi_{P_n//K}|_{\mathcal{S}_r}$ is continuous, compact and

$$\frac{pk_n}{c(n+1)!}\left(\frac{pr}{c}\right)^{\frac{n-\alpha}{\alpha}} < 1, \tag{5.10}$$

then $H_{P,K}$ is globally well-posed and is bounded on $\mathcal{S}_{(1-pq)r}$, where

$$q = \frac{k_n}{c(n+1)!}\left(\frac{pr}{c}\right)^{\frac{n-\alpha}{\alpha}}.$$

Let's consider an example where the operator $P : L^\infty(\mathbb{R}^+) \to L^\infty(\mathbb{R}^+)$ is given by

$$P : u \mapsto y,\ x' = x^2 + \sin(x) + u, x(0) = 0, \quad y = x.$$

Then $P = ND^{-1}$ with

$$\begin{aligned}
D : v \mapsto u, \quad & x' = -2x + \sin(x) + v, \quad x(0) = 0, \\
& u = -2x - x^2 + v, \\
N : v \mapsto y, \quad & x' = -2x + \sin(x) + v, \quad x(0) = 0, \\
& y = x.
\end{aligned}$$

Both D and N are compositions of memoryless operators and the input-to-state operator given by the equation $x' = -2x + \sin(x) + v, x(0) = 0$. Using the conclusions obtained in Section 4, we see that the first order approximations for D and N are

$$\begin{aligned}
D'(0) : v \mapsto u, \quad z' = -z + v, \quad z(0) = 0, \quad u = -2z + v, \\
N'(0) : v \mapsto y, \quad z' = -z + v, \quad z(0) = 0, \quad y = z
\end{aligned}$$

and the inverse of $D'(0)$ is

$$[D'(0)]^{-1} : u \mapsto v,\ z' = z + u,\ z(0) = 0, \quad v = 2z + u.$$

Hence, the first order approximation of P is given as

$$P_1 : u \mapsto y,\ x' = x + u,\ x(0) = 0, \quad y = x.$$

6. CONCLUSION

We have presented results which permit the systematic reduction of nonlinear systems to a series of simpler approximants. Linearisation is the simplest case, and thereafter the technique can be considered to be introducing higher order corrections to the linearisation to improve accuracy. These approximations are derived in a principled manner based on an input/output perspective, and give rise to systems of a particular structure: namely recursive linear systems, with inputs constructed by polynomial combinations of the earlier recursively constructed states and the actual input. We have illustrated the technique with simple examples for clarity, however, the complexity reduction itself arrives when the technique is applied to more complex systems; since the build up of the polynomial terms in the approximants is independent of the underlying system.

REFERENCES

[1] Bian W. and French M., Graph topologies, gap metrics and robust stability for nonlinear systems. SIAM Journal of Control and Optimization, 44 (2005) 418-443

[2] Verma M. S. and Hunt L.R., Right coprime factorizations and stabilization for nonlinear systems. IEEE Trans. Auto. Control, 38 (1993) 222-231

[3] French M. and Mueller M., Nonlinear high gain separation principles and fast sampling results ensuring robust stability. Preprint.

[4] French M. and Bian W., A biased approach to nonlinear robust stability with applications, Proceeding of 48th conf. on Decision and Control, Shanghai, 2009, 1393-1398

[5] Akerkar R., Nonlinear Functional Analysis. Narosa, 1999

[6] Suhubi E. S., Functional Analysis. Kluwer Academic Publishers, 2003

[7] Georgiou T. and Smith M. C., Robustness analysis of nonlinear feedback systems: an input-output approach, IEEE Transaction on Automatic Control, 42(1997) 1200-1221

[8] Georgiou T. and Smith M. C., Biased norms and robustness analysis for nonlinear feedback systems, Proceeding of 36th conf. on Decision and Control, San Diego, 1997, 642-643

[9] Howlett P. G., Pearce C. and Torokhti A. P., On nonlinear operator approximation with preassigned accuracy, J. Comput. Anal. Appl., 5(2003) 273-297

[10] Bruno V. J., A Weierstrass approximation theorem for topological vector spaces, J. Approx. Theory, 42(1984) 1-3.

[11] Torokhti A. P. and Howlett P. G., On the constructive approximation of nonClinear operators in the modeling of dynamical systems, J. Austral. Math. Soc., Ser. B 39 (1997) 1-27.

[12] Sontag E. D., Mathematical Control Theory: Deterministic Finite Dimensional Systems. Springer, New York, 1998.

[13] Isidori A., Nonlinear Control Systems. Springer Verlag, Berlin, Heidelberg, 2nd edition, 1989.

[14] Krstić M., Kanellakopoulos I. and Kokotović P. V., Nonlinear and Adaptive Control Design. Wiley, New York, 1995.