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INDUCTED CURRENTS ANALYSIS IN MULTIPLY CONNECTED CONDUCTORS USING RELUCTANCE – RESISTANCE NETWORKS

Rafał Wojciechowski[1], Andrzej Demenko[1], Jan K. Sykulski[2]

[1] Poznań University of Technology, Piotrowo 3, 60-965 Poznań, Poland
Email: rafal.wojciechowski@put.poznan.pl, andrzej.demenko@put.poznan.pl

[2] University of Southampton, Southampton SO17 1BJ, UK, email: jks@soton.ac.uk

Abstract* – The paper develops a reluctance-resistance network (RRN) formulation for determining the induced current distributions in a 3D space of multiply connected conducting systems. The proposed method has been applied to solve Problem No. 7 of the International TEAM Workshops. The induced currents in the conductive plate with an asymmetrically situated 'hole' have been analysed. The RRN equations have been formed by means of the finite element method using the magnetic vector potential A and the electric vector potentials T and T_0. The block relaxation method (BRM) combined with the Cholesky decomposition procedure has been applied to solve the resultant RRN equations. Selected results of the analysis are presented and discussed.

Introduction

Many devices operate by utilising conduction currents created by electromotive forces, known as induced currents. Systems using such currents may be categorised as (a) simply connected regions with solid conductors, e.g. the solid part of a magnetic core, (b) multiply connected regions with thin (filamentary) conductors, e.g. windings composed of stranded conductors, (c) multiply connected regions with a form-wound multi-conductor windings, e.g. a squirrel cage winding of an induction motor, and (d) multiply connected regions with a solid core with holes [1]. We focus here on the case (d); the considered system consists of a conducting plate with an asymmetrically positioned hole and a coil excited using an alternating current (Fig. 1), as described by Problem No. 7 of the TEAM Workshops [2].

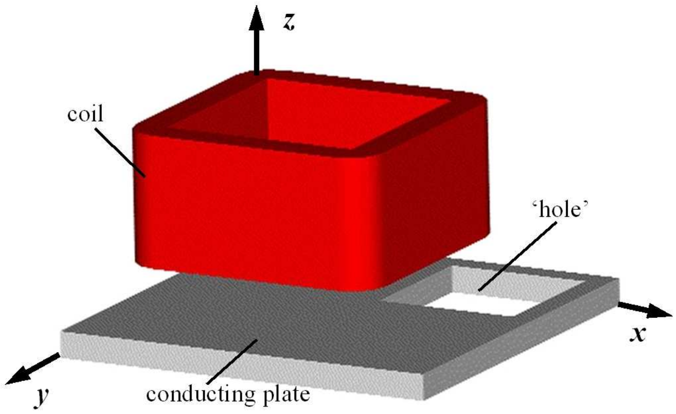

Fig. 1. The system considered containing induced currents - TEAM Workshops Problem No. 7

In the paper, a novel formulation of the reluctance-resistance network (RRN) approach is proposed. The reluctance-resistance network is obtained by coupling, via sources, two networks: (a) a reluctance network, and (b) a resistance network. The RRN equations have been derived by means of the edge element (EE) method. The use of vector potentials has been considered. The EE equations for the magnetic potential ***A*** represent the loop equations of the reluctance network, while the EE equations for the electric potential ***T*** correspond to the loop equations of the resistance network. However, the classical ***A***-***T*** formulation of the reluctance-resistance network method is not capable of treating multiply connected regions with solid conductors, such as aluminium bars in a cage rotor of an induction motor [3]. To rectify this shortcoming of the standard RRN formulation, the authors propose to complement the description of the RRN method by introducing supplementary equations in terms of $\boldsymbol{T}_0$ representing the induced current distribution in the region around the holes [3, 4].

Reluctance Network ($R_\mu N$)

A reluctance network ($R_\mu N$) may be constructed by applying an edge element formulation in terms of magnetic vector potential [5, 6]. In the reluctance model of an element, the network branches connect the centres of the facets with the centres of the element. A reluctance model of a hexahedron is shown in Fig. 2a. The branch fluxes ϕ_b passing through the facets of elements are related to the facet values of the flux density vector $\boldsymbol{B}$ in the element [5, 6]; their distribution may be described by

$$\boldsymbol{u}_\Omega = \boldsymbol{R}_\mu \boldsymbol{\phi}_b - \boldsymbol{\theta}, \tag{1}$$

where $\boldsymbol{R}_\mu$ is the matrix of branch reluctances, $\boldsymbol{u}_\Omega$ the vector of branch magnetic potential differencies, and $\boldsymbol{\theta}$ represents the vector of branch magnetomotive forces (*mmf*s). The branch reluctances $R_\mu N$ may be established using interpolating functions applied to the facet element

$$R_{\mu\, p,q} = \iiint_{V_e} \boldsymbol{w}_{fp}^T \boldsymbol{\nu} \boldsymbol{w}_{fq} \mathrm{d}V_e, \tag{2}$$

where $\boldsymbol{w}_{fp}$ and $\boldsymbol{w}_{fq}$ are the interpolating functions of the facet element for facets S_p i S_q [5, 7], V_e is the volume of the element, and $\boldsymbol{\nu}$ is the matrix of reluctivities. It should be noted that, in contrast to the classical formulation of the reluctance network, in the network constructed using the edge element method (EEM) mutual reluctances will normally appear [5, 6]. Next, having accounted for the structure of element connections and by introducing a full mesh (loop) matrix $\boldsymbol{k}_e$, the branch fluxes may be expressed in terms of mesh (loop) fluxes φ circulating around element edges

$$\boldsymbol{\phi}_b = \boldsymbol{k}_e \boldsymbol{\varphi}. \tag{3}$$

The loop fluxes represent the edge values of the magnetic vector potential. For example, the flux φ_{37} of Fig. 2a may be associated with the value of the vector potential $\boldsymbol{A}$ for the edge P_3P_7. Substituting (3) into (1), while imposing the condition that the sum of magnetic potential differences in a loop is zero ($\boldsymbol{k}_e^T \boldsymbol{u}_\Omega = \boldsymbol{0}$), yields the following loop equation for the loop fluxes φ of the reluctance network

$$\boldsymbol{k}_e^T \boldsymbol{R}_\mu \boldsymbol{k}_e \boldsymbol{\varphi} = \boldsymbol{k}_e^T \boldsymbol{\theta}. \tag{4}$$

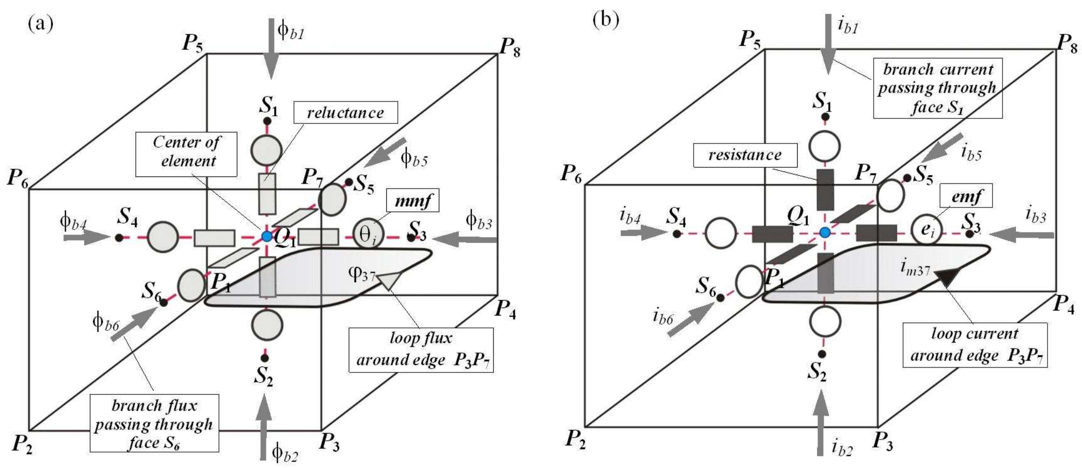

Fig. 2. Network models of a hexahedron: (a) reluctance, (b) resistance

In (4) the product $\boldsymbol{k}_e^T \boldsymbol{\theta}$ represents the vector of loop *mmf*s Θ, that is a vector of the sum of the branch *mmf*s in a loop of the reluctance network. The loop *mmf*s form the vector on the right hand side of (4), which may be related to the vector of branch currents $\boldsymbol{i}_e$ associated with the edges of elements, that is currents flowing through the loops of the reluctance network [6], (see also Fig. 4a)

$$\boldsymbol{\Theta} = \boldsymbol{k}_e^T \boldsymbol{\theta} = \boldsymbol{i}_e. \tag{5}$$

In electromagnetic field analysis the following circumstances may arise where magnetic field is created by: (a) magnetisation currents in permanent magnets, (b) conduction currents in filament windings, or (c) eddy currents induced in massive conductors or bulk materials. In the case of permanent magnets the matrix Θ is calculated from the distribution of the magnetisation vector $\boldsymbol{T}_m$ [8]. For filament currents (b), the sources are defined by the edge values of current density $\boldsymbol{J}$ [9]. For the case (c) of induced currents, the method of establishing the source term depends on the formulation used; for the approach presented in this paper, based on $\boldsymbol{A}$-$\boldsymbol{T}$-$\boldsymbol{T}_0$ and associated RRN method, the vector $\boldsymbol{i}_e$ is obtained from the distribution of edge values of vector potentials $\boldsymbol{T}$ and $\boldsymbol{T}_0$ of the relevant resistance network. Further details related to the description of sources are provided later in the article.

Resistance Network ($R_\sigma N$)

A resistance network ($R_\sigma N$) may be developed by applying the edge element method formulated in terms of the electric vector potential $\boldsymbol{T}$ [6]. The network branches, in a similar way as in the previously described reluctance network, are associated with the connections between the centres of the facets with the centres of the element (Fig. 2b). The branch currents $\boldsymbol{i}_b$ associated with the facets of elements are linked to the facet values of the current density $\boldsymbol{J}$ [6], and their distribution may be described by

$$\boldsymbol{u}_e = \boldsymbol{R}_\sigma \boldsymbol{i}_b - \boldsymbol{e} \, , \tag{6}$$

where $\boldsymbol{R}_\sigma$ is the matrix of branch resistances derived from the interpolating functions of the facet element, $\boldsymbol{u}_e$ is the vector of branch electric potential differences (voltage drops), and $\boldsymbol{e}$ represents the vector of branch electromotive forces (*emf*s). As before for the reluctance network, by taking account of the structure of element connections and introducing the matrix $\boldsymbol{k}_e$, the branch currents may be expressed in terms of mesh (loop) currents $\boldsymbol{i}_m$ around the element edges

$$\boldsymbol{i}_b = \boldsymbol{k}_e \boldsymbol{i}_m \, . \tag{7}$$

Substituting (7) to (6) and incorporating the Kirchhoff's voltage law that the sum of branch potential drops in a loop is equal to zero ($\boldsymbol{k}_e^T \boldsymbol{u}_e = \boldsymbol{0}$) leads to the following expression for loop currents

$$\boldsymbol{k}_e^T \boldsymbol{R}_\sigma \boldsymbol{k}_e \boldsymbol{i}_m = \boldsymbol{k}_e^T \boldsymbol{e} \, . \tag{8}$$

The branch resistances $\boldsymbol{R}_\sigma$ may be derived form the interpolating functions of the facet element as

$$R_{\sigma\, p,q} = \iiint\limits_{V_e} \boldsymbol{w}_{fp}^T \boldsymbol{\sigma} \boldsymbol{w}_{fq} \mathrm{d}V_e \, , \tag{9}$$

where the vectors $\boldsymbol{w}_{fp}$ are $\boldsymbol{w}_{fq}$ are the interpolating functions of the facet element for the facets S_p and S_q, V_e is the volume of the element, and $\boldsymbol{\sigma}$ is a matrix of resistivities of the conducting materials. As in the case of the reluctance networks, the resistance networks will contain mutual resistances [6]. The sum of branch *emf*s, represented by the product $\boldsymbol{k}_e^T \boldsymbol{e}$, is equal to the vector of loop *emf*s $\boldsymbol{e}_o$ in the loops of the resistance network. The vector of loop *emf*s is described by the right hand side of (8), which is established by time differentiation of branch fluxes $\boldsymbol{\phi}_e$ associated with element edges (Fig. 4b), that is fluxes passing through the loops of the resistance network [6]. Hence

$$\boldsymbol{e}_o = \boldsymbol{k}_e^T \boldsymbol{e} = -\frac{\mathrm{d}\boldsymbol{\phi}_e}{\mathrm{dt}} \, . \tag{10}$$

The procedure described above provides a very efficient method but is applicable – in the form presented – only to the analysis of singly-connected regions. Until recently it was argued that it would not be possible to use the electric vector potential $\boldsymbol{T}$ in cases of multiply-connected regions, that is regions containing 'holes' [3, 4]. However, the majority of conducting components in practical electromechanical devices contain such multiply-connected regions; examples include cage rotors of induction motors, windings made of multi-turn 'rods', coils made of thin filaments, bulk conducting elements in the rotor with holes introduced to limit losses due to eddy currents. A direct application of the network formulation $R_\sigma N$ using the electric vector potential $\boldsymbol{T}$ leads to the loop equations (8), which refer only to the loops containing currents induced around the element edges. Although the number of such equations is usually higher than the number of independent loops, it has been found that for multiply-connected regions it is impossible to create a set of fundamental loops necessary for achieving a unique solution – the reason is because the equations do not contain information about the current flow in the loops around the holes [4]. It is therefore necessary to supplement the loop equations with additional conditions expressed in terms of the potential $\boldsymbol{T}_0$ describing current flow around the multiply-connected regions [3, 4].

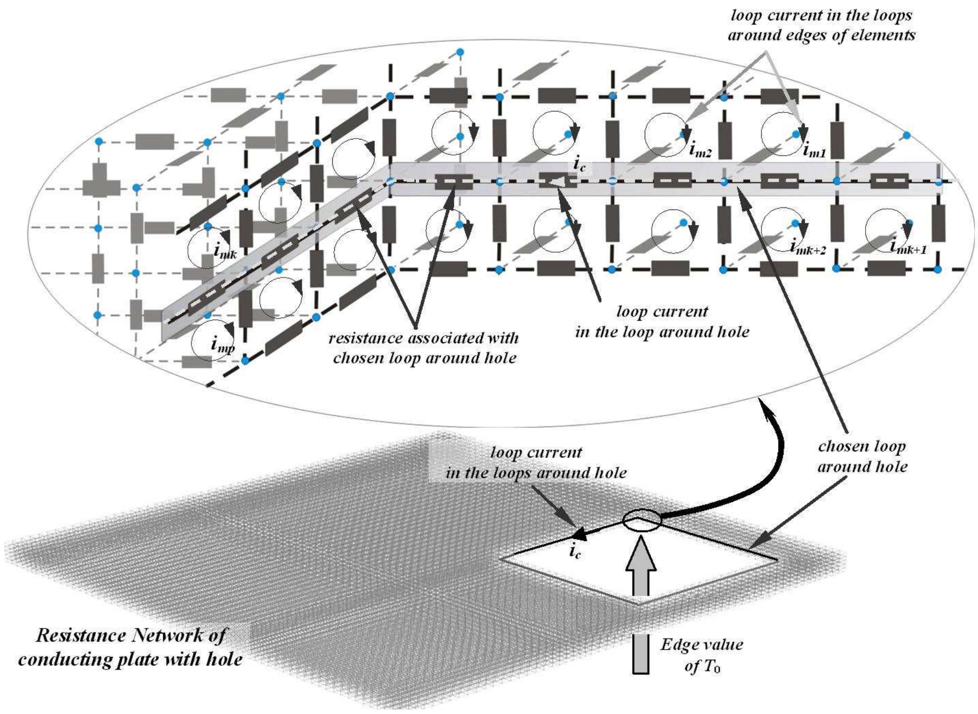

Fig. 3. Selecting the loop current i_c representing the edge value of potential $\boldsymbol{T}_0$

In the TEAM Workshop problem No. 7 considered here, the conducting plate is a doubly-connected region with asymmetrically positioned internal hole (Fig. 1). Following the argument presented above, in order to analyse the induced current distribution in the plate using the resistance network, it was necessary to expand the loop equations (8) by adding a supplementary equation describing the distribution of current i_c around the hole, which has in fact created another loop. The selection of this supplementary loop is guided by the requirement that any loop must contain the hole within itself; the appropriate procedure for our example case is depicted in Fig. 3. This allows a matrix $\boldsymbol{z}_e$ of 'cuts' between loop surfaces and element edges to be established, known as a surface-edge or S-E approach – further details of this technique may be found in [3, 4]. In the special case of Fig. 3 the matrix $\boldsymbol{z}_e$ has just one column; it enables specification of those resistances of the network which belong to the auxiliary loop and through which the current i_c will flow. Having incorporated the additional loop into the $\mathrm{R}_\sigma\mathrm{N}$ model, the branch currents $\boldsymbol{i}_b$ may be expressed as a sum of two terms: the first containing all loop currents around element edges, and the second expressed in terms of the current in the additional loop (or additional loops in the general case of multiply-connected regions). Expression (7) becomes

$$\boldsymbol{i}_b = \boldsymbol{k}_e \boldsymbol{i}_m + \boldsymbol{z}_e i_c \, . \tag{11}$$

As a consequence, the loop equations (8), supplemented by the condition arising through the introduction of the additional loop equation, may now be written as

$$\begin{bmatrix} \boldsymbol{k}_e^T \boldsymbol{R}_\sigma \boldsymbol{k}_e & \boldsymbol{k}_e^T \boldsymbol{R}_\sigma \boldsymbol{k}_e \boldsymbol{z}_e \\ \boldsymbol{z}_e^T \boldsymbol{k}_e^T \boldsymbol{R}_\sigma \boldsymbol{k}_e & \boldsymbol{z}_e^T \boldsymbol{k}_e^T \boldsymbol{R}_\sigma \boldsymbol{k}_e \boldsymbol{z}_e \end{bmatrix} \cdot \begin{bmatrix} \boldsymbol{i}_m \\ i_c \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}_o \\ e_{co} \end{bmatrix}, \tag{12}$$

where e_{co} is the *emf* in the loop around the hole [4] as given by

$$e_{co} = \boldsymbol{z}_e^T \boldsymbol{e}_o = -\boldsymbol{z}_e^T \frac{\mathrm{d}\phi_e}{\mathrm{d}t} \, . \tag{13}$$

Reluctance-Resistance Model

The reluctance-resistance network (RRN) can now be developed by appropriate coupling – via sources, that is through loop *mmf*s and *emf*s – of a reluctance network and a resistance network

(Fig. 2). The loop *mmf*s in the network R$_\mu$N are equal to the branch currents $\boldsymbol{i}_e$ associated with the element edges. These currents can be calculated from branch currents $\boldsymbol{i}_b$ associated with element facets of the R$_\sigma$N network as

$$\boldsymbol{\Theta} = \boldsymbol{i}_e = \boldsymbol{K}^T \boldsymbol{i}_b \,, \tag{14}$$

where $\boldsymbol{K}$ is the transposition matrix converting branch quantities associated with facets to branch quantities associated with edges of the elements [3]. The relationship between branch currents $\boldsymbol{i}_b$ of the resistance network and the currents passing through a single loop of the reluctance network and its relevant *mmf* is depicted in Fig. 4 for the case of a parallelepiped elements. The current $i_{e\,i,j}$ shown in Fig. 4a is calculated by adding (with appropriate signs and weights) the currents i_{bk} (Fig. 4b); for example, for a hexahedron each weight equals 1/8 [6]. When the branch currents $\boldsymbol{i}_b$ are expressed by loop currents $\boldsymbol{i}_m$ and the current i_c - expression (11), then the currents $\boldsymbol{i}_e$ can be calculated as follows

$$\boldsymbol{i}_e = \boldsymbol{K}^T \boldsymbol{k}_e \boldsymbol{i}_m + \boldsymbol{K}^T \boldsymbol{k}_e \boldsymbol{z}_e i_c \,. \tag{15}$$

The loop *emf*s in the resistance network are calculated as time derivatives of branch fluxes $\boldsymbol{\phi}_e$. To find the fluxes $\boldsymbol{\phi}_e$, first – on the basis of the loop fluxes – the fluxes $\boldsymbol{\phi}_b$ are established, which are at the same time branch fluxes in the reluctance network. The relationship (3) can be used for this purpose. Next, by using the matrix $\boldsymbol{K}$, that is after summing up of branch fluxes (again with appropriate signs and weights) [4, 6], the fluxes $\boldsymbol{\phi}_e$ passing through the loops of the resistance network may be found (Fig. 4) as

$$\boldsymbol{\phi}_e = \boldsymbol{K}^T \boldsymbol{\phi}_b = \boldsymbol{K}^T \boldsymbol{k}_e \boldsymbol{\varphi} \,, \tag{16}$$

Merging loop equations (4) of R$_\mu$N with loop equations (12) R$_\sigma$N, while incorporating the conditions describing the coupling between the two networks via the loop *mmf*s and *emf*s, the final set of equations for the system of Fig. 1 has been derived as

$$\begin{bmatrix} \boldsymbol{k}_e^T \boldsymbol{R}_\mu \boldsymbol{k}_e & -\boldsymbol{K}^T \boldsymbol{k}_e & -\boldsymbol{K}^T \boldsymbol{k}_e \boldsymbol{z}_e \\ \mathrm{p}\boldsymbol{K}^T \boldsymbol{k}_e & \boldsymbol{k}_e^T \boldsymbol{R}_\sigma \boldsymbol{k}_e & \boldsymbol{k}_e^T \boldsymbol{R}_\sigma \boldsymbol{k}_e \boldsymbol{z}_e \\ \mathrm{p}\boldsymbol{z}_e^T \boldsymbol{K}^T \boldsymbol{k}_e & \boldsymbol{z}_e^T \boldsymbol{k}_e^T \boldsymbol{R}_\sigma \boldsymbol{k}_e & \boldsymbol{z}_e^T \boldsymbol{k}_e^T \boldsymbol{R}_\sigma \boldsymbol{k}_e \boldsymbol{z}_e \end{bmatrix} \begin{bmatrix} \boldsymbol{\varphi} \\ \boldsymbol{i}_m \\ i_c \end{bmatrix} = \begin{bmatrix} \boldsymbol{\Theta}_z \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} . \tag{17}$$

where p is the differential operator (p=d/dt) and the vector $\boldsymbol{\Theta}_z$ represents the loop *mmf*s set up by the flow of current through the winding above the conducting plate (Fig. 1).

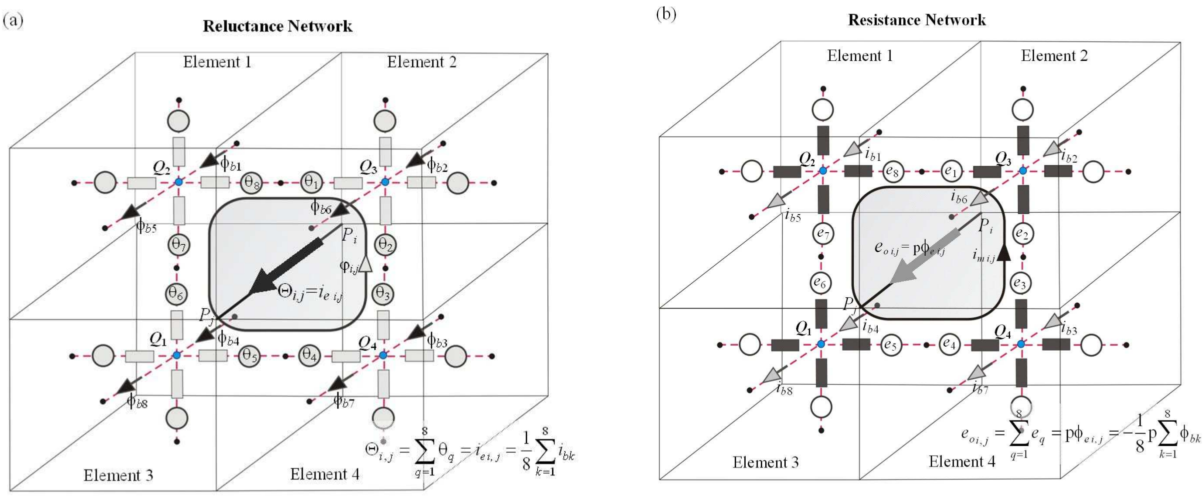

Fig. 4. The coupling in the RRN formulation between the networks: (a) reluctance, (b) resistance

Results and Conclusions

Based on the presented enhanced RRN formulation, a dedicated software algorithm has been developed for calculation of transient electromagnetic fields in 3D space. The block relaxation method (BRM), combined with Cholesky decomposition procedure, has been applied to solve the model equations. The software allows the determination of induced current distributions in multiply connected conducting regions. As an example, the test problem No. 7 of the International TEAM Workshops has been examined. The system consists of a coil and a conductive plate with an asymmetrically situated 'hole' – see, Fig. 1. The relevant space has been subdivided into about 150 000 hexahedron elements. The total number of RRN equations is approximately 450 000. The calculations have been performed for the value of the current density in the coil J_{cu}=1,0968 A/mm^2 and the source frequency f = 50Hz. Selected results are shown in Figs 5 and 6. Fig. 5a,b,c presents the distributions of the components of the magnetic flux densities (B_x, B_y, B_z) on the surface parallel to the conducting plate, half the distance between the plate and the coil. The corresponding distributions of the components of the vector current densities (J_x, J_y, J_z) on the upper surface of the plate are shown in Fig. 6a,b,c, respectively. The distributions have been determined assuming the maximum value of the current in the coil.

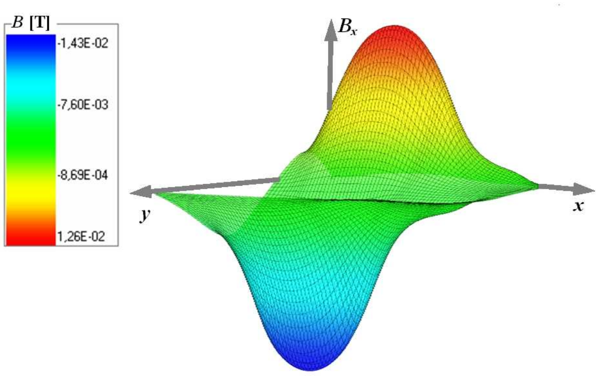

Fig. 5a. Distribution of the x component of the magnetic flux density (B_x)

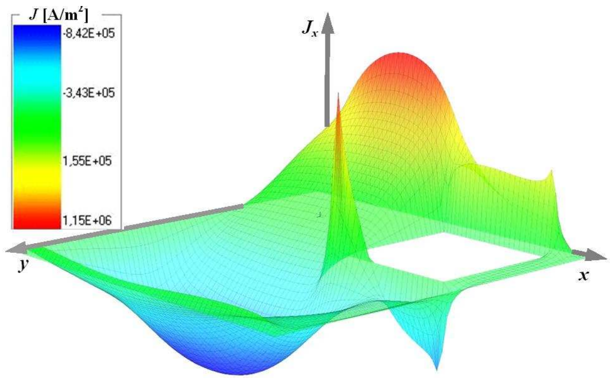

Fig. 6a. Distribution of the x component of the current density (J_x)

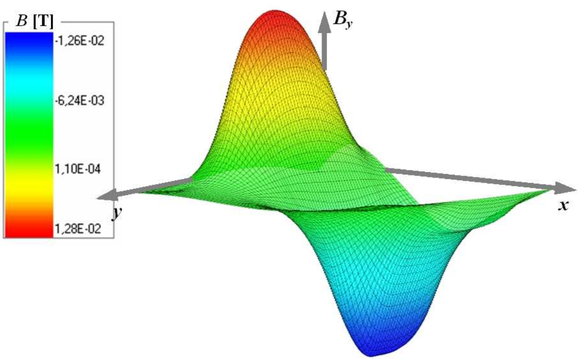

Fig. 5b. Distribution of the y component of the magnetic flux density (B_y)

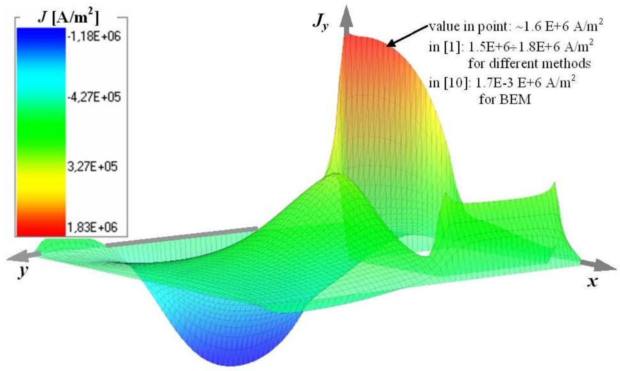

Fig. 6b. Distribution of the y component of the current density (J_y)

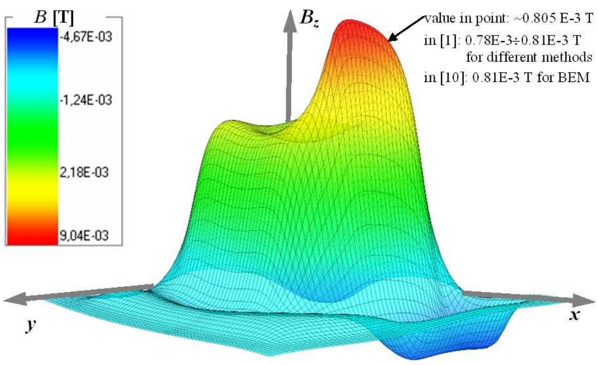

Fig. 5c. Distribution of the z component of the magnetic flux density (B_z)

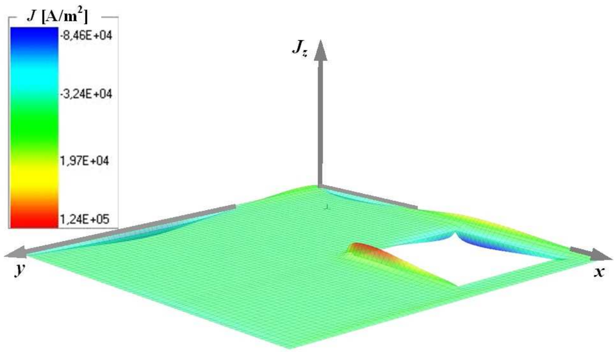

Fig. 6c. Distribution of the z component of the current density (J_z)

A comparison with the results published in [1, 10] and [11] reveals very close agreement and thus high accuracy of the proposed RRN computational scheme. The small differences for a selected point are marked on the figures. The total computational time using BRM and imposed error threshold of 10^{-6} was typically about an hour, which should be compared with 6.5 hours needed to achieve the same accuracy using a reluctance-conductance network described in [12].

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