Congestion games with load-dependent failures: Identical resources ⋆

Michal Penn,

Faculty of Industrial Engineering and Management, Technion, Haifa 32000, Israel

Maria Polukarov *,**,

School of Electronics and Computer Science, University of Southampton SO17 1BJ, Southampton, UK

Moshe Tennenholtz

Microsoft Israel R&D Center, 13 Shenkar Street, Herzeliya 46725, Israel
Faculty of Industrial Engineering and Management, Technion, Haifa 32000, Israel

Abstract

We define a new class of games, *congestion games with load-dependent failures* (CGLFs). In a CGLF each player can choose a subset of a set of available resources in order to try and perform his task. We assume that the resources are identical but that players' benefits from successful completion of their tasks may differ. Each resource is associated with a cost of use and failure probability which are load-dependent. Although CGLFs in general do not have a pure strategy Nash equilibrium, we prove the existence of a pure strategy Nash equilibrium in every CGLF with nondecreasing cost functions. Moreover, we present a polynomial time algorithm for computing such an equilibrium.

Key words: Congestion games, Load-dependent resource failures, Pure strategy Nash equilibrium, Algorithms
JEL classification: C72—Noncooperative Games

⋆ A preliminary version of this paper has been presented at EC-2007 (Penn et al., 2007).

* Corresponding author.
**The work was done when the author was a Ph.D. student at the Technion, Haifa 32000, Israel.

Email addresses: mpenn@ie.technion.ac.il (Michal Penn), pmasha@tx.technion.ac.il, mp3@ecs.soton.ac.uk (Maria Polukarov), moshet@microsoft.com, moshet@ie.technion.ac.il (Moshe Tennenholtz).

1 Introduction

Congestion games have become a major issue of study for the interplay between game theory and computer science, and are widely discussed in the multi-agent systems and the electronic commerce literature. In a classical congestion game, as defined by Rosenthal (1973), there exists a set of n players and a set of m resources. A strategy for a player is associated with a subset of the resources. In general, each player has his own set of possible strategies. Notice that a strategy is associated with a subset of the resources and not with a particular resource. Each resource is associated with a resource utility function, which determines the utility of a player who selected this resource as a function of the number of players using it. Given a strategy profile, a single strategy for each player, it determines the number of players who will be using each resource. The payoff for a player will be the sum of his utilities from the resources he has selected. In many cases the term "resource utility function" is replaced by the term "resource cost function", to reflect the nature of the particular application; the definition however remains as the one discussed above. In many applications discussed in the literature the resource utility function is decreasing as a function of the number of users (or, the resource cost function is increasing). This may reflect situations where a resource is a service provider whose costs per user are increasing due to competition on internal resources. In other applications, such as cost sharing, the resource cost functions are decreasing, reflecting cooperation among the users.

Consider the application of congestion games to a service industry. Let us assume that each resource is a law firm. A client pays the firm on an hourly basis. The law firm employs many lawyers that have access to a small number of shared resources (starting from local printers, and ending up with sources of information). When a client arrives he is assigned to a particular lawyer, where each lawyer can handle only a fixed small capacity of clients. Hence, when clients arrive the lawyers find themselves competing on a set of available resources, the time spent per client increases as a function of the number of clients of the firm, and the cost per client is monotonically increasing in the number of clients. Needles to say, this situation is typical to many canonical examples in the service industry, such as law firms, accounting firms, and private detective agencies.

Consider now congestion games as discussed above with the following addition. Assume that each resource may fail with probability f, where this probability may depend on the number of players who have selected it: the more players have selected this resource the higher the probability it will fail to deliver. Player i which has a set of available strategies Σ_i (recall that each strategy is a subset of the resources), is paid an additional utility v_i if at least one of the resources in his selected strategy, $\sigma_i \in \Sigma_i$, does not fail. Obviously, this defines a strict extension of congestion games; classical congestion games are obtained when selecting $f = 1$ or $v_i = 0$ for every player. In this paper we consider a restriction on this strict generalization of congestion games: all resources are taken to be identical, and all possible subsets of resources are available as possible strategies for each player. We call this model "Congestion Games with Load Dependent Failures" [CGLFs] [1] .

Let us now return to the motivation behind CGLFs. Recall the application of congestion games to the service industries, and consider for example the situation in the private detective agencies sector. In this case, a typical client may approach several private detective agencies, and will be charged by all; moreover, typically, the client is interested in the verification of a particular question, and will benefit as long as at least one of the agencies is able to deliver the answer; needless to say, delivery by an agency may fail, and may depend on the number of clients served by the agency. Such situations fit squarely into the CGLF setting. Of course, one may wish to re-visit the model, and consider various possible modifications; CGLFs however are the first attempt to handle such fundamental issues. The reader should be careful about the use of the term "resource"; in the service industry examples each resource is a firm in the formal model; the fact that resource cost functions may be increasing is a result of the competition on internal resources within the firm, which are not part of the model.

In a CGLF, the resource cost function may be increasing as in the service industry example, or decreasing. The latter typically reflects situations when there is price reduction due to economy of scale (for example, when the resources in the game represent buyer clubs which enable users to share their costs). An important modeling issue is whether costs are incurred by resources which fail to deliver; that is, would one need to pay for the service regardless of whether a success has been declared, or only in the case of success. Indeed, both options are reasonable. For simplicity, we will assume one of the options, and later discuss the other one. Both options lead to similar results.

In a previous paper we discussed the model of "Congestion Games with Failures" [CGFs] (Penn et al., 2005). Although the terms CGF and CGLF may sound similar, these models refer to very different classes of situations. In a CGF players care about the delay caused by using a set of alternative resources, and therefore the payoff of a player is determined by the minimum of delays of the set of selected resources (and by incompletion costs). Notice that this model does not define an extension of congestion games since it does not consider the additvity of costs across selected resources. Therefore, CGFs are an interesting model that model very different situations than CGLFs,

[1] Although one may wish to denote general congestion games with load-dependent failure by CGLFs, and use a different notation for the special case of identical resources, for simplicity of exposition we have chosen to use CGLF for the special case of identical resources studied in this paper.

have very different motivation, and do not refer to an extension of congestion games. Indeed, the proof techniques discussed for CGFs and CGLFs are completely different. In fact, if we take the probability of failure to be a constant, as is done in CGFs, then CGLFs will possess a potential function and therefore can be viewed as congestion games (Monderer and Shapley, 1996). While this is a technical observation, the major differences stem from the fundamentally different assumptions discussed above. It is also worth noticing that CGLFs should not be viewed as an extension of simple network games, where the network is a set of parallel identical links. They indeed refer to a strict extension of congestion games, although our study deals only with the identical-resources complete strategy spaces version.

In this paper we introduce CGLFs and show that:

- CGLFs and, in particular, CGLFs with nondecreasing cost functions, do not always admit a potential function. Therefore, they are not isomorphic to congestion games. Nevertheless, if the failure probabilities are constant (do not depend on the congestion) then a potential function is guaranteed to exist.
- CGLFs and, in particular, CGLFs with decreasing cost functions, do not always possess pure strategy Nash equilibria. However, as we show in our main result, there exists a pure strategy Nash equilibrium in any CGLF with nondecreasing cost functions. Moreover, we present an efficient algorithm for constructing such an equilibrium in any given CGLF with nondecreasing costs. The time complexity of our algorithm is $O(n^2m + nm^2)$, where n and m represent the number of players and resources, respectively.

While the work on CGFs is the most related to our current study, our work can be viewed as part of the literature extending upon congestion games. In particular, Leyton-Brown and Tennenholtz (2003) extended the class of (simple) congestion games to the class of *local-effect games*. Note that in simple congestion games, the strategy set of each player consists of singleton sets. In a local-effect game, each player's payoff is effected not only by the number of players who have chosen the same resources as he has chosen, but also by the number of players who have chosen neighboring resources (in a given graph structure). The authors (Leyton-Brown and Tennenholtz, 2003) showed both theoretically and empirically that these games often (but not always) have pure strategy Nash equilibria. Congestion games were also generalized to *weighted congestion games* in which each player has a weight, and the cost of the resource depends on the total weight of players. These games do not possess a pure strategy Nash equilibrium (Fotakis et al., 2005). However, they have a pure equilibrium in the case of simple games, in which the strategy set of every player is a singleton set, with nondecreasing cost functions (Even-Dar et al., 2003). Monderer (2007) dealt with another generalization of congestion games, in which the resource cost functions are not universal but *player-specific* – *PS-congestion games*. He defined PS-congestion games of type q – *q-congestion* *games*, where q is a positive number, and showed that every game in strategic form is a q-congestion game for some q. Perhaps the most known extension of congestion games is Milchtaich's work on player-specific resource cost functions (Milchtaich, 1996). Milchtaich showed that simple and strategy-symmetric PS-congestion games with nondecreasing cost functions do not admit a potential function, but always possess a Nash equilibrium in pure strategies. Recall that CGLFs are not simple congestion games, nor an extension of simple congestion games; indeed, the complexity of the analysis of CGLFs stems from the ability to select several resources.

The rest of the paper is organized as follows. In Section 2 we define our model. In Sections 3 – 5 we present our results. In 3 we show that CGLFs, in general, do not have pure strategy Nash equilibria. In 4 and 5 we focus on CGLFs with nondecreasing cost functions (nondecreasing CGLFs). We show that these games do not admit a potential function. However, in our main results we prove the existence of a pure strategy Nash equilibrium in non-decreasing CGLFs, and present an efficient algorithm for computing such an equilibrium. In Section 6 we consider a modified model of CGLFs in which the players are required to pay only for non-faulty resources they use. Section 7 presents a short summary and discussion of future research.

2 The model

We consider a finite set of players where each player has a task that can be carried out by any element of a set of identical resources (service providers). Each player simultaneously chooses a subset of the resources in order to perform his task, and his aim is to maximize his own expected payoff, as described in the sequel.

Let $N = \{1, \ldots, n\}$ be a set of n players and let $M = \{e_1, \ldots, e_m\}$ be a set of m resources. Each resource is associated with a *cost* and a *failure probability*, each of which depends on the number of players who use this resource. We assume that the failure probability of a resource $e \in M$ is a monotone nondecreasing function $f : \{1, \ldots, n\} \to [0, 1)$ of the congestion experienced by e, and that the failure or success of a particular resource is *independent* of the failure or success of other resources. The cost of utilizing resource $e \in M$ is a nonnegative function $c : \{1, \ldots, n\} \to \mathbb{R}_+$ of the congestion on e.

Player $i \in N$ chooses a strategy $\sigma_i \in \Sigma_i$ which is a (possibly empty) subset of the resources. That is, Σ_i is the power set of the set of resources: $\Sigma_i = P(M)$. Given a subset $S \subseteq N$ of the players, the set of strategy combinations of the members of S is denoted by $\Sigma_S = \times_{i \in S} \Sigma_i$, and the set of strategy combinations of the complement subset of players is denoted by Σ_{-S} ($\Sigma_{-S} = \Sigma_{N \smallsetminus S} = \times_{i \in N \smallsetminus S} \Sigma_i$). The set of pure strategy profiles of all the players is denoted by Σ ($\Sigma = \Sigma_N$). Let $\sigma = (\sigma_1, \ldots, \sigma_n) \in \Sigma$ be a pure strategy profile. The (m-

dimensional) *congestion vector* that corresponds to σ is $h(\sigma) = (h_e(\sigma))_{e \in M}$, where $h_e(\sigma) = \left|\{i \in N \,:\, e \in \sigma_i\}\right|$.

The *outcome* for player $i \in N$ is denoted by $x_i \in \{S, F\}$, where S and F, respectively, indicate whether the task execution succeeded or failed. We say that the execution of player i's task *succeeds* if the task of player i is successfully completed by *at least one* of the resources chosen by him. The execution of the player i's task *fails* if none of the i's selected resources completes it successfully. The *benefit* of player i from his outcome x_i is denoted by $V_i(x_i)$, where $V_i(S) = v_i$, a given (nonnegative) value, and $V_i(F) = 0$. W.l.o.g., we assume that the players' benefit values are given in descending order, i.e. $v_1 \geq v_2 \geq \ldots \geq v_n$.

The *utility* of player i from a strategy profile σ and his outcome x_i, $u_i(\sigma, x_i)$, is the difference between his benefit from the outcome, $V_i(x_i)$, and the sum of the costs of the resources he has used:

$$u_i(\sigma, x_i) = V_i(x_i) - \sum_{e \in \sigma_i} c(h_e(\sigma)).$$

The *expected utility* of player i from the strategy profile σ, $U_i(\sigma)$, is therefore,

$$U_i(\sigma) = \left(1 - \prod_{e \in \sigma_i} f(h_e(\sigma))\right) v_i - \sum_{e \in \sigma_i} c(h_e(\sigma)),$$

where $1 - \prod_{e \in \sigma_i} f(h_e(\sigma))$ denotes the probability of a successful completion of player i's task. We use the convention that $\prod_{e \in \varnothing} f(h_e(\sigma)) = 1$. Hence, if player i chooses an empty set $\sigma_i = \varnothing$ (does not assign his task to any resource), then his expected utility, $U_i(\varnothing, \sigma_{-i})$, equals zero.

Remark 1 *In the above model, a player is required to pay for any resource he uses, regardless of its success or failure. Such a scenario, for example, may take place when the success or the failure of every resource cannot be observed, and only the success or the failure of the player's task can be detected. In Section 6 of this paper we extend our model to deal with observable resources, where the players are required to pay only for the non-faulty resources they selected.*

3 CGLFs with no pure strategy Nash equilibrium

We start by showing that the class of CGLFs, and in particular the subclass of CGLFs with decreasing cost functions, does not in general possess Nash equilibria in pure strategies.

Consider a CGLF with two players $N = \{1, 2\}$ and two resources $M = \{e_1, e_2\}$. The cost function of each resource is given by $c(1) = 1$ and $c(2) = \frac{1}{4}$ and the failure probabilities are $f(1) = 0.01$ and $f(2) = 0.26$. The benefits for the players from a successful task completion are $v_1 = 1.1$ and $v_2 = 4$. In Figure 1 we present the payoff matrix of the game. By exploring Figure 1, it can be

		Player 2			
		$\varnothing$	$\{e_1\}$	$\{e_2\}$	$\{e_1, e_2\}$
Player 1	$\varnothing$	$U_1 = 0$ $U_2 = 0$	$U_1 = 0$ $U_2 = 2.96$	$U_1 = 0$ $U_2 = 2.96$	$U_1 = 0$ $U_2 = 1.9996$
	$\{e_1\}$	$U_1 = 0.089$ $U_2 = 0$	$U_1 = 0.564$ $U_2 = 2.71$	$U_1 = 0.089$ $U_2 = 2.96$	$U_1 = 0.564$ $U_2 = 2.7396$
	$\{e_2\}$	$U_1 = 0.089$ $U_2 = 0$	$U_1 = 0.089$ $U_2 = 2.96$	$U_1 = 0.564$ $U_2 = 2.71$	$U_1 = 0.564$ $U_2 = 2.7396$
	$\{e_1, e_2\}$	$U_1 = -0.90011$ $U_2 = 0$	$U_1 = -0.15286$ $U_2 = 2.71$	$U_1 = -0.15286$ $U_2 = 2.71$	$U_1 = 0.52564$ $U_2 = 3.2296$

Figure 1. Example for non-existence of a pure strategy Nash equilibrium in CGLFs.

easily seen that for every pure strategy profile σ in this game there exists a player i and a strategy $\sigma'_i \in \Sigma_i$ such that $U_i(\sigma_{-i}, \sigma'_i) > U_i(\sigma)$. That is, every pure strategy profile in this game is not in equilibrium.

The following two sections focus on the subclass of CGLFs with nondecreasing cost functions (henceforth, *nondecreasing CGLFs*). Nondecreasing CGLFs do not, in general, admit a potential function. Therefore, these games are not congestion games. Nevertheless, we prove that all such games possess a pure strategy Nash equilibrium and develop an efficient algorithm for computing such an equilibrium.

4 Potential functions

Monderer and Shapley (1996) introduced the notions of potential function and potential game, where a potential game is defined to be a game that possesses a potential function. A potential function is a real-valued function over the set of pure strategy profiles, with the property that the gain (or loss) of a player shifting to another strategy while the other players' strategies are kept unchanged, equals to the corresponding increment of the potential function. The authors (Monderer and Shapley, 1996) showed that the classes of finite potential games and congestion games coincide.

Here we show that the class of CGLFs, and in particular the subclass of nondecreasing CGLFs, does not admit a potential function, and therefore is not included in the class of congestion games. However, for the special case of constant failure probabilities, a potential function is guaranteed to exist. To prove these statements we use the following characterization of potential games (Monderer and Shapley, 1996).

A *path* in Σ is a sequence $\tau = (\sigma^0 \to \sigma^1 \to \cdots)$ of strategy profiles such that for every $k \geq 1$ there exists a unique player, say player i, such that $\sigma^k = (\sigma^{k-1}_{-i}, \sigma'_i)$

for some $\sigma_i' \neq \sigma_i^{k-1}$ in Σ_i. A finite path $\tau = (\sigma^0 \to \sigma^1 \to \cdots \to \sigma^K)$ is *closed* if $\sigma^0 = \sigma^K$. It is a *simple* closed path if in addition $\sigma^l \neq \sigma^k$ for every $0 \leq l \neq k \leq K-1$. The *length* of a simple closed path is defined to be the number of distinct points in it; that is, the length of $\tau = (\sigma^0 \to \sigma^1 \to \cdots \to \sigma^K)$ is K.

Theorem 2 (Monderer and Shapley, 1996) *Let G be a game in strategic form with a vector $U = (U_1, \ldots, U_n)$ of utility functions. For a finite path $\tau = (\sigma^0 \to \sigma^1 \to \cdots \to \sigma^K)$, let $U(\tau) = \sum_{k=1}^{K}[U_{i_k}(\sigma^k) - U_{i_k}(\sigma^{k-1})]$, where i_k is the unique deviator at step k. Then, G is a potential game if and only if $U(\tau) = 0$ for every simple closed path τ of length 4.*

4.1 Constant Failure Probabilities

Based on Theorem 2, we show below that CGLFs with constant failure probabilities always possess a potential function.

Theorem 3 *CGLFs with constant failure probabilities are potential games.*

Proof: Assume we are given a game G with constant failure probabilities. Let $\tau = (\alpha \to \beta \to \gamma \to \delta \to \alpha)$ be an arbitrary simple closed path of length 4. Let i and j denote the active players (deviators) in τ and $z \in \Sigma_{-\{i,j\}}$ be a fixed strategy profile of the other players. Let $\alpha = (x_i, x_j, z)$, $\beta = (y_i, x_j, z)$, $\gamma = (y_i, y_j, z)$, $\delta = (x_i, y_j, z)$, where $x_i, y_i \in \Sigma_i$ and $x_j, y_j \in \Sigma_j$. Then,

$$\begin{aligned} U(\tau) &= U_i(x_i,x_j,z) - U_i(y_i,x_j,z) + U_j(y_i,x_j,z) - U_j(y_i,y_j,z) \\ &\quad + U_i(y_i,y_j,z) - U_i(x_i,y_j,z) + U_j(x_i,y_j,z) - U_j(x_i,x_j,z) \\ &= \left(1 - f^{|x_i|}\right) v_i - \sum_{e \in x_i} c(h_e^{(x_i,x_j,z)}) - \ldots - \left(1 - f^{|x_j|}\right) v_j + \sum_{e \in x_j} c(h_e^{(x_i,x_j,z)}) \\ &= \left[\left(1 - f^{|x_i|}\right)v_i - \ldots - \left(1 - f^{|x_j|}\right)v_j\right] - \left[\sum_{e \in x_i} c(h_e^{(x_i,x_j,z)}) - \ldots - \sum_{e \in x_j} c(h_e^{(x_i,x_j,z)})\right]. \end{aligned}$$

Notice that $\left[\left(1 - f^{|x_i|}\right) v_i - \ldots - \left(1 - f^{|x_j|}\right) v_j\right] = 0$, as a sum of a telescope series. The remaining sum equals 0, by applying Theorem 2 to congestion games, which are known to possess a potential function. Thus, by Theorem 2, G is a potential game. □

Remark 4 *We note that the existence of a potential function holds also for more general settings of games with constant failure probabilities. For instance, for non-identical resources having different failure probabilities and cost functions and general cost functions (not necessarily monotone and/or nonnegative). This follows from the fact that if the failure probabilities are constant, then the expected benefit (revenue) of each player does not depend on the choices of the other players. In addition, for each player, the sum of the costs over his chosen subset of resources, equals the payoff of a player choosing* *the same strategy in the corresponding congestion game.*

However, as presented in the next paragraph, if the failure probabilities are not constant then even player- and resource-symmetric CGLFs do not admit a potential function. As a result, the class of CGLFs is not isomorphic to the class of congestion games.

4.2 Load-Dependent Failures

We show that no CGLF with $n \geq 2$ players, $m \geq 2$ resources and load-dependent failure probabilities, in which the players' benefits from a successful task completion are strictly positive and identical, is a potential game. Note that since there is no potential function in player-symmetric CGLFs, its absence is a result of the added features of the CGLF's setting (namely, the resource failures) and not due to the player-specific utility functions. Moreover, in some cases CGLFs with player-specific utilities may possess a potential function.

Proposition 5 *No CGLF with non-constant failures, $n, m \geq 2$ and $v_i = v > 0$ for all $i \in N$, possesses a potential function.*

Proof: Let G be a player-symmetric CGLF with $n, m \geq 2$ and $v_i = v > 0$ for all $i \in N$. Let $k \in \{1, \ldots, n-1\}$ be an arbitrary integer satisfying $f(k) < f(k+1)$,[2] and let $z \in \Sigma_{-\{1,2\}}$ be a combination of the strategies of all the players excluding players 1 and 2, satisfying $h^z_{e_1} = h^z_{e_2} = k - 1$.[3] Consider the simple closed path of length 4 which is formed by $\alpha = (\varnothing, \{e_2\}, z)$, $\beta = (\{e_1\}, \{e_2\}, z)$, $\gamma = (\{e_1\}, \{e_1, e_2\}, z)$, $\delta = (\varnothing, \{e_1, e_2\}, z)$, where players 1 and 2 change their strategies, while the others play z. The expected utilities of the deviators (players 1 and 2) on the path $\alpha \to \beta \to \gamma \to \delta \to \alpha$ are presented in Figure 2. Exploring Figure 2 we get

		Player 2: $\{e_2\}$	Player 2: $\{e_1, e_2\}$
Player 1	$\varnothing$	$U_1 = 0$ $U_2 = (1 - f(k))\, v - c(k)$	$U_1 = 0$ $U_2 = \left(1 - f(k)^2\right) v - 2c(k)$
	$\{e_1\}$	$U_1 = (1 - f(k))\, v - c(k)$ $U_2 = (1 - f(k))\, v - c(k)$	$U_1 = (1 - f(k+1))\, v - c(k+1)$ $U_2 = (1 - f(k)f(k+1))\, v - c(k) - c(k+1)$

Figure 2. The non-existence of a potential function in CGLFs.

$$\begin{aligned} &U_1(\alpha) - U_1(\beta) + U_2(\beta) - U_2(\gamma) + U_1(\gamma) - U_1(\delta) \\ &+U_2(\delta) - U_2(\alpha) = v\,(1 - f(k))\,(f(k) - f(k+1)) < 0, \end{aligned} \tag{1}$$

[2] If there is no such k then f is constant and, as has been shown in the previous paragraph, a potential function is guaranteed to exist.

[3] That is, e_1 and e_2 are chosen by exactly $k-1$ players, excluding players 1 and 2. For instance, let $k-1$ players in $N \setminus \{1,2\}$ play $\{e_1, e_2\}$ and all the others play $\varnothing$.

which implies the non-existence of a potential function. □

As a result, CGLFs are not congestion games. Note that this result appears to be independent of the resource cost functions (see (1)); therefore, it applies in particular to the special cases of nondecreasing and constant costs.

5 Pure strategy Nash equilibria in nondecreasing CGLFs

Here we present our main results on CGLFs – the proof of existence and efficient construction of a pure strategy Nash equilibrium in nondecreasing CGLFs. In Section 6, to follow, we point out that similar results hold also for the observable resource case.

5.1 Existence of a pure strategy Nash equilibrium

Our existence proof is based on the idea of using some basic operations on single resources ("A-, D- and S-moves", referred to as "single moves", to be defined below) to reduce the size of the set of possible player deviations from a given strategy profile, that we need to examine. We prove that there exists a "post-addition D-stable strategy profile" (to be defined in the sequel), and by applying a series of single moves to it, we guarantee to reach an equilibrium.

5.1.1 Single moves and the single profitable move property

The simplest deviations from a strategy profile in a CGLF involve moves with single resources, referred to as *single moves.*

Definition 6 *For any strategy profile $\sigma \in \Sigma$ and for any player $i \in N$, the operation of adding precisely one resource to his strategy, σ_i, is called an **A-move** of i from σ. Similarly, the operation of dropping a single resource is called a **D-move**, and the operation of switching one resource with another is called an **S-move**.*

Below we give intuitive and technical characterizations of single moves in CGLFs, which we will make use of in our results. Let $\sigma \in \Sigma$, $i \in N$ and $a \in \sigma_i$. We say that a D-move with a is *profitable* for i if $U_i\left(\sigma_i \setminus \{a\}, \sigma_{-i}\right) > U_i(\sigma)$. That is,

$$v_i\left(1 - \prod_{e\in\sigma_i\setminus\{a\}} f(h_e(\sigma))\right) - \sum_{e\in\sigma_i\setminus\{a\}} c(h_e(\sigma)) > v_i\left(1 - \prod_{e\in\sigma_i} f(h_e(\sigma))\right) - \sum_{e\in\sigma_i} c(h_e(\sigma)),$$

which is equivalent to

$$c(h_a(\sigma)) > v_i(1 - f(h_a(\sigma))) \prod_{e \in \sigma_i \setminus \{a\}} f(h_e(\sigma)). \tag{2}$$

Note that the right hand side in the above inequality stands for the expected benefit of player i from utilizing resource a at σ, if a is the only successful resource in his chosen set of resources. Clearly, if this value is less than $c(h_a(\sigma))$, the cost of using resource a at σ, then dropping a is a profitable move for player i. For simplicity of exposition, we use the following notation.

- For $i \in N$, $a \in M$ and $\sigma \in \Sigma$, let $\mathbf{v}_i^a(\sigma) = v_i(1-f(h_a(\sigma))) \prod_{e \in \sigma_i \setminus \{a\}} f(h_e(\sigma))$ denote the *marginal benefit* of i from a at σ. Note that due to the monotonicity of $f(\cdot)$, $\mathbf{v}_i^a(\cdot)$ (weakly) decreases with the congestion on a and increases with the congestion on each of the other resources.

Using the above notation, (2) can be rewritten as $c(h_a(\sigma)) > \mathbf{v}_i^a(\sigma)$. This means that, relative to σ, it is profitable for i to drop resource a if (and only if) the cost of using a is greater than its marginal benefit.

Similar inequalities can be derived also for A- and S-moves as follows. An A-move from σ with resource $b \notin \sigma_i$ is profitable for i, i.e. $U_i(\sigma_i \cup \{b\}, \sigma_{-i}) > U_i(\sigma)$, if and only if

$$v_i(1 - f(h_b(\sigma) + 1)) \prod_{e \in \sigma_i} f(h_e(\sigma)) > c(h_b(\sigma) + 1).$$

The above inequality indicates that the A-move with b is profitable for i if and only if the marginal benefit from b at the resulting profile, $(\sigma_i \cup \{b\}, \sigma_{-i})$, is greater than its cost, i.e. $\mathbf{v}_i^b(\sigma_i \cup \{b\}, \sigma_{-i}) > c(h_b(\sigma) + 1)$. In a similar way, we conclude that an S-move from $a \in \sigma_i$ to $b \notin \sigma_i$ is profitable for i if and only if the marginal benefit from switching from a to b is greater than the corresponding difference of their costs, i.e. $\mathbf{v}_i^b(\sigma_i \cup \{b\} \setminus \{a\}, \sigma_{-i}) - \mathbf{v}_i^a(\sigma) > c(h_b(\sigma)+1) - c(h_a(\sigma))$. We also note that, due to the monotonicity of $f(\cdot)$ and $c(\cdot)$, the above inequality holds if and only if $h_b(\sigma)+1 < h_a(\sigma)$ and $f(\cdot)$ or $c(\cdot)$ strictly increases from $h_b(\sigma) + 1$ to $h_a(\sigma)$. We summarize the above discussion in the following observation.

Observation 7 *Let σ be a strategy profile with $a \in \sigma_i$ and $b \notin \sigma_i$ for some $i \in N$. Then,*
(1) *A D-move with a is profitable for i if and only if $c(h_a(\sigma)) > \mathbf{v}_i^a(\sigma)$.*
(2) *An A-move with b is profitable for i if and only if $\mathbf{v}_i^b(\sigma_i \cup \{b\}, \sigma_{-i}) > c(h_b(\sigma) + 1)$.*
(3) *An S-move from a to b is profitable for i if and only if $h_b(\sigma) + 1 < h_a(\sigma)$ and $f(\cdot)$ or $c(\cdot)$ strictly increases from $h_b(\sigma) + 1$ to $h_a(\sigma)$.*

The following lemma implies that any strategy profile in which no player wishes unilaterally to apply a single A-, D- or S-move, is a Nash equilibrium. More precisely, we show that if there exists a player who benefits from a unilateral deviation from a given strategy profile, then there exists a single A-, D- or S-move which is profitable for him as well. This property is called the *single profitable move property*.

Lemma 8 (The single profitable move property) *Given a nondecreasing CGLF, let $\sigma \in \Sigma$ be a strategy profile which is not in equilibrium, and let $i \in N$ be a player for which a profitable deviation from σ is available. Then, i has a profitable A-, D- or S-move from σ.*

The outline of the proof is as follows. Assume on the contrary that σ possesses only non-single-move deviations. Each such deviation can be decomposed into a series of single moves. Consider such a deviation, say σ', with a decomposition consisting of a minimal number of single moves. Inverting any of these single moves is strictly non-profitable with respect to σ' (otherwise, it could have been omitted from the original deviation to result a shorter sequence of single moves). This, as we show below, implies that this move from σ was beneficial by itself. The formal proof is given below.

Proof: Let σ be a non-equilibrium strategy profile and let $i \in N$ be a player who can benefit from a unilateral deviation from σ. Let $PD_i(\sigma)$ denote the set of all profitable deviations of i from σ, that is

$$PD_i(\sigma) = \{x_i \in \Sigma_i \, : \, U_i(\sigma_{-i}, x_i) > U_i(\sigma)\}.$$

For any pair of sets A and B, let $\mu(A, B) = \max\{|A \setminus B|, |B \setminus A|\}$. Clearly, if player i deviates from strategy σ_i to strategy x_i by applying a single A-, D- or S-move, then $\mu(x_i, \sigma_i) = 1$, and vice versa, if $\mu(x_i, \sigma_i) = 1$ then x_i is obtained from σ_i by applying exactly one such move. Let $y_i \in \arg\min_{x_i \in PD_i(\sigma)} \mu(x_i, \sigma_i)$, and assume that $\mu(y_i, \sigma_i) > 1$.

It is convenient to consider separately each of the following three cases: (i) $|y_i \setminus \sigma_i| = 0$, (ii) $|\sigma_i \setminus y_i| = 0$, and (iii) both $|y_i \setminus \sigma_i|$ and $|\sigma_i \setminus y_i|$ are positive.

In the first case (i), in which $y_i \subsetneq \sigma_i$, let $a \in \sigma_i \setminus y_i$, and consider the strategy profile $y_i' = y_i \cup \{a\}$, obtained by inverting the D-move with a from σ. Clearly, $\mu(y_i', \sigma_i) = |\sigma_i \setminus y_i'| < |\sigma_i \setminus y_i| = \mu(y_i, \sigma_i)$. Hence, by the choice of y_i, $U_i(y_i', \sigma_{-i}) < U_i(y_i, \sigma_{-i})$, implying by Observation 7 that $c(h_a(\sigma)) > \mathbf{v}_i^a(y_i', \sigma_{-i})$. Since $y_i \subsetneq \sigma_i$ and $a \in \sigma_i \setminus y_i$ then $y_i \subseteq \sigma_i \setminus \{a\}$, implying $y_i' \subseteq \sigma_i$. Therefore, the above yields $c(h_a(\sigma)) > \mathbf{v}_i^a(\sigma)$, implying by Observation 7 that $U_i(\sigma_i \setminus \{a\}, \sigma_{-i}) > U_i(\sigma)$. That is, a single D-move with a from σ is profitable for player i.

In the second case (ii), in which $\sigma_i \subsetneq y_i$, let $b \in y_i \setminus \sigma_i$, and consider the strategy profile $y_i'' = y_i \setminus \{b\}$, obtained by inverting the A-move with b from σ. Clearly, $\mu(y_i'', \sigma_i) = |y_i'' \setminus \sigma_i| < |y_i \setminus \sigma_i| = \mu(y_i, \sigma_i)$. Hence, by the choice of y_i, $U_i(\sigma_{-i}, y_i'') < U_i(\sigma_{-i}, y_i)$, implying by Observation 7 that $c(h_b(\sigma) + 1) < \mathbf{v}_i^b(y_i, \sigma_{-i})$. Since $\sigma_i \subsetneq y_i$ and $b \in y_i \setminus \sigma_i$ then $\sigma_i \cup \{b\} \subseteq y_i$. Therefore, the above yields $c(h_b(\sigma)+1) < \mathbf{v}_i^b(\sigma_i \cup \{b\}, \sigma_{-i})$, implying $U_i(\sigma_i \cup \{b\}, \sigma_{-i}) > U_i(\sigma)$.

That is, a single A-move with b from σ is profitable for player i.

In the latter case (iii), both $|y_i \setminus \sigma_i|$ and $|\sigma_i \setminus y_i|$ are positive, hence let $a \in \sigma_i \setminus y_i$ and $b \in y_i \setminus \sigma_i$ and consider the strategy profile $y_i''' = y_i \setminus \{b\} \cup \{a\}$, obtained by inverting the S-move from a to b. Clearly, $|y_i''' \setminus \sigma_i| < |y_i \setminus \sigma_i|$ and $|\sigma_i \setminus y_i'''| < |\sigma_i \setminus y_i|$, yielding $\mu(y_i''', \sigma_i) < \mu(y_i, \sigma_i)$. Hence, by the choice of y_i, $U_i(\sigma_{-i}, y_i''') < U_i(\sigma_{-i}, y_i)$, implying by Observation 7 that $h_b(\sigma) + 1 < h_a(\sigma)$ and $f(\cdot)$ or $c(\cdot)$ strictly increases from $h_b(\sigma)+1$ to $h_a(\sigma)$. This, in turn, yields that $U_i(\sigma_i \setminus \{a\} \cup \{b\}, \sigma_{-i}) > U_i(\sigma)$. That is, a single S-move from a to b is profitable for i. This completes the proof. □

5.1.2 Stability under single moves

Based on the single profitable move property, in order to prove the existence of a pure strategy Nash equilibrium, it suffices to present a strategy profile for which no player wishes unilaterally to apply a single move. This motivates the following definition.

Definition 9 *A strategy profile σ is said to be **A-stable** (resp., **D-stable**, **S-stable**) if there are no players with a profitable A- (resp., D-, S-) move from σ.*

Our goal is to find an A-, D- and S-stable strategy profile, implying this profile is in equilibrium (by the single profitable move property). To describe how we achieve this, we define the notions of light and heavy resources as well as even and nearly-even strategy profiles. These concepts play a central role in the proof of our main results. In particular, we will show that the obtained equilibrium is either an even or a nearly-even strategy profile.

Definition 10 *Given a strategy profile σ, resource e' is called σ-**light** if $e' \in \arg\min_{e \in M} h_e(\sigma)$ and σ-**heavy** otherwise. A strategy profile σ with no heavy resources will be termed **even**. An even strategy profile with a common congestion of k on the resources will be termed **k-even**. A strategy profile σ satisfying $|h_e(\sigma) - h_{e'}(\sigma)| \leq 1$ for all $e, e' \in M$ will be termed **nearly-even**.*

Obviously, every even strategy profile is nearly-even. In addition, in a nearly-even strategy profile all heavy resources (if such exist) have the same congestion. As is shown in the following lemma, this implies the S-stability of that profile.

Lemma 11 *In a nondecreasing CGLF, every nearly-even strategy profile is S-stable.*

Proof: Let σ be a profile with a nearly even congestion on the resources. Then, $h_a(\sigma) \leq h_b(\sigma) + 1$ for any $a, b \in M$, implying by Observation 7 that $U_i(\sigma) \geq U_i(\sigma_i \setminus \{a\} \cup \{b\}, \sigma_{-i})$ for any $i \in N$ with $a \in \sigma_i$ and $b \notin \sigma_i$. Hence, σ is S-stable. □

One can easily check that the opposite direction holds whenever $f(\cdot)$ or $c(\cdot)$ is strictly increasing.

We observe that the profile $\sigma^0 = (\varnothing, \ldots, \varnothing)$ is D-stable and 0-even. Hence, by Lemma 11, σ^0 is D- and S-stable, termed *DS-stable*, and the subset of even, DS-stable strategy profiles is not empty. Based on this observation, one may hope to achieve a Nash equilibrium by applying a series of A-moves to an even, DS-stable strategy profile. The two types of A-moves (”one- and two-step additions”) defined below, keep the nearly-evenness property (and therefore, the S-stability) if applied to a nearly-even strategy profile. However, as we show in the sequel, an A-move (and, in particular, a one- or two-step addition), may destroy the D-stability of the newly obtained profile and cause a chain of D-moves by different players, leading to a reduced congestion on the resources. This implies that an A-move dynamics from an arbitrary initial point does not guarantee the convergence to an equilibrium (see the discussion after Lemma 14 in the sequel).

5.1.3 One- and two-step additions

Given a strategy profile σ, for each player $i \in N$ with $\sigma_i \subsetneq M$ let $e^i \in \arg\min_{e \in M \smallsetminus \sigma_i} h_e(\sigma)$. That is, e^i is a lightest resource not chosen previously by i. Then,

Observation 12 *If there exists a profitable A-move for player i, then the A-move with e^i, a lightest resource not chosen previously by i, is profitable for i as well.*

This is since if player i wishes to unilaterally add a resource, say $a \in M \smallsetminus \sigma_i$, then by Observation 7 and the monotonicity of $f(\cdot)$ and $c(\cdot)$,

$$\mathbf{v}_i^{e^i}(\sigma_i \cup \{e^i\}, \sigma_{-i}) \geq \mathbf{v}_i^a(\sigma_i \cup \{a\}, \sigma_{-i}) > c(h_a(\sigma) + 1) \geq c(h_{e^i}(\sigma) + 1).$$

If no player wishes to change his strategy in this manner, i.e. if $U_i(\sigma) \geq U_i(\sigma_i \cup \{e^i\}, \sigma_{-i})$ for all $i \in N$, then by Observation 12, $U_i(\sigma) \geq U_i(\sigma_i \cup \{a\}, \sigma_{-i})$ for all $i \in N$ and $a \in M \smallsetminus \sigma_i$. Hence, σ is A-stable.

Definition 13 *Assume σ is not A-stable and let $N(\sigma)$ denote the subset of all players for which there exists e^i such that a unilateral addition of e^i is profitable. Let $\mathbf{a} \in \arg\min_{e^i : i \in N(\sigma)} h_{e^i}(\sigma)$ be such a resource of minimum congestion. Let also $i \in N(\sigma)$ be the player for which $e^i = \mathbf{a}$. If $\mathbf{a}$ is σ-light, then let $\sigma' = (\sigma_i \cup \{\mathbf{a}\}, \sigma_{-i})$. In this case we say that σ' is obtained from σ by a **one-step addition** of resource $\mathbf{a}$, and $\mathbf{a}$ is called the **added** resource. If $\mathbf{a}$ is σ-heavy then there exists a σ-light resource $\mathbf{b}$ and a player j such that $\mathbf{a} \in \sigma_j$ and $\mathbf{b} \notin \sigma_j$. Then let $\sigma' = \left(\sigma_i \cup \{\mathbf{a}\}, \sigma_j \smallsetminus \{\mathbf{a}\} \cup \{\mathbf{b}\}, \sigma_{-\{i,j\}}\right)$. In this case we say that σ' is obtained from σ by a **two-step addition** of resource $\mathbf{b}$, and $\mathbf{b}$ is called the **added** resource.*

We notice that, in both cases, the congestion of each resource in σ' is the same as in σ, except for the added resource, for which its congestion in σ' has increased by 1. Thus, if σ is nearly-even then σ' is also nearly-even (since the added resource is σ-light). Then, Lemma 11 implies the S-stability of σ'. However, for D-stable strategy profiles, their D-stability after an addition operation is not necessarily preserved (see Figure 3 below), and a (possibly long) chain of D-moves may follow (such an example is presented at the end of this paragraph). Nevertheless, as we show in Lemma 14 below, if σ is a

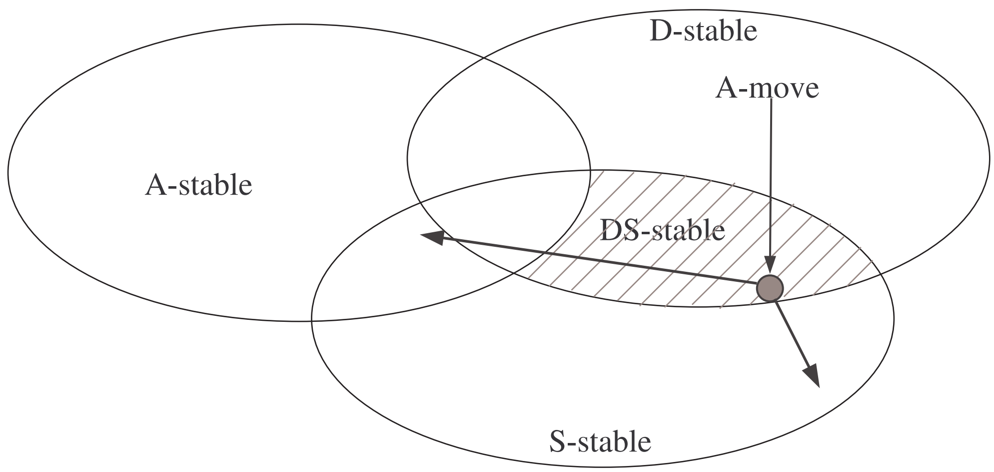

Figure 3. Applying an A-move to a DS-stable profile may destroy the D-stability.

D-stable and nearly-even strategy profile then the only potential cause for an in-D-stability of σ', which is obtained from σ by a one- or two-step addition, is the existence of a player who used the added resource before the addition operation and wishes to drop it after it had been added by another player.

Lemma 14 *Let σ be a nearly-even and D-stable strategy profile of a given nondecreasing CGLF, and let σ' be obtained from σ by a one- or two-step addition of resource* $\mathbf{a}$. *Then, there are no profitable D-moves for any player $i \in N$ with $\sigma'_i \neq \sigma_i$. For $i \in N$ with $\sigma'_i = \sigma_i$, the only possible profitable D-move (if exists) is to drop the added resource,* $\mathbf{a}$.

The outline of the proof is as follows. Note that the congestion on the resources in σ' are the same as in σ, expect for the added resource for which the congestion has been increased by 1. Therefore, for players that play the same strategy in both profiles the statement follows directly from the D-stability of σ and the non-decreaseness of $f(\cdot)$ and $c(\cdot)$. As for the other players, the proof uses the profitability of the addition operation and Observation 7. Before we present the formal proof, we need the following notation.

- For $i \in N$, $a \in M$ and $\sigma \in \Sigma$, let $\bar{\mathbf{v}}_i^a(\sigma) = \frac{\mathbf{v}_i^a(\sigma)}{1-f(h_a(\sigma))}$.
- For $k = 1, \ldots, n$, let $\bar{c}(k) = \frac{c(k)}{1-f(k)}$.

Remark 15 *Note that for any* $i \in N$, $a \in M$ *and* $\sigma \in \Sigma$, $\bar{\mathbf{v}}_i^a(\sigma)$ *is a nondecreasing function of the congestion on each resource in* $\sigma_i \setminus \{a\}$, *and* $\bar{c}(k)$ *is a nondecreasing function of* k $(k = 1, \ldots, n)$. *Thus, the above notation is technically more useful than that of* $\mathbf{v}_i^a(\sigma)$ *and* $c(k)$. *In addition, throughout this paper, the marginal benefit function,* $\mathbf{v}(\cdot)$, *and the cost function,* $c(\cdot)$, *could have been equivalently replaced by* $\bar{\mathbf{v}}(\cdot)$ *and* $\bar{c}(\cdot)$, *respectively. For exposition reasons though, we use the notions of cost and marginal benefit whenever possible, to keep the proofs more intuitive.*

Now we are ready to present the proof of Lemma 14.

Proof: We prove the lemma for the one- and the two-step addition, separately.

One-step addition. Suppose σ' is obtained from σ by a one-step addition of a by player i, that is, $\sigma' = (\sigma_i \cup \{a\}, \sigma_{-i})$. Then, $U_i(\sigma') > U_i(\sigma) \Rightarrow U_i(\sigma') > U_i(\sigma_i' \setminus \{a\}, \sigma_{-i}')$, i.e. dropping resource a is not a profitable policy for i. This, in turn, implies that the marginal benefit of i from a at σ' is greater than its cost, i.e.

$$\mathbf{v}_i^a(\sigma') > c(h_a(\sigma')). \tag{3}$$

Assume that there is a profitable D-move for player i, that is, there exists a resource $b \in \sigma_i' \setminus \{a\}$ such that $U_i\left(\sigma_i' \setminus \{b\}, \sigma_{-i}'\right) > U_i(\sigma')$, implying

$$c(h_b(\sigma')) > \mathbf{v}_i^b(\sigma'). \tag{4}$$

Since σ is nearly-even, $h_a(\sigma') = h_a(\sigma) + 1$ and $h_e(\sigma') = h_e(\sigma)$ for all $e \in M \setminus \{a\}$, implying $h_a(\sigma') \geq h_b(\sigma')$. By the monotonicity of $c(\cdot)$, (3) and (4) yield $\mathbf{v}_i^b(\sigma') < \mathbf{v}_i^a(\sigma')$, which, coupled with the monotonicity of $f(\cdot)$, implies $h_a(\sigma') < h_b(\sigma')$, a contradiction. Hence, no profitable D-move from σ' is available for i.

Let $k \neq i$. Then, $\sigma_k' = \sigma_k$. We have to show that dropping any resource $b \neq a$ is not profitable for player k. Assume otherwise, i.e. $U_k\left(\sigma_k' \setminus \{b\}, \sigma_{-k}'\right) > U_k(\sigma')$. Then,

$$c(h_b(\sigma')) > \mathbf{v}_k^b(\sigma_k'). \tag{5}$$

By the D-stability of σ,

$$c(h_b(\sigma)) \leq \mathbf{v}_k^b(\sigma_k). \tag{6}$$

Since $h_b(\sigma') = h_b(\sigma)$ and $h_e(\sigma') \geq h_e(\sigma)$ for all $e \in M$, by the monotonicity of $f(\cdot)$, (6) contradicts (5).

Two-step addition. Now suppose that σ' is obtained by a two-step addition of a. More precisely, let $\sigma' = \left(\sigma_i \cup \{b\}, \sigma_j \setminus \{b\} \cup \{a\}, \sigma_{-\{i,j\}}\right)$, where b is σ-heavy and a is σ-light. Then, $U_i(\sigma_i \cup \{b\}, \sigma_{-i}) > U_i(\sigma)$ implies

$$\mathbf{v}_i^b(\sigma_i \cup \{b\}, \sigma_{-i}) > c(h_b(\sigma) + 1), \tag{7}$$

or, equivalently, $\bar{\mathbf{v}}_i^b(\sigma_i \cup \{b\}, \sigma_{-i}) > \bar{c}(h_b(\sigma) + 1)$. Now, since b is σ-heavy, $h_b(\sigma') = h_b(\sigma) \geq h_e(\sigma') \geq h_e(\sigma)$, for all $e \in M$. Hence, for all $e' \in \sigma_i'$, (7) yields the following (note that $h_e\left(\sigma'_j \setminus \{a\}, \sigma_{-j}\right) = h_e\left(\sigma_i \cup \{b\}, \sigma_j \setminus \{b\}, \sigma_{-\{i,j\}}\right) = h_e(\sigma)$ for all $e \in M$):

$$\begin{aligned}&\bar{\mathbf{v}}_i^{e'}(\sigma') \geq \bar{\mathbf{v}}_i^{e'}(\sigma'_j \setminus \{a\}, \sigma'_{-j}) = \bar{\mathbf{v}}_i^{e'}(\sigma_i \cup \{b\}, \sigma_j \setminus \{b\}, \sigma_{-\{i,j\}})\\ &\geq \bar{\mathbf{v}}_i^{b}(\sigma_i \cup \{b\}, \sigma_{-i}) > \bar{c}(h_b(\sigma)+1) \geq \bar{c}(h_{e'}(\sigma')),\end{aligned}$$

which implies $U_i(\sigma') > U_i(\sigma'_i \setminus \{e'\}, \sigma'_{-i})$ for all $e' \in \sigma'_i$. That is, no profitable D-move from σ' is available for i.

Consider now player j. We have to show that no profitable D-move from σ' is available for j. We demonstrate below that the required follows directly from the D-stability of the original strategy profile, σ. Namely, we show that $U_j(\sigma) \geq U_j(\sigma_j \setminus \{e\}, \sigma_{-j})$ for all $e \in \sigma_j$ implies $U_j(\sigma') \geq U_j(\sigma'_j \setminus \{e'\}, \sigma'_{-j})$ for all $e' \in \sigma'_j$.

Since b is σ-heavy, a is σ-light and σ is nearly-even, $h_a(\sigma') = h_a(\sigma)+1 = h_b(\sigma)$. Recall that $b \in \sigma_j$. Then, $U_j(\sigma) \geq U_j(\sigma_j \setminus \{b\}, \sigma_{-j})$ implies $\mathbf{v}_i^b(\sigma) \geq c(h_b(\sigma))$ $\Rightarrow$ $\mathbf{v}_i^a(\sigma') \geq c(h_a(\sigma'))$. Now, since $h_a(\sigma') \geq h_e(\sigma')$ for all $e \in M$, the above yields $\mathbf{v}_i^{e'}(\sigma') \geq c(h_{e'}(\sigma'))$, which implies $U_j(\sigma') \geq U_j(\sigma'_{-j}, \sigma'_j \setminus \{e'\})$ for all $e' \in \sigma'_j$.

For any player $k \neq i, j$, the proof we provided for the first case, is valid here as well. □

By Lemma 14, applying a one- or two-step addition operation to a nearly-even[4], D-stable strategy profile, may only cause a D-move with the added resource. However, even this seemingly simple in-D-stability may be hard to fix. As we show in the following example, a (possibly long) chain of D-moves by different players may follow after the first D-move with the added resource.

Example 16 *Consider a CGLF with* 7 *players* $N = \{1, 2, 3, 4, 5, 6, 7\}$ *and* 8 *resources* $M = \{e_1, \ldots, e_8\}$. *The cost function of each resource is given by* $c(1) = 9$, $c(2) = 16$, $c(3) = c(4) = c(5) = c(6) = c(7) \geq 16$, *and the failure probabilities are* $f(1) = 0.1$, $f(2) = 0.2$ *and* $f(3) = f(4) = f(5) = f(6) = f(7) \geq 0.2$. *The benefits for the players from a successful task completion are* $v_1 = 7500$, $v_2 = 1500$, $v_3 = v_4 = v_5 = v_6 = 150$, *and* $v_7 = 15$. *Consider the strategy profile* σ *in which* $\sigma_1 = \{e_1, e_2, e_5\}$, $\sigma_2 = \{e_2, e_3, e_8\}$, $\sigma_3 = \{e_3, e_4\}$, $\sigma_4 = \{e_4, e_5\}$, $\sigma_5 = \sigma_6 = \{e_6\}$ *and* $\sigma_7 = \{e_7\}$*:*

e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
1	1	2	3	4	5	7	2
	2	3	4	1	6		

Note that $h_{e_1}(\sigma) = h_{e_7}(\sigma) = h_{e_8}(\sigma) = 1$ *and* $h_{e_2}(\sigma) = h_{e_3}(\sigma) = h_{e_4}(\sigma) =$

[4] Recall that every nearly-even profile is S-stable (see Lemma 11).

$h_{e_5}(\sigma) = h_{e_6}(\sigma) = 2$; that is, σ is nearly-even and therefore S-stable (by Lemma 11). In addition, for any player $i \in N$ and for any resource $e \in \sigma_i$, $\bar{\mathbf{v}}_i^e(\sigma) \geq \bar{c}(h_e(\sigma))$ [5] :

$$\bar{\mathbf{v}}_1^{e_1}(\sigma) = 7500 \cdot 0.2^2 = 300 > 150 = 7500 \cdot 0.1 \cdot 0.2 = \bar{\mathbf{v}}_1^{e_2}(\sigma) = \bar{\mathbf{v}}_1^{e_5}(\sigma) >$$
$$\bar{c}(h_{e_2}(\sigma)) = \bar{c}(h_{e_5}(\sigma)) = \bar{c}(2) = \frac{16}{1-0.2} = 20 > 10 = \frac{9}{1-0.1} = \bar{c}(1) = \bar{c}(h_{e_1}(\sigma)),$$
$$\bar{\mathbf{v}}_2^{e_8}(\sigma) = 1500 \cdot 0.2^2 = 60 > 30 = 1500 \cdot 0.1 \cdot 0.2 = \bar{\mathbf{v}}_2^{e_2}(\sigma) = \bar{\mathbf{v}}_2^{e_3}(\sigma) >$$
$$\bar{c}(h_{e_2}(\sigma)) = \bar{c}(h_{e_3}(\sigma)) = \bar{c}(2) = 20 > 10 = \bar{c}(1) = \bar{c}(h_{e_8}(\sigma)),$$
$$\bar{\mathbf{v}}_3^{e_3}(\sigma) = \bar{\mathbf{v}}_3^{e_4}(\sigma) = 150 \cdot 0.2 = 30 > 20 = \bar{c}(2) = \bar{c}(h_{e_3}(\sigma)) = \bar{c}(h_{e_4}(\sigma)),$$
$$\bar{\mathbf{v}}_4^{e_4}(\sigma) = \bar{\mathbf{v}}_4^{e_5}(\sigma) = 30 > 20 = \bar{c}(2) = \bar{c}(h_{e_4}(\sigma)) = \bar{c}(h_{e_5}(\sigma)),$$

$$\bar{\mathbf{v}}_5^{e_6}(\sigma) = 150 > 20 = \bar{c}(2) = \bar{c}(h_{e_6}(\sigma)),$$
$$\bar{\mathbf{v}}_6^{e_6}(\sigma) = 150 > 20 = \bar{c}(2) = \bar{c}(h_{e_6}(\sigma)),$$
$$\bar{\mathbf{v}}_7^{e_7}(\sigma) = 15 > 10 = \bar{c}(1) = \bar{c}(h_{e_7}(\sigma)),$$

implying by Observation 7 the D-stability of σ. Thus, σ is a DS-stable profile.

Now, since

$$\bar{\mathbf{v}}_1^{e_7}(\sigma_1 \cup \{e_7\}, \sigma_{-1}) = 7500 \cdot 0.1 \cdot 0.2^2 = 30 > 20 = \bar{c}(2) = \bar{c}(h_{e_7}(\sigma) + 1),$$

it follows by Observation 7 that player 1 wishes to apply an A-move with resource e_7. Let $\sigma^{\mathrm{i}} = (\sigma_1 \cup \{e_7\}, \sigma_{-1})$ (note that e_7 is σ-light, i.e. σ^{i} is obtained from σ by the one-step addition of e_7 by player 1) as presented below:

e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
1	1	2	3	4	5	7	2
	2	3	4	1	6	1	

Notice that the profile σ^{i} is not D-stable, since player 7 wishes to apply a D-move from the added resource, e_7 (by 14, this is the only possible profitable D-move):

$$\bar{\mathbf{v}}_7^{e_7}(\sigma^{\mathrm{i}}) = 15 < 20 = \bar{c}(2) = \bar{c}(h_{e_7}(\sigma^{\mathrm{i}})),$$

and let $\sigma^{\mathrm{ii}} = (\sigma_7 \setminus \{e_7\}, \sigma_{-7})$:

e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
1	1	2	3	4	5	1	2
	2	3	4	1	6		

[5] See the notation following Lemma 14.

Observe that profile σ^{ii} is not D-stable as well, since player 1 wishes to drop resource e_2:

$$\bar{\mathbf{v}}_1^{e_2}(\sigma^{\text{ii}}) = 7500 \cdot 0.1^2 \cdot 0.2 = 15 < 20 = \bar{c}(2) = \bar{c}(h_{e_2}(\sigma^{\text{ii}})),$$

and let $\sigma^{\text{iii}} = (\sigma_1 \setminus \{e_2\}, \sigma_{-1})$:

e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
1	2	2	3	4	5	1	2
		3	4	1	6		

Note that the moves of player 1 demonstrate the following chain of preferences: utilizing 2 resources of congestion 2 and 1 of congestion 1 is preferred by using 3 resources of congestion 2 and 1 of congestion 1 (his A-move operation); now, after the 7's D-move, using 2 resources of congestion 2 and 2 of congestion 1 is preferred by using 2 resources of congestion 1 and 1 of congestion 2 (the 1's D-move).

Now observe that the D-move with e_3 is profitable for player 2:

$$\bar{\mathbf{v}}_2^{e_3}(\sigma^{\text{iii}}) = 1500 \cdot 0.1^2 = 15 < 20 = \bar{c}(2) = \bar{c}(h_{e_3}(\sigma^{\text{iii}})),$$

and let $\sigma^{\text{iv}} = (\sigma_2 \setminus \{e_3\}, \sigma_{-2})$:

e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
1	2	3	3	4	5	1	2
			4	1	6		

Player 3 now wishes to drop resource e_4:

$$\bar{\mathbf{v}}_3^{e_4}(\sigma^{\text{iv}}) = 150 \cdot 0.1 = 15 < 20 = \bar{c}(2) = \bar{c}(h_{e_4}(\sigma^{\text{iv}})),$$

and let $\sigma^{\text{v}} = (\sigma_3 \setminus \{e_4\}, \sigma_{-3})$:

e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
1	2	3	4	4	5	1	2
				1	6		

Profile σ^{v} is also in-D-stable, since player 4 wishes to drop resource e_5:

$$\bar{\mathbf{v}}_4^{e_5}(\sigma^{\text{v}}) = 150 \cdot 0.1 = 15 < 20 = \bar{c}(2) = \bar{c}(h_{e_5}(\sigma^{\text{v}})),$$

and let $\sigma^{\text{vi}} = (\sigma_4 \setminus \{e_5\}, \sigma_{-4})$*:*

e_1	e_2	e_3	e_4	e_5	e_6	e_7	e_8
1	2	3	4	1	5	1	2
					6		

Although the resulting profile, σ^{vi}*, is DS-stable (no more profitable D-moves are available), it is not a Nash equilibrium, since players* 5 *and* 6 *would now benefit from an A-move with any of the resources in* $M \setminus \{e_6\}$*, i.e.*

$$\bar{\mathbf{v}}_i^e(\sigma^{\text{vi}}) = 150 \cdot 0.2 = 30 > 20 = \bar{c}(2) = \bar{c}(h_e(\sigma^{\text{vi}}))$$

for $i = 5, 6$*,* $e \in M \setminus \{e_6\}$*, and a new chain of moves will begin.*

As one may learn from the above example, the addition of a resource a by player i may force player j to drop the added resource, a. This, in turn, may cause player i to drop another resource, say b. This is since the D-move of j from a has decreased the failure probability of a; hence, the i's marginal benefit from b has been decreased. Moreover, the modified marginal benefit from b might be smaller than its cost (clearly, b is σ-heavy; otherwise, the addition of a would be non-profitable for i). For the same reason, the D-move of i from b may cause a chain of D-moves from other heavy resources by different players. Although the length of such a chain is bounded by m, the number of resources, it is not clear whether the one-/two-step addition dynamics converges to an equilibrium if it initializes with an arbitrary DS-stable profile. [6] This observation motivates us to look for a DS-stable profile that would remain DS-stable after applying to it a one- or two-step addition.

5.1.4 Post-addition D-stability

Based on Lemma 14, given a strategy profile $\sigma \in \Sigma$, we say that σ is *post-addition D-stable* if any strategy profile σ' obtained from σ by applying a one- or two-step addition operation, does not admit profitable D-moves with the added resource. More precisely, using Observation 7, we define post-addition D-stability as follows.

Definition 17 *Let* σ *be a strategy profile of a given nondecreasing CGLF. A profile* σ *will be termed* ***post-addition D-stable*** *if for every player* $i \in N$ *with* $|\sigma_i| > 0$ *and for every* σ*-light resource* $\mathbf{a} \in \sigma_i$*,*

$$\bar{\mathbf{v}}_i^a(\sigma) \geq \bar{c}(h_a(\sigma) + 1). \tag{8}$$

[6] However, we do not have examples for loops in such a dynamics, implying that the question of the existence of an *ordinal* potential function (see (Monderer and Shapley, 1996)) in CGLFs has not yet been resolved.

Given a post-addition D-stable strategy profile σ, let σ' be obtained from σ by applying a one- or two-step addition of a σ-light resource a by one of the players (note that σ' is nearly-even whenever σ is nearly-even). Then, for any player i with $a \in \sigma_i$, (8) is essentially equivalent to the non-profitability of a D-move with a from σ' (since $\bar{\mathbf{v}}_i^a(\sigma') = \bar{\mathbf{v}}_i^a(\sigma)$ and $\bar{c}(h_a(\sigma')) = \bar{c}(h_a(\sigma) + 1)$). Then, Lemma 14 implies the D-stability of σ' whenever σ is nearly-even and D-stable. We also notice that any nearly-even, post-addition D-stable strategy profile is D-stable. Therefore, if σ is post-addition D-stable and nearly-even then σ' is D-stable and nearly-even.

At this point, two questions arise.

- First, is a nearly-even, post-addition D-stable strategy profile guaranteed to exist in CGLFs?
- Second, would the existence of such a profile guarantee the convergence to a Nash equilibrium? Namely, could we preserve the post-addition D-stability while applying one-/two-step addition operations sequentially? Would such a dynamics reach an equilibrium in a reasonable number of steps (if ever)? (See Figure 4 below.)

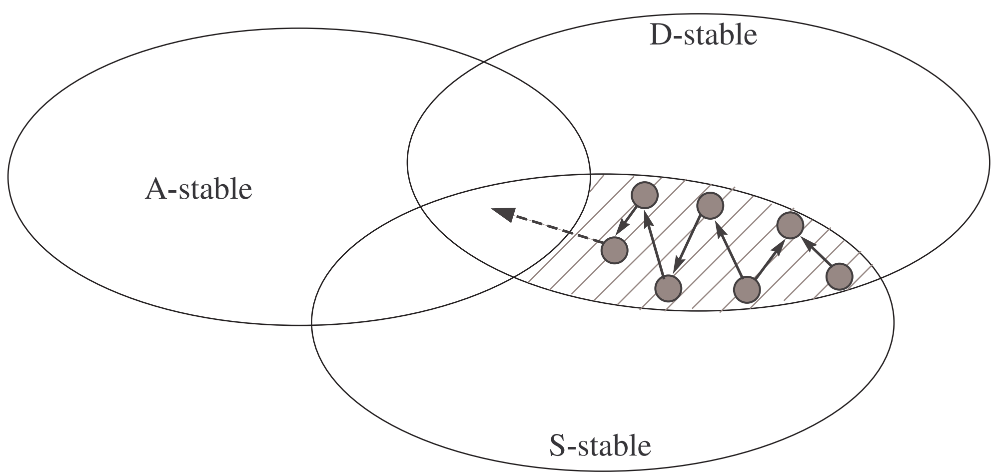

Figure 4. Convergence of the one-/two-step addition dynamics to an equilibrium.

Lemmas 18 and 19 below, which are central to the proof of our main result, provide affirmative answers to the above questions. They show the existence of an even, post-addition D-stable strategy profile, and that applying to it a polynomial series of one- and two-step addition operations leads to a Nash equilibrium. Let us provide some notations first.

- Let $\Sigma^0 \subseteq \Sigma$ denote the subset of all D-stable strategy profiles, and let $\Sigma^1 \subseteq \Sigma^0$ be the subset of all even, D-stable strategy profiles. By Lemma 11, every profile in Σ^1 (if such exists) is S-stable.
- For any even strategy profile σ, let h^σ denote the common congestion on

the resources,[7] and let $\Sigma^2 \subseteq \Sigma^1$ be the subset of Σ^1 consisting of all those profiles with maximum congestion on the resources. That is, $\Sigma^2 = \arg\max_{\sigma \in \Sigma^1} h^\sigma$.

Lemma 18 *Given a nondecreasing CGLF, there exists a strategy profile $\sigma \in \Sigma^2$ that is either a pure strategy Nash equilibrium or post-addition D-stable.*

Lemma 19 *Given a nondecreasing CGLF, let σ be a nearly-even, post-addition D-stable strategy profile, and let σ' be obtained from σ by applying to it a one- or two-step addition operation. If $\min_{e \in M} h_e(\sigma') = \min_{e \in M} h_e(\sigma)$ then σ' is post-addition D-stable.*

For exposition reasons, we leave the proofs of Lemmas 18 and 19 to the end of this subsection.

Assume Σ^2 does not contain an equilibrium profile. Then, by Lemma 18, there is a post-addition D-stable profile σ' that lies in Σ^2. Furthermore, Lemma 19 implies the existence of a pure strategy Nash equilibrium and that this equilibrium can be achieved by applying sequentially less than m one- or two-step addition operations to σ'. This is because otherwise, after m addition operations we obtain an even, D-stable strategy profile σ'' with $h^{\sigma''} > h^{\sigma'}$, contradicting $\sigma' \in \Sigma^2$. Therefore, combining Lemmas 18 and 19 yields the following theorem.

Theorem 20 *Every nondecreasing CGLF possesses a Nash equilibrium in pure strategies.*

It is worthwhile mentioning that our existence proof is "almost" algorithmic. Namely, in Lemma 18 we prove the existence of an even, post-addition D-stable profile, and point to a subset of profiles, Σ^2, that contains such a profile. Furthermore, Lemma 19 states that applying the one-/two-step addition dynamics to that profile will result in a Nash equilibrium profile. Therefore, to make the proof algorithmic, we only need to find constructively a post-addition D-stable profile in Σ^2. This goal is achieved by our CGLF-algorithm presented in the sequel.

We turn now to prove Lemmas 18 and 19. To simplify the exposition of the following proofs, we use the notations below.

- For $i \in N$, $x \in \{0, \ldots, m\}$ and $k \in \{1, \ldots, n\}$ let $\mathbf{v}_i^k(x) = v_i f(k)^x (1 - f(k))$. Note that $\mathbf{v}_i^k(x)$ denotes the marginal benefit of i from using any of his $x+1$ chosen resources at a k-even strategy profile. That is, $\mathbf{v}_i^k(x) = \mathbf{v}_i^a(\sigma)$ for any k-even profile σ with $|\sigma_i| = x + 1$, and $a \in \sigma_i$.
- For $i \in N$, $x \in \{0, \ldots, m\}$ and $k \in \{1, \ldots, n\}$, let $\bar{\mathbf{v}}_i^k(x) = \frac{\mathbf{v}_i^k(x)}{(1-f(k))}$. Note that $\bar{\mathbf{v}}_i^k(x)$ weakly increases in k (since $f(\cdot)$ is nondecreasing) and strictly decreases in x (since $f(\cdot) < 1$). Recall that $\bar{c}(k) = \frac{c(k)}{(1-f(k))}$ is a nondecreasing

[7] That is, σ is h^σ-even.

function of k.

Proof of Lemma 18: If Σ^2 includes a pure strategy Nash equilibrium strategy profile, we are done. Otherwise, let $U_N(\sigma) = \sum_{i \in N} U_i(\sigma)$ denotes the *group utility* of the players, and let $\Sigma^3 \subseteq \Sigma^2$ be the subset of all profiles in Σ^2 with maximum group utility. That is, $\Sigma^3 = \arg\max_{\sigma \in \Sigma^2} \sum_{i \in N} U_i(\sigma) = \arg\max_{\sigma \in \Sigma^2} U_N(\sigma)$. We show below that there exists a post-addition D-stable strategy profile in Σ^3. That is, by Definition 17 and the above notation, we show that there is $\sigma \in \Sigma^3$ such that for all $i \in N$ with $|\sigma_i| > 0$,

$$\bar{\mathbf{v}}_i^{h^\sigma}(|\sigma_i| - 1) \geq \bar{c}(h^\sigma + 1). \tag{9}$$

If $\max_{\sigma \in \Sigma^1} h^\sigma = 0$ then, obviously, $\Sigma^1 = \Sigma^2 = \Sigma^3 = \{\sigma^0 = (\varnothing, \ldots, \varnothing)\}$ and $|\sigma_i^0| = 0$ for all $i \in N$. Hence, clearly, σ^0 is post-addition D-stable. Now assume that $\max_{\sigma \in \Sigma^1} h^\sigma > 0$. Let $\sigma \in \Sigma^3$ and let $M(\sigma)$ be the subset of all resources for which there exists a player for whom adding the resource is profitable. First, we show that (9) holds for all $i \in N$ such that $\sigma_i \cap M(\sigma) \neq \varnothing$, i.e. for all those players with one of their resources being desired by another player.

Let $a \in M(\sigma)$, and let σ' be obtained from σ by the one-step addition of a by player i. Assume there is a player, j, with $a \in \sigma_j$ such that

$$\bar{\mathbf{v}}_j^{h^\sigma}(|\sigma_j| - 1) < \bar{c}(h^\sigma + 1). \tag{10}$$

Let $\sigma'' = (\sigma'_j \setminus \{a\}, \sigma'_{-j})$. Below we demonstrate that σ'' is a D-stable strategy profile and, since σ'' and σ correspond to the same congestion vector, we conclude that σ'' lies in Σ^2. In addition, we show that $U_N(\sigma'') > U_N(\sigma)$, contradicting the fact that $\sigma \in \Sigma^3$.

To show that $\sigma'' \in \Sigma^0$ we note that since $h^{\sigma''} = h^\sigma$ and $\sigma \in \Sigma^0$, there are no profitable D-moves for any player $k \neq i, j$. It remains to show that there are no profitable D-moves for players i and j as well. Since $U_i(\sigma') > U_i(\sigma)$, by Observation 7 we get

$$\bar{\mathbf{v}}_i^{h^\sigma}(|\sigma_i|) > \bar{c}(h^\sigma + 1), \tag{11}$$

which implies $\bar{\mathbf{v}}_i^{h^{\sigma''}}(|\sigma''_i| - 1) = \bar{\mathbf{v}}_i^{h^\sigma}(|\sigma_i|) > \bar{c}(h^\sigma + 1) \geq \bar{c}(h^\sigma) = \bar{c}(h^{\sigma''})$, yielding $U_i(\sigma'') > U_i(\sigma''_i \setminus \{b\}, \sigma''_{-i})$ for all $b \in \sigma''_i$. Thus, there are no profitable D-moves for player i. By the D-stability of σ, for player j and for all $b \in \sigma_j$ we have that $\bar{\mathbf{v}}_j^{h^\sigma}(|\sigma_j| - 1) \geq \bar{c}(h^\sigma)$. Then, $\bar{\mathbf{v}}_j^{h^{\sigma''}}(|\sigma''_j| - 1) > \bar{\mathbf{v}}_j^{h^{\sigma''}}(|\sigma''_j|) = \bar{\mathbf{v}}_j^{h^\sigma}(|\sigma_j| - 1) \geq \bar{c}(h^\sigma) = \bar{c}(h^{\sigma''})$, implying $U_j(\sigma'') > U_j(\sigma''_j \setminus \{b\}, \sigma''_{-j})$ for all $b \in \sigma_j$. Therefore, σ'' is D-stable and lies in Σ^2.

To show that $U_N(\sigma'')$, the group utility of σ'', satisfies $U_N(\sigma'') > U_N(\sigma)$, we note that $h^{\sigma''} = h^\sigma$, and thus $U_k(\sigma'') = U_k(\sigma)$, for all $k \in N \setminus \{i, j\}$. Therefore, we have to show that $U_i(\sigma'') + U_j(\sigma'') > U_i(\sigma) + U_j(\sigma)$ or, equivalently, that $U_i(\sigma'') - U_i(\sigma) > U_j(\sigma) - U_j(\sigma'')$. By (10) and (11), $\bar{\mathbf{v}}_i^{h^\sigma}(|\sigma_i|) > \bar{\mathbf{v}}_j^{h^\sigma}(|\sigma_j| - 1)$, which is equivalent to $\mathbf{v}_i^{h^\sigma}(|\sigma_i|) > \mathbf{v}_j^{h^\sigma}(|\sigma_j| - 1)$. Thus,

$$
\begin{aligned}
&U_i(\sigma'') - U_i(\sigma) \\
&= \left(1 - f(h^\sigma)^{|\sigma_i|+1}\right) v_i - (|\sigma_i| + 1)\, c(h^\sigma) - \left[\left(1 - f(h^\sigma)^{|\sigma_i|}\right) v_i - |\sigma_i| c(h^\sigma)\right] \\
&= v_i f(h^\sigma)^{|\sigma_i|} \left(1 - f(h^\sigma)\right) - c(h^\sigma) = \mathbf{v}_i^{h^\sigma}(|\sigma_i|) - c(h^\sigma) \\
&> \mathbf{v}_j^{h^\sigma}(|\sigma_j| - 1) - c(h^\sigma) = v_j f(h^\sigma)^{|\sigma_j|-1} \left(1 - f(h^\sigma)\right) - c(h^\sigma) \\
&= \left(1 - f(h^\sigma)^{|\sigma_j|}\right) v_j - |\sigma_j| c(h^\sigma) - \left[\left(1 - f(h^\sigma)^{|\sigma_j|-1}\right) v_j - (|\sigma_i| - 1)\, c(h^\sigma)\right] \\
&= U_j(\sigma) - U_j(\sigma'').
\end{aligned}
$$

Therefore, σ'' lies in Σ^2 and satisfies $U_N(\sigma'') > U_N(\sigma)$, in contradiction to $\sigma \in \Sigma^3$.

Hence, if $\sigma \in \Sigma^3$ then (9) holds for all $i \in N$ such that $\sigma_i \cap M(\sigma) \neq \varnothing$. It remains to show that there exists $\sigma \in \Sigma^3$ such that (9) holds for all the players. For that, choose a player $i \in \arg\min_{k \in N} \bar{\mathbf{v}}_k^{h^\sigma}(|\sigma_k|)$. If $\sigma_i \cap M(\sigma) \neq \varnothing$ then (9) holds for i, implying by the choice of i the correctness of (9) for any player $k \in N$. Otherwise, if no resource of σ_i lies in $M(\sigma)$, then let $a \in \sigma_i$ and $b \in M(\sigma)$. Since $a \in \sigma_i$, $b \notin \sigma_i$ and $h_a^\sigma = h_b^\sigma$, there exists a player j such that $b \in \sigma_j$ and $a \notin \sigma_j$. Consider the strategy profile $\sigma' = \left(\sigma_i \smallsetminus \{a\} \cup \{b\}, \sigma_j \smallsetminus \{b\} \cup \{a\}, \sigma_{-\{i,j\}}\right)$ which is obtained from σ by applying sequentially two S-moves with a and b, in opposite directions, by players i and j. Obviously, $U_k(\sigma') = U_k(\sigma)$ for any $k \in N$, and therefore σ' lies in Σ^3 as well as σ. Thus, σ' satisfies (9) for player i, and therefore, for any player $k \in N$. □

Proof of Lemma 19: Using (8) with respect to σ, for any player k with $\sigma'_k = \sigma_k$ and for any σ'-light resource $e' \in \sigma'_k$, we get $\bar{\mathbf{v}}_k^{e'}(\sigma') \geq \bar{\mathbf{v}}_k^{e'}(\sigma) \geq \bar{c}(h_{e'}(\sigma) + 1) = \bar{c}(h_{e'}(\sigma') + 1)$, as required. Now let us consider the rest of the players. Assume σ' is obtained by the one-step addition of resource a by player i. In this case, i is the only player with $\sigma'_i \neq \sigma_i$. The required property for player i follows directly from $U_i(\sigma') > U_i(\sigma)$ and Observation 7.

In the case of a two-step addition, let $\sigma' = \left(\sigma_i \cup \{b\}, \sigma_j \smallsetminus \{b\} \cup \{a\}, \sigma_{-\{i,j\}}\right)$, where b is a σ-heavy resource. Consider player i. Since $U_i(\sigma_i \cup \{b\}, \sigma_{-i}) > U_i(\sigma)$, by Observation 7, $\bar{\mathbf{v}}_i^b(\sigma_i \cup \{b\}, \sigma_{-i}) > \bar{c}(h_b(\sigma) + 1)$. Note that $h_b(\sigma) \geq h_e(\sigma')$ for all $e \in M$ and, in particular, for all σ'-light resources. Hence,

$$
\bar{\mathbf{v}}_i^b(\sigma_i \cup \{b\}, \sigma_{-i}) > \bar{c}(h_{e'}(\sigma) + 1) \tag{12}
$$

for any σ'-light resource e'. Now, since $h_e(\sigma') \geq h_e(\sigma)$ for all $e \in M$ and b is σ-heavy, it follows that $\bar{\mathbf{v}}_i^{e'}(\sigma') \geq \bar{\mathbf{v}}_i^{e'}(\sigma'_j \smallsetminus \{a\}, \sigma'_{-j}) = \bar{\mathbf{v}}_i^{e'}\left(\sigma_i \cup \{b\}, \sigma_j \smallsetminus \{b\}, \sigma_{-\{i,j\}}\right)$ $\geq \bar{\mathbf{v}}_i^b\left(\sigma_i \cup \{b\}, \sigma_j \smallsetminus \{b\}, \sigma_{-\{i,j\}}\right) = \bar{\mathbf{v}}_i^b(\sigma_i \cup \{b\}, \sigma_{-i})$ for any σ'-light resource e'. This, coupled with (12), yields $\bar{\mathbf{v}}_i^{e'}(\sigma') > \bar{c}(h_{e'}(\sigma') + 1)$ for any σ'-light resource a, as required. For player j we just use (8) with respect to σ and the equality $h_b(\sigma) = h_a(\sigma')$. For any σ'-light resource e', $\bar{\mathbf{v}}_j^{e'}(\sigma') = \bar{\mathbf{v}}_j^{e'}(\sigma) \geq \bar{c}(h_{e'}(\sigma) + 1) = \bar{c}(h_{e'}(\sigma') + 1)$, as required. □

5.2 Computation of a pure strategy Nash equilibrium

We now present our next result – the CGLF-algorithm that constructs a pure strategy Nash equilibrium in a given nondecreasing CGLF. As we previously mentioned, the algorithm is based on Lemmas 18 and 19. It first finds a strategy profile in Σ^2 which is either a pure strategy Nash equilibrium or post-addition D-stable. If it is post-addition D-stable then the algorithm applies to it at most $m-1$ one-/two-step addition operations to reach an equilibrium. Note that the proof of Lemma 18 insures the existence of a post-addition D-stable profile in $\Sigma^3 \subseteq \Sigma^2$, the subset of profiles in Σ^2 that maximizes the group utility. We note though that, based on Lemma 18, the CGLF-algorithm is designed to find a post-addition D-stable profile in Σ^2, which does not necessarily belong to Σ^3. We start with a brief explanation of the logic behind the algorithm, and then proceed to its detailed description.

- First, the algorithm determines a value $k^* = \max_{\sigma \in \Sigma^1} h^\sigma$ that represents the common congestion on the resources for any strategy profile in Σ^2.
- To find k^* as above, the algorithm uses a variable k initiated with the value $k = n$ that gradually decreases until k^* is found (Steps [0] – [1]).
- For $k = n$, the only even strategy profile with n being its common congestion is $\sigma = (M, \ldots, M)$, which is obviously A- and S-stable. If σ is also D-stable then $k^* = n$, and the algorithm outputs σ and halts after Step [0]. Otherwise, $k^* < n$ and the algorithm proceeds with $k = n-1$ (Step [1]).
- Given $0 < k < n$, for every player $i \in N$ the algorithm determines the maximum number of resources, $x^i_D(k)$, that i would keep without dropping, in a k-even strategy profile. If $\sum_{i \in N} x^i_D(k) < km$ then there is no k-even, D-stable strategy profile. This means that $k^* < k$, and the algorithm proceeds to the next value of k (repeating Step [1]). Otherwise, $k^* = k$.
- If $k^* = 0$ then the algorithm constructs a strategy profile σ in which every player uses precisely $|\sigma_i| = x^i_D(1)$ resources and every resource is used by at most one player (Step [2]). As we show in the proof of Theorem 21, σ is a Nash equilibrium.
- Otherwise, $k^* > 0$. In this case, for every player $i \in N$ the algorithm determines the minimum number of resources, $x^i_A(k^*)$, for which i will not apply an A-move, if the common congestion on the resources is k^* (Step [3]). If $\sum_{i \in N} x^i_A(k^*) \leq k^* m$ and $x^i_A(k^*) \leq x^i_D(k^*)$ for all $i \in N$ then the algorithm constructs a strategy profile σ in which every player uses at least $x^i_A(k^*)$ and at most $x^i_D(k^*)$ resources, and the congestion on each resource is k^* (Step [4]). This profile is a Nash equilibrium.
- Otherwise, if $\sum_{i \in N} x^i_A(k^*) > k^* m$ or $\exists i \in N$ with $x^i_A(k^*) > x^i_D(k^*)$, then Σ^2 does not include a Nash equilibrium strategy profile. In this case Lemma 18 implies the existence of a post-addition D-stable strategy profile in Σ^2. For every player $i \in N$ the algorithm determines the maximum number of resources, $x^i(k^*)$, that i would keep without dropping, when the congestion

on one of his resources is k^*+1 and k^* on all the others (Step [5]). By the existence of a post-addition D-stable profile in Σ^2 and by the definition of $x^i(k^*)$, we get that $\sum_{i\in N} x^i(k^*) \geq k^*m$ (otherwise, such a profile does not exist). Then, the algorithm constructs a strategy profile $\sigma \in \Sigma^2$ in which every player uses at most $x^i(k^*)$ resources, implying that σ is post-addition D-stable (Step [5]). Given that, based on Lemma 19, a pure strategy Nash equilibrium is achieved by applying at most $m-1$ one-/two-step addition operations to σ (Steps [6] – [7]).

The CGLF-algorithm is presented below.

CGLF-algorithm

Step [0] If $\bar{\mathbf{v}}_n^n(m-1) \geq \bar{c}(n)$ then set $\sigma := (M,\dots,M)$ and QUIT;
Otherwise set $k := n-1$ and go to [1];
% if the condition holds then $k^* = n$;

Step [1] For all $i \in N$ set $X_D^i := \Big\{x \in \{1,\dots,m\} \,:\, \bar{\mathbf{v}}_i^k(x-1) \geq \bar{c}(k)\Big\}$;
If $X_D^i \neq \varnothing$ then set $x_D^i := \max_{x\in X_D^i} x$; Otherwise set $x_D^i := 0$;
If $\sum_{i\in N} x_D^i < km$ then set $k := k-1$ and go to [2]; Otherwise go to [3];
% finding $k^* < n$;

Step [2] If $k = 0$ then:
For $i = 1$ to n:
If $x_D^i > 0$ set $\sigma_i := \{e_r \in M \,:\, 1 \leq r \leq x_D^i\}$ and reorder the resources: for all $e_r \in M$ set $e_r := e_{r+x_D^i}$; Otherwise set $\sigma_i = \varnothing$;
QUIT;
Otherwise go to [1];
% constructing an equilibrium in the case of $k^* = 0$;

Step [3] For all $i \in N$ set $X_A^i := \Big\{x \in \{0,\dots,m-1\} \,:\, \bar{\mathbf{v}}_i^k(x) \leq \bar{c}(k+1)\Big\}$;
If $X_A^i \neq \varnothing$ then set $x_A^i := \min_{x\in X_A^i} x$; Otherwise set $x_A^i := m$;
If $\sum_{i\in N} x_A^i > km$ or $\exists i \in N$ such that $x_A^i > x_D^i$ then go to [5];
% checking the existence of a k^*-even equilibrium;

Step [4] For $i = 1$ to n:
Set $d^i := km - \sum_{j=1}^{i-1} |\sigma_j| - \sum_{j=i}^{n} x_A^j$, $\sigma_i := \{e_r \in M \,:\, 1 \leq r \leq \min\{x_D^i, x_A^i + d^i\}\}$ and reorder the resources: for all $e_r \in M$ set $e_r := e_{\left(r+\min\{x_D^i, x_A^i+d^i\}\right) \bmod m}$;
QUIT;
% constructing a k^*-even equilibrium;

Step [5] For all $i \in N$ set $X^i := \Big\{x \in \{1,\dots,m\} \,:\, \bar{\mathbf{v}}_i^k(x-1) \leq \bar{c}(k+1)\Big\}$;
If $X^i \neq \varnothing$ then set $x^i := \max_{x\in X^i} x$; Otherwise set $x^i := 0$;
For $i = 1$ to n:
If $km - \sum_{j=1}^{i-1} |\sigma_j| > 0$ then set $\sigma_i := \Big\{e_r \in M \,:\, 1 \leq r \leq \min\{x^i, km - \sum_{j=1}^{i-1} |\sigma_j|\}\Big\}$ and reorder the resources: for all $e_r \in M$ set $e_r := e_{\left(r+\min\{x^i, km - \sum_{j=1}^{i-1}|\sigma_j|\}\right) \bmod m}$;
Otherwise set $\sigma_i = \varnothing$;
% constructing a k^*-even, post-addition D-stable profile;

Step [6] For all $i \in N$, select $e^i \in \arg\min_{e \in M \setminus \sigma_i} h_e(\sigma)$;

Set $N(\sigma) := \{i \in N \,:\, U_i(\sigma_{-i}, \sigma_i \cup \{e^i\}) > U_i(\sigma)\}$;
If $N(\sigma) = \varnothing$ then QUIT;
% checking the A-stability;

Step [7] Set $M(\sigma) := \{e \in M \,:\, \exists i \in N(\sigma), e = e^i\}$;
Select $a^* \in \arg\min_{e \in M(\sigma)} h_e^\sigma$ and $i^* \in \{i \in N(\sigma) \,:\, e^i = a^*\}$;
If a^* is σ-light then set $\sigma_{i^*} := \sigma_{i^*} \cup \{a^*\}$ and go to [6];
Otherwise select a σ-light resource b^* and $j^* \in \{i \in N \,:\, a^* \in \sigma_i, b^* \notin \sigma_i\}$,
set $\sigma_{i^*} := \sigma_{i^*} \cup \{a^*\}$, $\sigma_{j^*} := (\sigma_{j^*} \setminus \{a^*\}) \cup \{b^*\}$, and go to [6].
% performing a one-/two-step addition.

Theorem 21 *The CGLF-algorithm finds a pure strategy Nash equilibrium in any given nondecreasing CGLF and its time complexity is $O(n^2m + nm^2)$.*

Proof: First we prove that the CGLF-algorithm finds a pure strategy Nash equilibrium in a given nondecreasing CGLF, and then proceed to the proof of its time complexity.

Validity. Recall that $\Sigma^1 \subseteq \Sigma^0$ represents the subset of all even, D-stable strategy profiles, with h^σ being the common congestion on the resources for every $\sigma \in \Sigma^1$. Let k^* represent the maximum possible common congestion on the resources in an even, D-stable strategy profile, that is in a strategy profile in Σ^2. Thus, $k^* = \arg\max_{\sigma \in \Sigma^1} h^\sigma$.

- $k^* = n$*: construction of an equilibrium.*

In the simplest case where

$$\bar{\mathbf{v}}_n^n(m-1) \geq \bar{c}(n), \tag{13}$$

the CGLF-algorithm terminates after Step [0] with the outcome $\sigma = (M, \ldots, M)$, which is obviously A- and S-stable. By Observation 7, (13) yields the D-stability of σ.[8] Hence, $k^* = n$ and Lemma 8 implies that σ is a Nash equilibrium. Otherwise, if (13) does not hold, then $k^* < n$ and the algorithm proceeds to Step [1] with $k = n - 1$.

- *Finding* $k^* < n$*.*

At Step [1], for every player $i \in N$ the algorithm determines the value $x_D^i(k)$ which is the maximum integer between 1 and m satisfying $\bar{\mathbf{v}}_i^k(x_D^i(k)-1) \geq \bar{c}(k)$ (if for all $x \in \{1, \ldots, m\}$ we have $\bar{\mathbf{v}}_i^k(x-1) < \bar{c}(k)$ then the algorithm determines $x_D^i(k) = 0$). Hence, by Observation 7, for any $i \in N$, $x_D^i(k)$ represents the maximum number of resources that i would keep without dropping, in a k-even strategy profile. We note that since $\bar{\mathbf{v}}_i^k(x-1)$ decreases in x and $x \leq m$,

[8] Recall that, w.l.o.g., it is assumed that $v_1 \geq v_2 \geq \ldots \geq v_n$, implying in particular that $\bar{\mathbf{v}}_1^n(m-1) \geq \bar{\mathbf{v}}_2^n(m-1) \geq \ldots \geq \bar{\mathbf{v}}_n^n(m-1)$. Thus, (13) yields an analogous inequality for any player $i \in N$.

then $x^i_D(k)$ is uniquely defined and the condition $\bar{\mathbf{v}}^k_i(x-1) \geq \bar{c}(k)$ holds for any $x < x^i_D(k)$. If $\sum_{i\in N} x^i_D(k) < km$ then there is no k-even, D-stable strategy profile. This means that $k^* < k$, and the algorithm proceeds to the next value of k (repeating Step [1]). Otherwise, $k^* = k$ (recall that $k^* = h^\sigma$ for all $\sigma \in \Sigma^2$).

- $k^* = 0$*: construction of an equilibrium.*

Consider first the case in which $\sum_{i\in N} x^i_D(k) < km$ for all $0 < k \leq n-1$ (Step [1] has been repeated $n-1$ times). By the definition of $x^i_D(k)$, this means that the single even, D-stable strategy profile is $(\varnothing, \ldots, \varnothing)$, and any even profile with a higher congestion is not D-stable. That is, $k^* = 0$. In this case, the CGLF-algorithm terminates after Step [2]. At Step [2], the algorithm assigns to any player $i \in N$ exactly $x^i_D(1)$ lightest resources, and $\sum_{i\in N} x^i_D(1) < m$. Therefore, in the resulting strategy profile σ, any player $i \in N$ uses precisely $|\sigma_i| = x^i_D(1)$ resources, and every resource is used by at most one player. Below we demonstrate that σ is a Nash equilibrium. By Lemma 8, it suffices to prove the A-, D- and S-stability of σ. As we mentioned before, $h_e(\sigma) \leq 1$, for all $e \in M$. Hence, σ is a nearly-even profile. Then, by Lemma 11, σ is S-stable. For any $i \in N$, either $|\sigma_i| = 0$, or $|\sigma_i| > 0$ and $\bar{\mathbf{v}}^1_i(|\sigma_i| - 1) = \bar{\mathbf{v}}^1_i(x^i_D(1) - 1) \geq \bar{c}(1)$ (by the definition of $x^i_D(1)$). Hence, by Observation 7, no profitable D-moves from σ are available. In addition (again, by the definition of $x^i_D(1)$), for any $i \in N$, $\bar{\mathbf{v}}^1_i(|\sigma_i|) = \bar{\mathbf{v}}^1_i\left(x^i_D(1)\right) = \bar{\mathbf{v}}^1_i\left((x^i_D(1)+1)-1\right) < \bar{c}(1)$, implying by Observation 7 the A-stability of σ.

- $0 < k^* \leq n-1$*: checking the existence of a* k^**-even equilibrium.*

In the complement case, the CGLF-algorithm calculates k^*, $0 < k^* \leq n - 1$. Recall that the fact that $\sum_{i\in N} x^i_D(k^*) \geq k^* m$ implies that the subset of k^*-even, D-stable strategy profiles is not empty. Now, at Step [3], for every player $i \in N$ the algorithm determines a value $x^i_A(k^*)$ which is the minimum integer between 0 and $m-1$ satisfying $\bar{\mathbf{v}}^{k^*}_i\left(x^i_A(k^*)\right) \leq \bar{c}(k^*+1)$ (if for all $x \in \{0, \ldots, m-1\}$ we have $\bar{\mathbf{v}}^{k^*}_i(x) > \bar{c}(k^*+1)$ then the algorithm determines $x^i_A(k^*) = m$). Thus, by Observation 7, for any $i \in N$, $x^i_A(k^*)$ represents the minimum number of resources for which i will not apply an A-move from a k^*-even strategy profile. We note that since $\bar{\mathbf{v}}^{k^*}_i(x)$ increases as x decreases and since $x \geq 0$, the value of $x^i_A(k^*)$ is uniquely defined, and the condition $\bar{\mathbf{v}}^{k^*}_i(x) \leq \bar{c}(k^*+1)$ holds for all $x > x^i_A(k^*)$.

If $x^i_A(k^*) \leq x^i_D(k^*)$ for all $i \in N$ and $\sum_{i\in N} x^i_A(k^*) \leq k^* m$ then, by the definition of $x^i_A(k^*)$, there exists a k^*-even, A- and D-stable strategy profile, and the algorithm terminates after Step [4].

Otherwise, if $\sum_{i\in N} x^i_A(k^*) > k^* m$ or $\exists i \in N$ such that $x^i_A(k^*) > x^i_D(k^*)$, then any k^*-even, D-stable strategy profile is not A-stable, and thus is not a Nash equilibrium. In this case, Lemma 18 guarantees the existence of a k^*-even, post-addition D-stable strategy profile, and the algorithm proceeds to Step [5].

. $0 < k^* \leq n-1$*: construction of a* k^**-even equilibrium.*

At Step [4], the algorithm constructs the required strategy profile by assigning each player at least $x^i_A(k^*)$ and at most $x^i_D(k^*)$ resources, where each resource is assigned to exactly k^* users. This is achieved in the following way. For any player i, let $d^i(k^*) = k^*m - \sum_{j=1}^{i-1}|\sigma_j| - \sum_{j=i}^{n} x^j_A(k^*)$. The algorithm proceeds gradually from player 1 to player n such that player i receives $|\sigma_i|$ least currently congested resources, where $|\sigma_i| = \{x^i_D(k^*), x^i_A(k^*) + d^i(k^*)\}$. That is, for player i, $d^i(k^*)$ denotes the number of resources available to him in addition to the $x^i_A(k^*)$ resources (which is the minimum number of resources that i should keep in a k^*-even, A-stable profile), such that each player j receives at least $x^j_A(k^*)$ resources, and the total number of assignments does not exceed k^*m. Since for any $i \in N$ we have $x^i_A(k^*) \leq x^i_D(k^*)$, $\sum_{i\in N} x^i_D(k^*) \geq k^*m$ and $\sum_{i\in N} x^i_A(k^*) \leq k^*m$, then for any i we have $d^i(k^*) \geq 0$ and $x^i_A(k^*) \leq |\sigma_i| \leq x^i_D(k^*)$. Moreover, there exists $1 \leq i^* \leq n+1$ such that $k^*m - \sum_{j=1}^{i^*-1}|\sigma_j| - \sum_{j=i^*}^{n} x^j_A(k^*) = 0$ implying that $\sum_{i\in N}|\sigma_i| = k^*m$. Also, since player i at each step adds only the least currently congested resources, the congestion on any resource in the resulting profile is exactly k^*. That is, the algorithm results with a k^*-even strategy profile σ, in which any player $i \in N$ uses at least $x^i_A(k^*)$ and at most $x^i_D(k^*)$ resources ($x^i_A(k^*) \leq |\sigma_i| \leq x^i_D(k^*)$). Since σ is even it is S-stable. In addition, by the definitions of $x^i_A(k^*)$ and $x^i_D(k^*)$, and Observation 7, σ is A- and D-stable. Therefore, by Lemma 8, σ is a Nash equilibrium strategy profile.

. $0 < k^* \leq n-1$*: construction of a* k^**-even, post-addition D-stable profile.*

At Step [5], the algorithm constructs a k^*-even, post-addition D-stable strategy profile. For any player $i \in N$ the algorithm determines a value $x^i(k^*)$ which is the maximum integer between 1 and m satisfying $\bar{\mathbf{v}}_i^{k^*}(x^i(k^*)-1) \geq \bar{c}(k^*+1)$ (if for all $x \in \{1,\ldots,m\}$ we have $\bar{\mathbf{v}}_i^{k^*}(x-1) < \bar{c}(k^*+1)$ then the algorithm determines $x^i(k^*) = 0$). That is, by Definition 17, for any $i \in N$, $x^i(k^*)$ represents the maximum number of resources that i would keep in a k^*-even, post-addition D-stable strategy profile. We note that since $\bar{\mathbf{v}}_i^{k^*}(x-1)$ decreases in x and $x \leq m$, the value of $x^i(k^*)$ is uniquely defined, and the condition $\bar{\mathbf{v}}_i^{k^*}(x-1) \geq \bar{c}(k^*+1)$ holds for all $x < x^i(k^*)$. By Lemma 18, the existence of a profile as described above implies that $\sum_{i\in N} x^i(k^*) \geq k^*m$ (otherwise, such a profile does not exist). Based on this, the algorithm constructs the required profile in the following way. The algorithm proceeds gradually from player 1 to player n such that player i receives $|\sigma_i|$ least currently congested resources, where $|\sigma_i|$ is the minimum between $x^i(k^*)$ and $k^*m - \sum_{j=1}^{i-1}|\sigma_j|$. Therefore, each player i receives at most $x^i(k^*)$ resources. Moreover, since $\sum_{i\in N} x^i(k^*) \geq k^*m$ then for any i, $k^*m - \sum_{j=1}^{i-1}|\sigma_j| \geq 0$, and there exists

$1 \leq i^* \leq n+1$ such that $k^*m - \sum_{j=1}^{i^*-1} |\sigma_j| = 0$ implying that $\sum_{i \in N} |\sigma_i| = k^*m$. Finally, since player i at each step adds only the least currently congested resources, in the resulting profile the congestion on any resource is exactly k^*. By the definition of $x^i(k^*)$, the resulting profile is post-addition D-stable. Moreover, $\bar{\mathbf{v}}_i^{k^*}(x^i(k^*)-1) \geq \bar{c}(k^*+1)$ yields $\bar{\mathbf{v}}_i^{k^*}(x^i(k^*)-1) \geq \bar{c}(k^*)$ for any $i \in N$. Hence, by Observation 7, the above profile is D-stable.

- *The one-/two step addition dynamics and the convergence to an equilibrium.*

Steps [6] – [7] describe the procedure of one- or two-step addition. The algorithm halts after Step [6] if and only if the current strategy profile is A-stable. We also note that the resulting strategy profile is nearly-even and hence S-stable. Recall that the algorithm begins the first iteration of Steps [6] – [7] with a k^*-even, post-addition D-stable strategy profile. Therefore, by Lemma 19, applying to it less than m one-/two-step addition operations, we preserve its post-addition D-stability. Moreover, the resulting profile of each iteration is nearly-even, which, coupled with the post-addition D-stability, implies its D-stability. Assume that the algorithm has not terminated after $m-1$ iterations of [6] – [7]. Then, at the m-th iteration the algorithm produces a k^*+1-even, D-stable strategy profile, contradicting the choice of k^* (recall that k^* represents the maximum possible common congestion on the resources, allowing the D-stability of an even strategy profile).

The above implies that the CGLF-algorithm terminates with a pure strategy Nash equilibrium of a given nondecreasing CGLF.

Complexity. Step [0] takes $O(1)$ operations and is repeated only once. Step [1] takes $O(nm)$ operations and can be repeated at most n times. Step [2] takes $O(nm)$ operations and can be repeated only once. Steps [3] – [5] take $O(nm)$ operations each and are performed at most once. Steps [6] – [7] take $O(nm)$ operations – $O(nm)$ at Step [6] and $O(n+m)$ at Step [7] – and can be repeated at most m times. Therefore, the complexity of CGLF-algorithm is $O(n^2m + nm^2)$. □

6 CGLFs with observable resource failures

In the CGLF-model discussed above, a player is required to pay for any resource he uses, regardless of its success or failure. Such a scenario, for example, may take place when the success or the failure of every resource cannot be observed but only the success or the failure of the player's task can be detected. This model can be modified to describe situations in which the success or failure of each resource can be detected, and the players are required to pay only for the non-faulty resources they selected. Both models are reasonable and worthwhile studying. It turns out that the two models hold similar results, as described in the sequel.

The expected utility for a player in the modified model with observable resource failures can be written as follows:

$$
\begin{aligned}
U_i(\sigma) &= \left(1 - \prod_{e\in\sigma_i} f(h_e(\sigma))\right) v_i - \sum_{e\in\sigma_i} c(h_e(\sigma))\,(1 - f(h_e(\sigma))) \\
&= \left(1 - \prod_{e\in\sigma_i} f(h_e(\sigma))\right) v_i - \sum_{e\in\sigma_i} c'(h_e(\sigma)),
\end{aligned}
$$

where $c'(\cdot) = c(\cdot)\,(1 - f(\cdot))$ is the modified cost function. Thus, the modified model can be viewed as a special case of the CGLF-model we discussed before. Therefore, all the results we obtained for CGLFs are valid in this case as well, with respect to the modified cost function. In particular, there is a pure strategy Nash equilibrium whenever $c'(\cdot)$ is nondecreasing.

However, the fact that the *original* cost function, $c(\cdot)$, is nondecreasing does not imply that the modified cost function, $c'(\cdot)$, is nondecreasing; hence, the modified CGLF with nondecreasing costs is *not* a special case of the original nondecreasing CGLF. One may notice though that $c(\cdot) = \frac{c'(\cdot)}{1-f(\cdot)} = \bar{c'}$ (recall our notation following Lemma 14). Therefore, if $c(\cdot)$ is nondecreasing then so is $\bar{c'}(\cdot)$. This implies that the technical approach we proposed in this paper to prove the existence of a pure strategy Nash equilibrium in CGLFs with unobservable resource failures (which uses the monotonicity of $\bar{c}(\cdot)$ rather than that of $c(\cdot)$) [9] is applicable to the case of observable failures as well. Since the proof techniques in this case are similar to those we used earlier, the detailed proofs are omitted from this paper.

7 Summary and future work

In this paper, we introduced and studied congestion settings with unreliable resources, where the resource failure probability depends on the congestion experienced by the resource. We defined a class of *congestion games with load-dependent failures* (CGLFs). In contrast with the work on CGFs (Penn et al. (2005)) where the players care about delays and therefore consider the minimum delay of their selected resources, in CGLFs players care about their profits, and therefore the model is additive with respect to the cost of the selected resources, as in classical congestion games. In a CGLF, a player aims to maximize the difference between his expected benefit from a successful task completion and the sum of costs over the resources he uses. We have studied

[9] See Remark 15.

the existence of a pure strategy Nash equilibrium and the existence of a potential function in the presented class of games. We showed that these games do not, in general, possess pure strategy equilibria. Nevertheless, if the resource cost functions are nondecreasing then such an equilibrium is guaranteed to exist, despite the non-existence of a potential function. We presented an efficient algorithm for constructing a pure strategy Nash equilibrium in CGLFs with nondecreasing costs.

The aim of this work was to initiate the study of CGLFs, which we believe to be a realistic model, capturing basic phenomena in e.g. the service industries. In future research we plan to consider various extensions of CGLFs. In particular, we wish to consider the more general situation where resources are not identical, and players are restricted and differ in the subsets of resources they may use. In addition, it is of interest to discuss the question of the existence of an ordinal potential function in CGLFs, stability under deviations by coalitions and social (in)efficiency of equilibria in CGLFs.

References

Even-Dar, E., Kesselman, A., Mansour, Y., 2003. Convergence time to Nash equilibria. In: Proc. of 30th Int. EATCS Coll. on Automata, Languages and Programming (ICALP). pp. 502–513.

Fotakis, D., Kontogiannis, S., Spirakis, P., 2005. Selfish unsplittable flows. Theor. Comput. Sci. 348 (2-3), 226–239.

Leyton-Brown, K., Tennenholtz, M., 2003. Local-effect games. In: Proceedings of the 18th International Joint Conference on Artificial Intelligence (IJCAI-03). pp. 772–777.

Milchtaich, I., 1996. Congestion games with player-specific payoff functions. Games and Economic Behavior 13, 111–124.

Monderer, D., 2007. Multipotential games. In: IJCAI-07. pp. 1422–1427.

Monderer, D., Shapley, L., 1996. Potential games. Games and Economic Behavior 14, 124–143.

Penn, M., Polukarov, M., Tennenholtz, M., 2005. Congestion games with failures. In: Proceedings of the 6th ACM Conference on Electronic Commerce (EC-05). pp. 259–268.

Penn, M., Polukarov, M., Tennenholtz, M., 2007. Congestion games with load-dependent failures: identical resources. In: Proceedings of the 8th ACM Conference on Electronic Commerce (EC-07). pp. 210–217.

Rosenthal, R., 1973. A class of games possessing pure-strategy Nash equilibria. International Journal of Game Theory 2, 65–67.