Counterexample to "Generalized eigenvalue-based stability tests for 2-D linear systems: Necessary and sufficient conditions" by Fu, P., Chen, J., and S.I. Niculescu [1]

P. Rapisarda [a], K. Takaba [b]

[a] *ISIS, School of Electronics and Computer Science, University of Southampton, SO171BJ Southampton, United Kingdom*

[b] *Dept. of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606–8501, Japan, Fax: +81-(0)75-753-5507*

Abstract

We present a counterexample contradicting the equivalence of statements $(i)-(ii)$ and $(iii)-(v)$ of Theorem 1 of [2].

Key words: 2-D system; Stability criteria; Matrix pencil.

1 Introduction

In [2] necessary and sufficient stability conditions for a two-variable polynomial were stated; we now briefly review them introducing first some notation. Consider the two-variable polynomial

$$a(z_1, z_2) = \sum_{i=0}^{n_1} \sum_{j=0}^{n_2} a_{i,j} z_1^i z_2^j, \tag{1}$$

where $a_{ij} \in \mathbb{R}$, $i = 0, \ldots, n_1$, $j = 0, \ldots, n_2$. The polynomial (1) is called *stable* if it has no roots in the closed bidisk, i.e. if

$$a(z_1, z_2) \neq 0 \text{ for all } z_1 \in \mathbb{D}, z_2 \in \mathbb{D}, \tag{2}$$

where $\mathbb{D} := \{z \in \mathbb{C} \mid |z| \leq 1\}$. In order to state the main result of [2] on the stability of (1), we need to introduce some more notation. Define from (1) the n_2+1 univariate polynomials $a_j(z_1) := \sum_{i=0}^{n_1} a_{ij} z_1^i$, and note that (1) can then be rewritten as $a(z_1, z_2) = \sum_{j=0}^{n_2} a_j(z_1) z_2^j$. Now let $z_1 \in \delta\mathbb{D}$, where $\delta\mathbb{D} := \{z \in \mathbb{C} \mid |z| = 1\}$ is the unit circle, and introduce the matrices

$$\Delta_1(z_1) := \begin{bmatrix} a_0(z_1) & 0 & \ldots & 0 \\ a_1(z_1) & a_0(z_1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n_2-1}(z_1) & a_{n_2-2}(z_1) & \ldots & a_0(z_1) \end{bmatrix}$$

$$\Delta_2(z_1) := \begin{bmatrix} a_{n_2}(z_1) & 0 & \ldots & 0 \\ a_{n_2-1}(z_1) & a_{n_2}(z_1) & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_1(z_1) & a_2(z_1) & \ldots & a_{n_2}(z_1) \end{bmatrix}$$

$$\Delta(z_1) := \begin{bmatrix} \Delta_1(z_1^{-1}) & \Delta_2(z_1) \\ \Delta_2(z_1^{-1})^\top & \Delta_1(z_1)^\top \end{bmatrix}$$

and the matrix

$$K(z_1) := \Delta_1(z_1)^\top \Delta_1(z_1^{-1}) - \Delta_2(z_1^{-1})^\top \Delta_2(z_1) \tag{3}$$

where $^\top$ denotes transposition. Moreover, for $i = 0, \ldots, n_1$ consider the matrix

$$P_i := \begin{bmatrix} P_{11}^{(i)} & P_{12}^{(i)} \\ P_{21}^{(i)} & P_{22}^{(i)} \end{bmatrix}$$

Email addresses: pr3@ecs.soton.ac.uk (P. Rapisarda), takaba@amp.i.kyoto-u.ac.jp (K. Takaba).

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where

$$P_{11}^{(i)} := \begin{bmatrix} a_{n_1-i,0} & 0 & \dots & 0 \\ a_{n_1-i,1} & a_{n_1-i,0} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{n_1-i,n_2-1} & a_{n_1-i,n_2-2} & \dots & a_{n_1-i,0} \end{bmatrix}$$

$$P_{12}^{(i)} := \begin{bmatrix} a_{i,n_2} & 0 & \dots & 0 \\ a_{i,n_2-1} & a_{i,n_2} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ a_{i,1} & a_{i,2} & \dots & a_{i,n_2} \end{bmatrix}$$

$$P_{21}^{(i)} := \begin{bmatrix} a_{n_1-i,n_2} & a_{n_1-i,n_2-1} & \dots & a_{n_1-i,1} \\ 0 & a_{n_1-i,n_2} & \dots & a_{n_1-i,2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{n_1-i,n_2} \end{bmatrix}$$

$$P_{22}^{(i)} := \begin{bmatrix} a_{i,0} & a_{i,1} & \dots & a_{i,n_2-1} \\ 0 & a_{i,0} & \dots & a_{i,n_2-2} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{i,0} \end{bmatrix}$$

and define the matrices P and Q as

$$P := \begin{bmatrix} 0 & I & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & I \\ -P_0 & -P_1 & \dots & -P_{n_1-1} \end{bmatrix}$$

$$Q := \begin{bmatrix} I & & & \\ & \ddots & & \\ & & I & \\ & & & P_{n_1} \end{bmatrix}$$

Moreover, for a matrix pair A, B define the set of its generalized eigenvalues as $\sigma(A, B) := \{\lambda \in \mathbb{C} \mid \det(A - \lambda B) = 0\}$.

Theorem 1 of [2] states the following equivalence.

Theorem 1 *The polynomial (1) is stable if and only if $a_0(z_1)$ is stable, and one of the following equivalent conditions holds:*

(*i*) $\|z_1^{n_1}\Delta_2(\frac{1}{z_1})\Delta_1(z_1)^{-1}\|_\infty < 1$;
(*ii*) $K(z_1) > 0$ *for all* $z_1 \in \delta\mathbb{D}$;
(*iii*) $\det(K(z_1)) \neq 0$ *for all* $z_1 \in \delta\mathbb{D}$, *and* $\det(K(1)) > 0$;
(*iv*) $\det(\Delta(z_1) \neq 0$ *for all* $z_1 \in \delta\mathbb{D}$, *and* $\det(\Delta(1)) > 0$;
(*v*) $\sigma(P, Q) \cap \delta\mathbb{D} = \emptyset$, *and* $\det(Q - P) > 0$.

Using a continuity argument it can be proved that if (*ii*) holds, then (*iii*) also holds. In the following section we show that (under the stability assumption on $a_0(z_1)$) the converse implication does not hold. The error is caused by the fact that the authors quote incorrectly a result in [1] in support of statement (*iii*) of Theorem 1. Using the notation introduced here, the result of [1], formula (3.40) p. 196, states that (1) is stable if and only if $K(1) > 0$ and $\det(K(z_1)) \neq 0$ for all $z_1 \in \delta\mathbb{D}$. In statement (*iii*) of Theorem 1 the condition $K(1) > 0$ has been erroneously replaced by $\det(K(1)) > 0$. Since statement (*iii*) of Theorem 1 is incorrect, but it is equivalent with statements (*iv*) and (*v*), it follows that also (*iv*) and (*v*) are incorrect.

2 The counterexample

Consider the polynomial

$$a(z_1, z_2) := \underbrace{(z_1 - 2)}_{=:a_0(z_1)} z_2^0 + \underbrace{0}_{=:a_1(z_1)} z_2^1 + \underbrace{\sqrt{3}(z_1 - 2)}_{=:a_2(z_1)} z_2^2 \quad (4)$$

Note that (4) can be rewritten as $a(z_1, z_2) = (z_1 - 2)(1 + \sqrt{3}\, z_2^2)$, and that consequently $a(z_1, z_2) = 0$ if and only if $z_1 = 2$ or $z_2 = \pm i\frac{1}{\sqrt[4]{3}}$. In particular, (4) has roots of the form $(\mu, \pm i\frac{1}{\sqrt[4]{3}})$, with $\mu \in \mathbb{D}$ and consequently it is not stable according to condition (2). The matrix (3) equals

$$K(z_1) = \begin{bmatrix} -2(z_1^{-1} - 2)(z_1 - 2) & 0 \\ 0 & -2(z_1^{-1} - 2)(z_1 - 2) \end{bmatrix} (5)$$

Note that $a_0(z_1) = z_1 - 2$ is stable, since all its roots are outside of the closed unit disk. Moreover, $\det(K(z_1)) \neq 0$ for z_1 on the unit circle, since it is easy to verify that $\det\left(K(e^{i\theta})\right) = 4(5 - 4\cos(\theta))$. Finally, $\det(K(1)) = 4 > 0$. Consequently, the polynomial (4) satisfies the condition on the stability of $a_0(z_1)$ and condition (*iii*) of Theorem 1. However, (4) is unstable. Observe that the matrix $K(z_1)$ in (5) is negative definite for z_1 on the unit circle, and that consequently condition (*ii*) of Theorem 1 is not satisfied. This clearly shows that conditions (*ii*) and (*iii*) of Theorem 1 are not equivalent, although as already mentioned in section 1, the implication (*ii*) $\Longrightarrow$ (*iii*) is correct.

References

[1] Bose, N. K. (1982). *Applied multidimensional systems theory*, New York: Van Nostrand Reinhold.

[2] Fu, P., & Chen, J., and S.I. Niculescu (2006). "Generalized eigenvalue-based stability tests for 2-D linear systems: Necessary and sufficient conditions", *Automatica*, 42(9) 1569–1576.