

Balanced state-space representations: a polynomial algebraic approach

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Abstract—We show how to compute a minimal Riccati-balanced state map and a minimal Riccati-balanced state space representation starting from an image representation of a strictly dissipative system. The result is based on an iterative procedure to solve a generalization of the Nevanlinna interpolation problem.

I. INTRODUCTION

Computing a balanced state-space representation for a system constitutes the first step of various model reduction methods, see [15], [12], [11]. These procedures invariably consider a given state-space representation of the system; however, usually in engineering practice a state-space model is *derived* from the equations describing the dynamics of the system (typically transfer functions, or higher-order differential equations), and it is not given *a priori*. Instrumental in this derivation is the concept of *state map* introduced in [21], that is a polynomial differential operator which, when applied to the variables of the system, induces a state variable, to which correspond an input-state-output representation

$$\begin{aligned} \frac{d}{dt}x &= Ax + Bu \\ y &= Cx + Du \end{aligned} \quad (1)$$

Algorithms for computing a state map and the matrices involved in a description (1) have been given in [21].

In this paper we define a state map to be balanced if it induces a representation (1) such that the minimal- and the maximal storage functions for the system with respect to a given supply rate are diagonal and the inverse of each other. We also investigate how to compute such a state map. Our starting point will be a polynomial matrix M describing the set of system trajectories $w \in \mathfrak{B}$ as $w = M\left(\frac{d}{dt}\right)\ell$. This representation exists for any controllable system (see sect. 6.6 of [19]) and can be shown to correspond in a natural way with a right-factorization of the transfer function of the system; hence, some of the results presented in this paper are also related to the problem of computing a balanced realization of a given transfer function.

The problem of obtaining a balanced state map from the matrix M has been considered before in the behavioral context in [26], where algorithms based on interpolation and numerical linear algebra methods were presented for the single-input, single-output case. The approach proposed in this paper is a different one, and can be applied to

the multiple-input, multiple-output case. It is based on an iterative procedure for modeling vector-exponential trajectories which has been also used in order to solve metric interpolation and J -spectral factorization problems (see [22], [14], [18]). This procedure assumes the strict-dissipativeness of the to-be-reduced system with respect to a given supply rate; and the knowledge of the spectral zeroes of the system, together with the associated directions.

The paper is organized as follows: in section II we have gathered the background material necessary in order to follow the exposition. In section III we illustrate the concept of Σ -unitary kernel representation. Section IV contains an algorithm to compute a sum-of-squares representation of a storage function. In section V we show how to compute minimal diagonalizing state maps. Section VI illustrates the computation of a minimal balanced state map and the corresponding input-state-output representation.

Notation. The space of n dimensional real, respectively complex, vectors is denoted by \mathbb{R}^n , respectively \mathbb{C}^n , and the space of $m \times n$ real, respectively complex, matrices, by $\mathbb{R}^{m \times n}$, respectively $\mathbb{C}^{m \times n}$. Whenever one of the two dimensions is not specified, a bullet \bullet is used; for example, $\mathbb{R}^{\bullet \times w}$ denotes the set of matrices with w columns and with an arbitrary finite number of rows. If $A \in \mathbb{C}^{p \times m}$, then $A^* \in \mathbb{C}^{m \times p}$ denotes its complex conjugate transpose. A signature matrix is of the form

$$\begin{bmatrix} I_{\sigma_+} & 0 \\ 0 & -I_{\sigma_-} \end{bmatrix} \quad (2)$$

where I_{σ_+} is a $\sigma_+ \times \sigma_+$ identity matrix, and analogously for I_{σ_-} . If $S = S^T$, then we denote with $\sigma_+(S)$ the number of positive eigenvalues, and with $\sigma_-(S)$ the number of negative eigenvalues, of S .

The ring of polynomials with real coefficients in the indeterminate ξ is denoted by $\mathbb{R}[\xi]$; the ring of two-variable polynomials with real coefficients in the indeterminates ζ and η is denoted by $\mathbb{R}[\zeta, \eta]$. The space of all $n \times m$ polynomial matrices in the indeterminate ξ is denoted by $\mathbb{R}^{n \times m}[\xi]$, and that consisting of all $n \times m$ polynomial matrices in the indeterminates ζ and η by $\mathbb{R}^{n \times m}[\zeta, \eta]$. Given a matrix $R \in \mathbb{R}^{n \times m}[\xi]$, we define $R^\sim(\xi) := R(-\xi)^* \in \mathbb{R}^{m \times n}[\xi]$.

We denote with $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w)$ the set of infinitely often differentiable functions from \mathbb{R} to \mathbb{R}^w . The set of infinitely differentiable functions with compact support is denoted with $\mathcal{D}(\mathbb{R}, \mathbb{R}^w)$. The exponential function whose value at t is $e^{\lambda t}$ will be denoted with \exp_λ .

II. BACKGROUND MATERIAL

The results illustrated in this section relate to systems with real trajectories in $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w)$. However, in the rest of the

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paper we will also use complex-valued trajectories; in this case it is necessary to consider straightforward generalizations of the concepts and results presented here. When a generalization is not straightforward, we discuss it in detail.

A. Linear differential systems and their representations

A subspace $\mathfrak{B} \subseteq \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w)$ is a *linear differential behavior* (LDI in the following) if it consists of the solutions of a system of linear, constant-coefficient differential equations. We denote with \mathfrak{L}^w the set of linear differential systems with w external variables. A complex LDI behavior $\mathfrak{B}_c \subseteq \mathcal{C}^\infty(\mathbb{R}, \mathbb{C}^w)$ can be obtained from a real LDI behavior \mathfrak{B} by *complexification*: $w \in \mathfrak{B}_c$ if and only if the real and the complex part of w belong to \mathfrak{B} .

If \mathfrak{B} is controllable (see section 5.2 of [19]) then it also admits an *image representation*, i.e.

$$\mathfrak{B} = \left\{ w \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w) \mid \exists \ell \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^1) \text{ s.t. } w = M \left(\frac{d}{dt} \right) \ell \right\}. \quad (3)$$

In the following we denote the set of controllable behaviors with w external variables by $\mathfrak{L}_{\text{cont}}^w$.

We call the behavior

$$\mathfrak{B}_{\text{full}} = \left\{ (w, \ell) \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{w+1}) \mid w = M \left(\frac{d}{dt} \right) \ell \right\}$$

the *full behavior* of the representation (3).

A state system is a special type of latent variable system, in which the latent variable, typically denoted with x , satisfies the *state property*, see section 3 of [21]. A state system is said to be *minimal* if the state variable has minimal number of components among all state representations that have the same manifest behavior. In [21] it was shown that a state variable for an image representation of \mathfrak{B} can be obtained as $x = X \left(\frac{d}{dt} \right) \ell$, for some $X \in \mathbb{R}^{\bullet \times 1}[\xi]$. In this case $X \left(\frac{d}{dt} \right)$ is called a *state map*; the definition of minimal state map follows in a straightforward manner. In [21] algorithms are stated to construct a state map from the equations describing the system.

The definitions of state property and that of state map in the case of complex behaviors are the same as in the real case, with the coefficients of the polynomial matrix X being constant complex matrices.

Important *integer invariants* associated with a behavior $\mathfrak{B} \in \mathfrak{L}^w$ are: the *input cardinality* denoted $\mathfrak{m}(\mathfrak{B})$; and the dimension of any minimal state variable for \mathfrak{B} , also called the *McMillan degree* of \mathfrak{B} , and denoted with $\mathfrak{n}(\mathfrak{B})$. It can be shown that $\mathfrak{m}(\mathfrak{B})$ is the number of columns of the matrix M in any *observable* image representation $\mathfrak{B} = \text{im } M \left(\frac{d}{dt} \right)$, i.e. one such that $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$. If $\mathfrak{m}(\mathfrak{B}) = 0$ then the behavior is called *autonomous*; it can be proved that in this case all its trajectories are vector polynomial-exponential ones.

B. Quadratic differential forms

Let $\Phi \in \mathbb{R}^{w \times w}[\zeta, \eta]$, written out in terms of its coefficient matrices $\Phi_{k,\ell}$ as the (finite) sum $\Phi(\zeta, \eta) = \sum_{k,\ell \in \mathbb{Z}_+} \Phi_{k,\ell} \zeta^k \eta^\ell$. It induces the map

$Q_\Phi : \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^w) \rightarrow \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$, defined by $Q_\Phi(w) = \sum_{k,\ell \in \mathbb{Z}_+} \left(\frac{d^k}{dt^k} w \right)^\top \Phi_{k,\ell} \left(\frac{d^\ell}{dt^\ell} w \right)$. This map is called the *quadratic differential form (QDF)* induced by Φ . (We understand that the term ‘quadratic differential’ is used in the theory of singularities; however we stick to the convention introduced in [27] and use this terminology anyway, apologizing for the confusion.) We can without loss of generality assume that Φ is symmetric, i.e. $\Phi(\zeta, \eta) = \Phi(\eta, \zeta)^\top$; we denote the set of such matrices with $\mathbb{R}_s^{w \times w}[\zeta, \eta]$. Any $\Phi \in \mathbb{R}_s^{w \times w}[\zeta, \eta]$ admits a *canonical symmetric factorization* $\Phi(\zeta, \eta) = M^\top(\zeta) \Sigma_\Phi M(\eta)$, see p. 1709 of [27]. These definitions hold *mutatis mutandis* for the case of complex two-variable polynomial matrices and associated functionals.

Some features of the calculus of QDFs which will be used in this paper are the following. The first one is the *del operator*, defined as $\partial : \mathbb{R}_s^{w \times w}[\zeta, \eta] \rightarrow \mathbb{R}^{w \times w}[\xi]$, with $\partial\Phi(\xi) := \Phi(-\xi, \xi)$. Observe that $\partial\Phi$ is a para-Hermitian matrix, i.e. $\partial\Phi(\xi) = \partial\Phi(-\xi)^\top$.

The functional $\frac{d}{dt} Q_\Phi$ defined by $\left(\frac{d}{dt} Q_\Phi \right)(w) := \frac{d}{dt} (Q_\Phi(w))$, is again a QDF, called the *derivative of Q_Φ* ; the two-variable polynomial matrix inducing it is $(\zeta + \eta)\Phi(\zeta, \eta)$.

Next, we introduce the notion of *integral of a QDF*. In order to make sure that the integral exists, we assume that the QDF acts on $\mathfrak{D}(\mathbb{R}, \mathbb{R}^w)$. The integral of Q_Φ is the form $\int Q_\Phi : \mathfrak{D}(\mathbb{R}, \mathbb{R}^w) \rightarrow \mathbb{R}$ defined by $\int Q_\Phi(w) := \int_{-\infty}^{\infty} Q_\Phi(w) dt$.

The notion of *observability of a QDF* (see section 7 of [27]) can be characterized in terms of a canonical symmetric factorization $M^\top(\zeta) \Sigma_\Phi M(\eta)$ of $\Phi(\zeta, \eta)$: Φ is observable if and only if $M(\lambda)$ has full column rank for all $\lambda \in \mathbb{C}$ (see Corollary 7.3 of [27]).

Finally, we show how to associate a QDF to a behavior $\mathfrak{B} \in \mathfrak{L}_{\text{cont}}^w$. Let $\mathfrak{B} = \text{im } M \left(\frac{d}{dt} \right)$, and let $\Phi \in \mathbb{R}_s^{w \times w}[\zeta, \eta]$. Define $\Phi' \in \mathbb{R}_s^{1 \times 1}[\zeta, \eta]$ as $\Phi'(\zeta, \eta) := M^\top(\zeta) \Phi(\zeta, \eta) M(\eta)$. If w and ℓ satisfy $w = M \left(\frac{d}{dt} \right) \ell$, then $Q_\Phi(w) = Q_{\Phi'}(\ell)$. The introduction of the two-variable matrix Φ allows to study the behavior Q_Φ along \mathfrak{B} in terms of properties of the QDF $Q_{\Phi'}$ acting on free trajectories of $\mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^1)$.

C. Dissipative behaviors

The definition of dissipative system is as follows.

Definition 1: Let $\mathfrak{B} \in \mathfrak{L}_{\text{cont}}^w$ and $\Sigma = \Sigma^\top \in \mathbb{R}^{w \times w}$. \mathfrak{B} is called *Σ -dissipative* if $\int_{\mathbb{R}} Q_\Sigma(w) dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^w)$. \mathfrak{B} is called *strictly Σ -dissipative* if there exists $\varepsilon > 0$ such that $\int_{\mathbb{R}} Q_\Sigma(w) dt \geq \varepsilon \int_{\mathbb{R}} w^\top w dt$ for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^w)$. \mathfrak{B} is called *strictly Σ -dissipative on \mathbb{R}_-* if there exists $\varepsilon > 0$ such that $\int_{\mathbb{R}_-} Q_\Sigma(w) dt \geq \varepsilon \int_{\mathbb{R}_-} w^\top w dt$ for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}_-, \mathbb{R}^w)$.

Dissipativity is related to the concept of storage function.

Definition 2: Let $\Sigma = \Sigma^\top \in \mathbb{R}^{w \times w}$ and $\mathfrak{B} \in \mathfrak{L}_{\text{cont}}^w$. Assume that \mathfrak{B} is Σ -dissipative; then a QDF Q_Ψ is a *storage function* if for all $w \in \mathfrak{B}$ it holds $\frac{d}{dt} Q_\Psi(w) \leq Q_\Sigma(w)$. A QDF Q_Δ is a *dissipation function* if $Q_\Delta(w) \geq 0$ for all $w \in \mathfrak{B}$, and for all $w \in \mathfrak{B} \cap \mathfrak{D}(\mathbb{R}, \mathbb{R}^w)$ it holds $\int_{\mathbb{R}} Q_\Sigma(w) = \int_{\mathbb{R}} Q_\Delta(w)$.

The following result holds.

Proposition 3: The following conditions are equivalent

- 1) \mathfrak{B} is Σ -dissipative,
- 2) \mathfrak{B} admits a storage function,
- 3) \mathfrak{B} admits a dissipation function.

Moreover, for every dissipation function Q_Δ there exists a unique storage function Q_Ψ , and for every storage function Q_Ψ there exists a unique dissipation function Q_Δ , such that $\frac{d}{dt}Q_\Psi(w) = Q_\Phi(w) - Q_\Delta(w)$ for all $w \in \mathfrak{B}$.

Proof: See [27, Proposition 5.4]. ■

Every storage function is a quadratic function of the state.

Proposition 4: Let $\Sigma = \Sigma^\top \in \mathbb{R}^{w \times w}$ and $\mathfrak{B} \in \mathfrak{L}_{\text{cont}}^w$ be Σ -dissipative. Let Q_Ψ be a storage function. Then Q_Ψ is a state function, i.e. for every X inducing a state map for \mathfrak{B} , there exists a real symmetric matrix K acting on ℓ such that

$$Q_\Psi(w) = \left(X \left(\frac{d}{dt} \right) \ell \right)^\top K \left(X \left(\frac{d}{dt} \right) \ell \right).$$

for every w and ℓ such that $w = M \left(\frac{d}{dt} \right) \ell$.

Proof: See Theorem 5.5 of [27]. ■

In general there exist an infinite number of storage functions; however, all of them lie between two extremal ones.

Proposition 5: Let \mathfrak{B} be Σ -dissipative; then there exist storage functions Ψ_- and Ψ_+ such that any storage function Ψ satisfies $Q_{\Psi_-} \leq Q_\Psi \leq Q_{\Psi_+}$ along \mathfrak{B} .

Proof: See [27, Theorem 5.7]. ■

If $\mathfrak{m}(\mathfrak{B}) = \sigma_+(\Sigma)$, then the nonnegativity of all storage functions is equivalent with the half-line Σ -dissipativity of \mathfrak{B} , as the following result shows.

Proposition 6: Let $\mathfrak{B} \in \mathfrak{L}_{\text{cont}}^w$ and $\Sigma = \Sigma^\top \in \mathbb{R}^{w \times w}$ be nonsingular. Assume that $\mathfrak{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Then the following statements are equivalent.

- 1) \mathfrak{B} is Σ -dissipative on \mathbb{R}_- ;
- 2) there exists a nonnegative storage function of \mathfrak{B} ;
- 3) all storage functions of \mathfrak{B} are nonnegative;
- 4) there exists a real symmetric matrix $K > 0$ such that

$$Q_K(w) := \left(X \left(\frac{d}{dt} \right) w \right)^\top K \left(X \left(\frac{d}{dt} \right) w \right)$$

is a storage function of \mathfrak{B} ;

- 5) there exists a storage function of \mathfrak{B} , and every real symmetric matrix $K > 0$ such that $Q_K(w) := \left(X \left(\frac{d}{dt} \right) w \right)^\top K \left(X \left(\frac{d}{dt} \right) w \right)$ is a storage function of \mathfrak{B} satisfies $K > 0$.

Proof: See [27, Proposition 6.4]. ■

D. Pick matrices

In order to introduce the notion of Pick matrix associated with a QDF, we must introduce the notion of Λ -set.

Definition 7: Let $\Gamma \in \mathbb{R}_s^{w \times w}[\xi]$ be nonsingular, para-Hermitian, and such that $\det(\Gamma(i\omega)) \neq 0$ for all $\omega \in \mathbb{R}$. A subset $S \subset \mathbb{C}$ is a Λ -set of Γ if:

- 1) there exists a factorization $cp(-\xi)p(\xi)$ of $\det(\Gamma(\xi))$ with $c \in \mathbb{R}$ and $p \in \mathbb{R}[\xi]$ such that the set of roots (counting multiplicities) of p equals S ;
- 2) $\{\lambda \in S\} \implies \{-\lambda \notin S\}$.

The determinant of a para-Hermitian matrix has always even degree, say $2n$. It follows from the definition that any Λ -set has exactly n elements. The number of distinct elements in a given Λ -set S is called the *cardinality* of S . Observe that if S is a Λ -set, then also the set

$$\bar{S} := \{\lambda \in \mathbb{C} \mid \det(\Gamma(\lambda)) = 0 \text{ and } \lambda \notin S\}$$

is a Λ -set; we call \bar{S} the *complementary* Λ -set of S .

The following result connecting Λ -sets of a para-Hermitian matrix and spectral factorizations holds.

Proposition 8: Let $\Gamma \in \mathbb{R}^{w \times w}[\xi]$ be a para-Hermitian matrix such that for all $\omega \in \mathbb{R}$ it holds $\det(\Gamma(i\omega)) > 0$. Let S be a Λ -set for Γ . Then there exists $F \in \mathbb{R}^{w \times w}[\xi]$ such that $\Gamma = F \sim F$ and the set of roots of $\det(F)$ equals S .

Proof: See [2]. ■

In the following, we define the notion of Pick matrix only for semisimple polynomial matrices. Let $G \in \mathbb{R}^{w \times w}[\xi]$; then G is *semisimple* if for every $\lambda \in \mathbb{C}$, the multiplicity of λ as a root of $\det(G)$ equals the dimension of the subspace $\ker G(\lambda)$.

Definition 9: Let $\Phi \in \mathbb{R}_s^{w \times w}[\zeta, \eta]$. Assume $\partial\Phi$ is semisimple and that $\det(\partial\Phi)$ has no roots on the imaginary axis. Let $S = \{\lambda_i\}_{i=1, \dots, n}$ be a Λ -set of $\partial\Phi$ with cardinality k , and let $\lambda_1, \dots, \lambda_k$ be the distinct elements of S . Denote with n_i the multiplicity of λ_i as a root of $\det(\partial\Phi)$. Let $V_i \in \mathbb{C}^{w \times n_i}$, $i = 1, \dots, k$, be full column rank matrices such that $\ker(\partial\Phi(\lambda_i)) = \text{im}(V_i)$. The Pick matrix of Φ associated with S is the $n \times n$ matrix

$$T_{\Phi, S} := \left[\frac{V_i^* \Phi(\bar{\lambda}_i, \lambda_j) V_j}{\bar{\lambda}_i + \lambda_j} \right]_{i, j=1, \dots, k}.$$

Observe that every Pick matrix is Hermitian. Note that since the definition of the basis matrices V_i for $\ker \partial\Phi(\lambda_i)$ is not unique, $T_{\Phi, S}$ also depends on the particular choice of the V_i , $i = 1, \dots, k$. However, since this nonuniqueness issue is of no consequence for the use we will make of Pick matrices, we will continue to talk about “the” Pick matrix, and denote it as $T_{\Phi, S}$.

In this paper, factorizations of the Pick matrix will play an important role. Instrumental in these factorizations is the notion of V -matrix, which we now introduce.

Definition 10: Let $\Phi(\zeta, \eta) = M^\top(\zeta)\Sigma M(\eta) \in \mathbb{R}_s^{1 \times 1}[\zeta, \eta]$ be observable. Assume that $\partial\Phi$ is semisimple and let $\mathfrak{B} = \text{im } M \left(\frac{d}{dt} \right)$ be strictly Σ -dissipative. Assume that $M(-\xi)^\top \Sigma M(\xi)$ is semisimple. Then any Λ -set has $n(\mathfrak{B})$ elements, counting multiplicities. Let $X \in \mathbb{R}^{n(\mathfrak{B}) \times 1}[\xi]$ be a minimal state map for \mathfrak{B} . Let $S = \{\lambda_i\}_{i=1, \dots, n}$ be a Λ -set of $\partial\Phi$; denote its cardinality with k . Let $V_i \in \mathbb{C}^{w \times n_i}$, $i = 1, \dots, k$, be full column rank matrices such that $\ker \partial\Phi(\lambda_i) = \text{im } V_i$. The V -matrix associated with S and X is the $n(\mathfrak{B}) \times n(\mathfrak{B})$ matrix

$$V := [X(\lambda_1)V_1 \quad \dots \quad X(\lambda_k)V_k]$$

It can be shown (see Theorem 7.1 of [23]), that the V -matrix associated with S and X is nonsingular.

The following result gives a relation between storage functions, Pick matrices, and V -matrices.

Proposition 11: Let $\Phi(\zeta, \eta) = M^\top(\zeta)\Sigma M(\eta) \in \mathbb{R}_s^{1 \times 1}[\zeta, \eta]$ be observable, and assume that $\partial\Phi$ is semisimple. Assume that $\mathfrak{B} = \text{im } M\left(\frac{d}{dt}\right)$ is strictly Σ -dissipative. Let $X \in \mathbb{R}^{n(\mathfrak{B}) \times 1}[\xi]$ be a minimal state map for \mathfrak{B} acting on the latent variable. Let $K \in \mathbb{R}^{n(\mathfrak{B}) \times n(\mathfrak{B})}$ be a symmetric matrix, and let S be a Λ -set for $\partial\Phi$. The following two statements are equivalent:

- 1) There exists F square, nonsingular, and such that the set of roots of $\det(F)$ equals S , such that $X^\top(\zeta)KX(\eta)$ is the storage function of Φ corresponding to the dissipation rate $F(\zeta)^\top F(\eta)$;
- 2) $K = (V^*)^{-1}T_{\Phi, S}V^{-1}$, with V the V -matrix of (S, X) and with $T_{\Phi, S}$ the Pick matrix of (S, Φ) .

Proof: See Theorem 7.1 of [23]. \blacksquare

If $\mathfrak{m}(\mathfrak{B}) = \sigma_+(\Sigma)$, then Proposition 6 and Proposition 11 imply the following result.

Proposition 12: Let $\mathfrak{B} \in \mathfrak{L}_{\text{cont}}^w$ and $\Sigma = \Sigma^\top \in \mathbb{R}^{w \times w}$ be nonsingular. Let $\mathfrak{B} = \text{im } M\left(\frac{d}{dt}\right)$ be strictly Σ -dissipative. Assume that $\mathfrak{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Let $\Phi(\zeta, \eta) = M^\top(\zeta)\Sigma M(\eta)$ be observable, and assume that $\partial\Phi$ is semisimple. Then for any Λ -set S , the Pick matrix associated with S is positive definite.

III. Σ -UNITARY KERNEL REPRESENTATIONS

Let $\Sigma \in \mathbb{R}^{w \times w}$ represent an involution, i.e. $\Sigma^2 = I_w$; in the following it will be often the case that Σ is a signature matrix. A polynomial matrix $R \in \mathbb{C}^{w \times w}[\xi]$ is called Σ -unitary if there exists $p \in \mathbb{C}[\xi]$, $p \neq 0$, such that

$$R\Sigma R^\sim = R^\sim \Sigma R = pp^\sim \Sigma.$$

If $\mathcal{V} \subset \mathbb{C}^w$ is a linear subspace and $\lambda \in \mathbb{C}$, we define $\mathcal{V} \exp_\lambda := \{v \exp_\lambda \mid v \in \mathcal{V}\}$. We also denote $\mathcal{V}^{\perp \Sigma} := \{v \in \mathbb{C}^w \mid v^* \Sigma v' = 0 \text{ for all } v' \in \mathcal{V}\}$.

The following result holds.

Theorem 13: For $i = 1, \dots, N$ let $\mathcal{V}_i \subset \mathbb{C}^w$ be linear subspaces, V_i be full column rank matrices such that $\text{im}(V_i) = \mathcal{V}_i$. Denote with S the set $S = \{\lambda_i \mid i = 1, \dots, N\}$ where the λ_i are distinct complex numbers not lying on the imaginary axis. Let \mathfrak{B} be the autonomous behavior

$$\mathfrak{B} := \text{span} \left(\bigcup_{i=1, \dots, N} \mathcal{V}_i \exp_{\lambda_i} \cup \mathcal{V}_i^{\perp \Sigma} \exp_{-\overline{\lambda_i}} \right)$$

Assume that the matrix

$$\left[\frac{V_i^* \Sigma V_j}{\lambda_i + \lambda_j} \right]_{i, j=1, \dots, N}$$

is nonsingular. Then \mathfrak{B} admits a Σ -unitary kernel representation.

IV. ITERATIVE COMPUTATION OF STORAGE FUNCTIONS

In this section we illustrate an iterative scheme to compute storage functions from knowledge of the spectral zeroes and associated directions. This algorithm can be used to derive algorithms for the computation of balanced state maps, but it is also of independent interest. The procedure is germane to that for J -spectral factorization presented in [18], which

in turn is related to the work of Georgiou and collaborators (see [1], [5], [6], [7], [8]) in the context of rational spectral factorization and finding solutions to the Riccati equation. In section IV-A, we first illustrate the procedure for the computation of polynomial spectral factors with zeroes in a pre-specified Λ -set S (in the following “ S -spectral factorization”). In section IV-B we show that this procedure also directly provides a sum-of-squares expression for the storage function corresponding to the S -spectral factor.

A. S -spectral factorization by iteration

Recall from [19] that given an image representation M , there exists a permutation matrix Π such that $\Pi M = \text{col}(D, N)$ with D nonsingular and ND^{-1} proper. The partition of the external variables associated with the permutation Π is then called an *input-output partition* for \mathfrak{B} (see [19]). If M is observable, then it follows directly from Proposition 8.4 of [21] that $\deg(\det(D)) = n(\mathfrak{B})$, the McMillan degree of $\mathfrak{B} = \text{im } M\left(\frac{d}{dt}\right)$.

Theorem 14: Let $\Sigma = \text{diag}(I_{\mathfrak{m}(\mathfrak{B})}, -I_{\mathfrak{p}(\mathfrak{B})})$, and $\mathfrak{B} \in \mathfrak{L}_{\text{cont}}^w$ be strictly Σ -dissipative. Let $M \in \mathbb{R}^{w \times \mathfrak{m}(\mathfrak{B})}[\xi]$ induce an observable image representation of \mathfrak{B} . Assume that $M(-\xi)^\top \Sigma M(\xi)$ is semisimple, and that $M = \text{col}(D, N)$ is such that ND^{-1} is strictly proper. Let S be a Λ -set of $M(-\xi)^\top \Sigma M(\xi)$ with cardinality \mathfrak{k} , and denote with λ_i , $i = 1, \dots, \mathfrak{k}$, the distinct elements of S . Assume that the Pick matrix $\left(\frac{M(\overline{\lambda_i})^\top \Sigma M(\lambda_j)}{\lambda_i + \lambda_j} \right)$ is nonsingular.

Define $K_0(\xi) := M(\xi)$, and consider the following recursion for $i = 1, \dots, \mathfrak{k}$:

- 1) $V_i :=$ full column-rank matrix such that $\text{im}(V_i) = \text{im}(K_{i-1}(\lambda_i))$;
- 2) $R_i(\xi) := (\xi + \overline{\lambda_i})I_w - V_i \left(\frac{V_i^* \Sigma V_i}{\lambda_i + \lambda_i} \right)^{-1} V_i^* \Sigma$;
- 3) $K_i(\xi) := \frac{R_i(\xi)K_{i-1}(\xi)}{\xi - \lambda_i}$;

Then:

- 1) $\frac{V_i^* \Sigma V_i}{\lambda_i + \lambda_i}$ is nonsingular for $i = 1, \dots, \mathfrak{k}$;
- 2) $K_i(\xi)$ is a polynomial matrix for $i = 0, \dots, \mathfrak{k}$;
- 3) $K_i(-\xi)^* \Sigma K_i(\xi) = K_{i-1}(-\xi)^* \Sigma K_{i-1}(\xi)$ for $i = 1, \dots, \mathfrak{k}$;
- 4) $K_{\mathfrak{k}}(\xi) = \text{col}(H(\xi), 0)$, with $H \in \mathbb{C}^{m \times m}[\xi]$ such that

$$M(-\xi)^\top \Sigma M(\xi) = H(-\xi)^\top H(\xi)$$

and the set of zeroes of $\det(H)$ is the complementary Λ -set \bar{S} .

Remark 15: The assumption on the strict properness of ND^{-1} is a technical one, needed to prove statement 4 of Theorem 14 (and later Theorem 17 of this paper).

B. Storage functions by iteration

The main result of this section shows that applying iterations similar to those of Theorem 14, the storage function corresponding to the \bar{S} -spectral factor H can be computed *directly*, without computing H explicitly. Moreover, as will be shown in section V, we can also compute a representation of the storage function as sum-of-squares.

Theorem 16: Assume that all assumptions of Theorem 14 hold. Define

$$\sum_{i=1}^k \Theta_i(\zeta, \eta) := \sum_{i=1}^k \frac{K_{i-1}(\zeta)^* \Sigma K_{i-1}(\eta) - K_i(\zeta)^* \Sigma K_i(\eta)}{\zeta + \eta}. \quad (4)$$

Then:

- 1) For all $i = 1, \dots, k$, Θ_i defined in (4) is a two-variable polynomial matrix, and it equals

$$\frac{K_{i-1}(\zeta)^*}{\zeta - \lambda_i} \left(\Sigma V_i T_{\{V_i, \lambda_i\}, \Sigma}^{-1} V_i^* \Sigma - (\bar{\lambda}_i + \lambda_i) \Sigma \right) \frac{K_{i-1}(\eta)}{\eta - \lambda_i}. \quad (5)$$

- 2) $\Theta_i(\zeta, \eta)$ can be factorized as

$$\left(\frac{K_{i-1}(\zeta)^*}{\zeta - \lambda_i} \Sigma W_i S_i^* \right) \Sigma_i \left(S_i W_i^* \Sigma \frac{K_{i-1}(\eta)}{\eta - \lambda_i} \right), \quad (6)$$

where $W_i \in \mathbb{C}^{w \times (w-m)}$ is a full column-rank matrix spanning $\text{im}(V_i)^{\perp \Sigma}$; S_i is a nonsingular matrix, and $\Sigma_i \in \mathbb{R}^{m \times m}$ is a signature matrix, such that $-(\lambda_i + \bar{\lambda}_i)(W_i^* \Sigma W_i)^{-1} = S_i^* \Sigma_i S_i$.

- 3) The two-variable polynomial matrix

$$\Psi_{\bar{S}}(\zeta, \eta) := \sum_{i=1}^k \frac{K_{i-1}(\zeta)^*}{\zeta - \lambda_i} \left(\Sigma V_i T_{\{V_i, \lambda_i\}, \Sigma}^{-1} V_i^* \Sigma - (\bar{\lambda}_i + \lambda_i) \Sigma \right) \frac{K_{i-1}(\eta)}{\eta - \lambda_i} \quad (7)$$

induces a \bar{S} -storage function for \mathfrak{B} .

The result of Theorem 16, coupled with the factorization (6), shows that the storage function is also *diagonalized* iteratively, since Σ is a signature matrix. We elaborate on this point in the next section.

V. MINIMAL DIAGONALIZING STATE MAPS BY ITERATION

The purpose of this subsection is to show how to compute a minimal state map $X_S(\xi)$ such that $X_S(\zeta)^* \Sigma' X_S(\eta) = \Psi_{\bar{S}}(\zeta, \eta)$, with Σ' a signature matrix. In order to do this, note that since $\Psi_{\bar{S}}(\zeta, \eta) = \sum_{i=1}^k \Theta_i(\zeta, \eta)$, using (6) we obtain

$$\Psi_{\bar{S}}(\zeta, \eta) =: \text{col}(P_i(\zeta))_{i=1, \dots, k}^{\top} \Sigma' \text{col}(P_i(\eta))_{i=1, \dots, k}$$

with $P_i(\xi) := S_i W_i^* \Sigma \frac{K_{i-1}(\xi)}{\xi - \lambda_i}$ and $\Sigma' := \text{diag}(\Sigma_i)_{i=1, \dots, k}$. The following result holds.

Theorem 17: Let $P_i(\xi) := S_i W_i^* \Sigma \frac{K_{i-1}(\xi)}{\xi - \lambda_i}$. Then $X_S(\xi) := \text{col}(P_i(\xi))_{i=1, \dots, k}$ is a state map for \mathfrak{B} .

Remark 18: From Theorem 17 it follows that if the spectral zeroes of $M(-\xi)^{\top} \Sigma M(\xi)$ are all real, then X_S is also a polynomial matrix with real coefficients.

Note that the state map of Theorem 17 is not *minimal* unless $m = p = 1$, i.e. in the single-input, single-output case, since only in that case is the number of rows of $X_S \left(\frac{d}{dt} \right) w$ equal to $\sum_{i=1}^k \gamma_i = n(\mathfrak{B})$. Given the special interest in

balanced state maps arising in model reduction algorithms, it is of interest to compute a minimal diagonalizing state map. We now show an off-line approach for deriving it.

Theorem 19: Assume that the conditions of Theorem 16 are satisfied, and let $S = \{\lambda_i\}_{i=1, \dots, k}$ be a Λ -set. Let U_i be full column rank matrices such that $\text{im } U_i = \ker M(-\bar{\lambda}_i) \Sigma M(-\lambda_i)$, $i = 1, \dots, N$. Then the Pick matrix

$$T_S = \left[\frac{U_i^* M(-\bar{\lambda}_i)^* \Sigma M(-\lambda_j) U_j}{-\bar{\lambda}_i - \lambda_j} \right]_{i,j}$$

is positive definite. Factorize T_S as $T_S = Z_1^* Z_1$. Denote with X_S the state map of Theorem 17, and with Z_2 the V -matrix associated to it:

$$Z_2 := [X_S(-\bar{\lambda}_1) U_1 \quad X_S(-\bar{\lambda}_2) U_2 \quad \dots \quad X_S(-\bar{\lambda}_k) U_k].$$

Then Z_2 has full column rank. Denote with Z_2^{\dagger} a left inverse of Z_2 . Define $F := Z_1 Z_2^{\dagger}$ and $X_S^{\min}(\xi) := F X_S(\xi)$. Then

$$M(\zeta)^{\top} \Sigma M(\eta) = (\zeta + \eta) X_S^{\min}(\zeta)^{\top} X_S^{\min}(\eta) + H(\zeta)^{\top} H(\eta)$$

where the set of roots $\det(H)$ equals the Λ -set \bar{S} . Equivalently, X_S^{\min} is a minimal diagonalizing state map for \mathfrak{B} with respect to the storage function associated with the Λ -set \bar{S} .

Remark 20: Based on the results of Theorem 17 and Theorem 19 an iterative algorithm to compute a minimal diagonalizing state map can be devised. We will not enter into these details here.

VI. APPLICATION: BALANCED STATE MAPS AND STATE REPRESENTATIONS

The concept of LQG balanced realization has been introduced in [13], and extended to the bounded-real and positive-real cases in [4], [10], [16], [17]. A behavioral point of view on balancing arbitrary quadratic measures on the external signals of a system has been put forward in [25]; only systems in which a state variable was given *a priori* were considered. In this section, we consider a controllable system described in image form, and we show how using the material presented in this paper one can compute *directly from these high-order equations* a minimal state variable inducing a balanced realization in the classical sense. In our framework it is minimal state maps acting on the trajectories of the system, rather than realizations, which are defined as being “balanced”. Moreover, the role of the supply rate associated with the Riccati equation is taken by a general supply rate induced by a matrix $\Sigma = \Sigma^{\top}$ with the property that the number of positive eigenvalues of Σ equals the input cardinality of the system. The following definition formalizes this point of view.

Definition 21: Let $\Sigma = \Sigma^{\top} \in \mathbb{R}^{w \times w}$ be nonsingular. Let $\mathfrak{B} \in \mathfrak{L}_{\text{cont}}^w$ be Σ -dissipative on \mathbb{R}^- , and assume $m(\mathfrak{B}) = \sigma_+(\Sigma)$. A minimal state map $X \left(\frac{d}{dt} \right)$ induced by $X \in \mathbb{C}^{n(\mathfrak{B}) \times \bullet}[\xi]$ is *balanced* if the maximal and minimal storage functions for \mathfrak{B} can be written as

$$\begin{aligned} \Psi_+(\zeta, \eta) &= X(\zeta)^{\top} \Delta X(\eta) \\ \Psi_-(\zeta, \eta) &= X(\zeta)^{\top} \Delta^{-1} X(\eta) \end{aligned}$$

for some diagonal matrix $\Delta \in \mathbb{R}^{n(\mathfrak{B}) \times n(\mathfrak{B})}$.

Later in this section we show exactly in what sense balanced state maps correspond to balanced realizations.

Remark 22: Using standard linear algebra techniques for the simultaneous diagonalization of Hermitian matrices and the results of section V an algorithm can be devised to yield a balanced state map in the sense of Definition 21. We will not enter into these details here.

We now discuss how to obtain a realization from a balanced state map, and moreover we also show that this realization is “balanced” in the classical sense. Let $\mathfrak{B} = \text{im } M \left(\frac{d}{dt} \right)$, with $M = \text{col}(D, N)$ such that ND^{-1} is proper. It follows from the material in [21] that if X is a state map for $\text{im} \left(M \left(\frac{d}{dt} \right) \right)$, then there exist matrices $A, B, C, G \in \mathbb{C}^{\bullet \times \bullet}$ such that

$$\begin{aligned} \xi X(\xi) &= AX(\xi) + BD(\xi) \\ N(\xi) &= CX(\xi) + GD(\xi) \end{aligned} \quad (8)$$

The computation of the matrices A, B, C, G can be efficiently performed by means of Gröbner basis manipulations, see [3].

Now assume that X is a balanced state map; then the realization associated with the matrices (A, B, C, G) satisfying (8) is balanced in the classical sense, i.e. the minimal- and maximal solutions to the Riccati equation are diagonal and one the inverse of the other.

Theorem 23: Let $\mathfrak{B} \in \mathfrak{L}_{\text{cont}}^w$, and assume that \mathfrak{B} is strictly Σ -dissipative. Let X be a minimal balanced state map for \mathfrak{B} . Let A, B, C, G be such that (8) holds. Then there exists a diagonal matrix Δ such that the minimal and maximal solutions of the ARE satisfy

$$K_- = K_+^{-1} = \Delta$$

VII. CONCLUSIONS

We have shown how one can compute a minimal state map X such that storage function associated with the spectral factor with zeroes in S is of the form $X(\zeta)^\top \Delta X(\eta)$, with Δ a constant diagonal matrix, directly from an image representation of a strictly dissipative system and a Λ -set S . Applying standard linear algebra procedures for the simultaneous diagonalization of Hermitian matrices, a balanced state map is easily obtained. From this balanced state map an input-state-output realization LQG-balanced in the classical sense can be obtained in a straightforward way.

ACKNOWLEDGMENTS

The authors would like to thank the anonymous referees for their careful reading of a previous version of this paper, and for their precious suggestions, remarks, and critiques.

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