Exact identification of lossless systems

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Open and Interconnected Systems
Modeling and Control
Brugge/Bruges, 16th-17th September 2009

Joint work with Harry L. Trentelman

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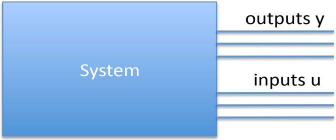
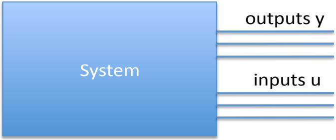
Outline

Problem statement



***u* and *y* exactly measured: no noise**

Problem statement



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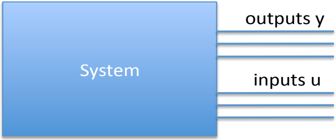

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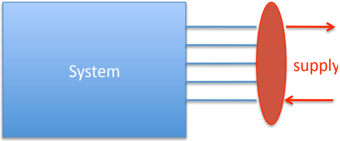

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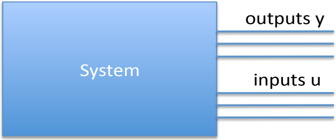

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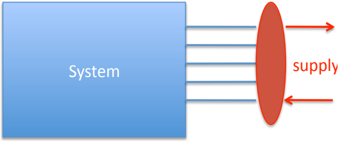

Supply rate: e.g. power

Lossless system: $\int_{\mathbb{R}}$ supply in $-$ $\int_{\mathbb{R}}$ supply out $= 0$
beginning at rest, ending at rest

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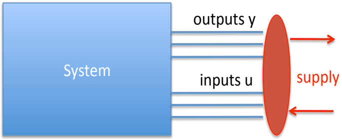

Given (infinite) exact measurements of u and y of system lossless w.r.t. (known) supply rate Q_Φ, find state representation

$$\begin{aligned} \sigma x &= Ax + Bu \\ y &= Cx + Du \end{aligned}$$

of a system compatible with the measurements.

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- Behaviors
- Losslessness

Main result

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System $\rightsquigarrow$ behavior $\mathfrak{B}$, set of trajectories compatible with the dynamical laws of the system

In this talk, $\mathfrak{B}$ is:

- **linear subspace of $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}} := \{w : \mathbb{Z} \to \mathbb{R}^{\mathtt{w}}\}$;**
- **shift-invariant: $\sigma\mathfrak{B} \subseteq \mathfrak{B}$, with σ the forward shift;**
- **complete:**
  $w \in \mathfrak{B} \Leftrightarrow w|_{\mathbb{Z}\cap[t_0,t_1]} \in \mathfrak{B}|_{\mathbb{Z}\cap[t_0,t_1]} \ \forall -\infty < t_0 \leq t_1 < \infty$
- **'controllable'**
  ($\Longrightarrow \mathfrak{B} \ni$ compact suppport trajectories)

Behaviors

System $\rightsquigarrow$ behavior $\mathfrak{B}$, set of trajectories compatible with the dynamical laws of the system

Let $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$, $C \in \mathbb{R}^{\mathtt{p}\times\mathtt{n}}$, $D \in \mathbb{R}^{\mathtt{p}\times\mathtt{m}}$, then

$$\mathfrak{B} = \{(u, y) \in \left(\mathbb{R}^{\mathtt{m}+\mathtt{p}}\right)^{\mathbb{Z}} \mid \exists\, x \in (\mathbb{R}^{\mathtt{n}})^{\mathbb{Z}} \text{ s.t. } \sigma x = Ax + Bu,\ y = Cx + Du\}$$

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$\mathfrak{B}$ is external behavior of the full behavior (w/ state)

$$\begin{aligned} \mathfrak{B}_f \;=\; & \{(u, y, x) \in \left(\mathbb{R}^{\mathtt{m}+\mathtt{p}+\mathtt{n}}\right)^{\mathbb{Z}} \mid \\ & \sigma x = Ax + Bu,\, y = Cx + Du\} \end{aligned}$$

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Let $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{p \times n}$, $D \in \mathbb{R}^{p \times m}$, then

$$\begin{aligned} \mathfrak{B} \;=\; & \{(u, y) \in \left(\mathbb{R}^{m+p}\right)^{\mathbb{Z}} \mid \exists\, x \in (\mathbb{R}^{n})^{\mathbb{Z}} \\ & \text{s.t. } \sigma x = Ax + Bu,\, y = Cx + Du\} \end{aligned}$$

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Minimal state: $\dim(x) = n(\mathfrak{B})$, McMillan degree of $\mathfrak{B}$.

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$f = \mathbb{Z}_+ \to \mathbb{R}^{\mathtt{f}}$ persistently exciting of order L if

$$\text{rank} \begin{bmatrix} f(0) & f(1) & \cdots & f(T-L) & \cdots \\ f(1) & f(2) & \cdots & f(T-L+1) & \cdots \\ \vdots & \vdots & & \vdots & \vdots \\ f(L-1) & f(L) & \cdots & f(T+1) & \cdots \end{bmatrix} = L\mathtt{f}$$

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Very nice consequences. Work of Ivan Markovsky.

Outline

Losslessness

Let $\Phi = \Phi^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. $\mathfrak{B}$ is lossless w.r.t. supply rate

$$Q_\Phi(w) := w^\top \Phi w$$

if exists $\{\Psi_{i,j} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}\}_{i,j=0,\ldots,L}$ such that

$$Q_\Psi(w) := \sum_{i,j=0}^{L} \sigma^i w^\top \Psi_{i,j} \sigma^j w$$

satisfies

$$\nabla Q_\Psi(w)(k) := Q_\Psi(w)(k+1) - Q_\Psi(w)(k) = Q_\Phi(w)(k)$$

for all $w \in \mathfrak{B}$, for all $k \in \mathbb{Z}$.

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Q_Ψ is called a storage function for $\mathfrak{B}$ w.r.t. Q_Φ.

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The Stein matrix

***Proposition*: Let $\mathfrak{B}$ controllable and Φ-lossless. Let $w = \text{col}(u, y) \in \mathfrak{B}_{|[0,\infty)} \cap \ell_2^{\mathtt{w}}$, and assume u persistently exciting of order $\mathtt{n}(\mathfrak{B})$.**

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Define the Stein matrix

$$[S(w)]_{i,j=0,\ldots,N} := \sum_{k=0}^{\infty} \left(\sigma^i w\right)(k)^\top \Phi \left(\sigma^j w\right)(k)$$

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Let $\mathfrak{B}_f$ be a minimal state system with state x and external behavior $\mathfrak{B}$. Then $\exists\ K = K^\top \in \mathbb{R}^{\mathtt{n}(\mathfrak{B}) \times \mathtt{n}(\mathfrak{B})}$ s.t.

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Proof: **"Every storage function is a quadratic function of the state." Thanks Jan, thanks Harry!**

The Stein matrix revisited

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$$\Phi = \begin{bmatrix} \Phi_{uu} & \Phi_{uy} \\ \Phi_{uy}^\top & \Phi_{yy} \end{bmatrix} \in \mathbb{R}^{(\mathtt{u}+\mathtt{y})\times(\mathtt{u}+\mathtt{y})} \; .$$

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E.g.

$$\Phi = \begin{bmatrix} I_{\mathtt{u}} & 0 \\ 0 & -I_{\mathtt{y}} \end{bmatrix}$$

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Φ_{yy} sign-definite $\Longrightarrow$ minimal state sequence computable from rank-revealing factorization of $S(w)$

$$S(w) = U^\top \Delta U \rightsquigarrow U = \begin{bmatrix} x(0) & x(1) & \cdots \end{bmatrix}$$

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State equations by solving for A, B, C, D

$$\begin{bmatrix} x(1) & x(0) & \dots \\ y(0) & y(1) & \dots \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} x(0) & x(1) & \dots \\ u(0) & u(1) & \dots \end{bmatrix}$$

since $\sigma x = Ax + Bu$, $y = Cx + Du$.

Refinements and work in progress

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- **If dissipation rate Q_Δ known, dissipative case tackled in analogous way:**

$$S(w) - \left[\sum_{k=0}^{\infty} L_\Delta(\sigma^i w, \sigma^j w)(k)\right]_{i,j}$$

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- **¿What if dissipation rate not known?**
- **Lossless model reduction from data**

$$S(w) \rightsquigarrow \text{lossless } (A, B, C, D)$$

$$\text{Approximation } S(w) \rightsquigarrow \text{lossless } (A_{\rm red}, B_{\rm red}, C_{\rm red}, D_{\rm red})$$

Buon compleanno!

Feliz cumpleaños!

С днем рождения

생일축하합니다

יום הולדת שמח

Herzlichen Glückwunsch zum Geburstag

Gelukkige verjaarsdag

お誕生日おめでとう

สุขสันต์วันเกิด

كل سنة و انت طيب

Feliz aniversário

Χρόνια Πολλά!

Честит рожден ден

Happy birthday!

生日快樂

Wszystkiego najlepszego!

सालगिरह की हार्दिक शुभकामनायें

Joyeux anniversaire!