A Behavioral Approach to Passivity and Bounded Realness Preserving Balanced Truncation with Error Bounds

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Abstract— **In this paper we revisit the problems of passivity and bounded realness preserving model reduction by balanced truncation. In the behavioral framework, these problems can be considered as special cases of balanced truncation of strictly half line dissipative system behaviors, where the number of input variables of the behavior is equal to the positive signature of the supply rate. Instead of input-state-output representations, the balancing algorithm uses normalized driving variable representations of the behavior. We show that the diagonal elements of the minimal solution of the balanced algebraic Riccati equation are the singular values of the map that assigns to each past trajectory the optimal storage extracting future continuation. Since the future behavior is only an indenite inner product space, the term singular values should be interpreted here in a generalized sense. We establish some new error bounds for this model reduction method.**

Keywords: Model reduction, strictly half line dissipative behaviors, balanced truncation, bounded real balancing, positive real balancing, normalized driving-variable representations.

I. Introduction

The method of balanced truncation is the most prominent method of model reduction for linear dynamical systems. The method is straightforward and simple, has a nice and convincing physical interpretation, preserves stability, and, last but not least, comes with simple and effective $\mathcal{H}_\infty$ error bounds.

Starting with the seminal paper [3] by Desai and Pal on stochastic model reduction, there has been also an interest in balanced truncation methods that preserve typical structural properties of the original system. The paper by Desai and Pal introduces a balanced truncation method to approximate a given positive real transfer matrix by a reduced order positive real transfer matrix. In [5] this problem was revisited, and it was shown that also stability and minimality are preserved under this balanced truncation method. In [14], and later in [2], $\mathcal{H}_\infty$ error bounds for balanced reduction of strictly positive real transfer matrices were found. The related problem of balanced truncation of *bounded real* transfer matrices, including $\mathcal{H}_\infty$ error bounds, was studied extensively in [8]. For a nice overview, we refer to [1].

Research on balanced reduction methods using ideas from stochastic model reduction can also be found in the the work of Weiland [15]. In [15], the problem of model reduction by balancing is put into a more general, behavioral, framework. It is shown that the system invariants that appear as diagonal elements in the solutions of the algebraic Riccati equations after balancing are, in fact, the nonzero singular values of a given map from past to future behavior. This map assigns to any past trajectory its optimal continuation, in an appropriate sense.

In the present paper we revisit the problems of reduction by balancing of positive real and bounded real systems *from a behavioral point of view*. Positive real (or passive) and bounded real (or contractive) systems are special cases of systems with a given input-output partition that are *dissipative on the negative half line, and whose number of input components is equal to the positive signature of the supply rate*. In this paper we study the general problem of model reduction by balancing for such strictly half line dissipative systems. In particular, the property of strict half line dissipativity should be preserved, and the input-output partition of the original system should be respected.

We find a number of frequency domain inequalities involving the error transfer matrix, i.e. the difference between the original and reduced order transfer matrix from driving variable to manifest variable. We study in what sense these inequalities can be interpreted as error bounds. In particular, for the special case of strictly bounded real systems we find a new error bound for bounded real balanced truncation.

Most of the proofs of the results in this paper are omitted. For these, we refer to a future, full version of the paper.

Notation and background material. $\mathfrak{C}^\infty(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ denotes the space of all infinitely often differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$. $\mathfrak{D}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ denotes its subspace of functions with compact support. For this space we use the shorthand notation $\mathfrak{D}$. We denote by $\mathfrak{L}_2^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ the space of all measurable functions w from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$ such that $\int_a^b \|w\|^2 dt < \infty$ for all $a, b \in \mathbb{R}$. $\mathfrak{L}_2(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ denotes the ambient space of all measurable functions w from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$ such that $\int_{-\infty}^{\infty} \|w\|^2 dt < \infty$. The $\mathfrak{L}_2$-norm of w is $\|w\|_2 := (\int_{-\infty}^{\infty} \|w\|^2 dt)^{1/2}$. We denote by $\mathbb{R}_-$ the set of negative real numbers, and by $\mathbb{R}_+$ the complementary set of nonnegative real numbers. $\mathfrak{L}_2(\mathbb{R}_-,\mathbb{R}^{\mathtt{w}})$ ($\mathfrak{L}_2(\mathbb{R}_+,\mathbb{R}^{\mathtt{w}})$) denotes the space of all measurable functions w from $\mathbb{R}_-$ ($\mathbb{R}_+$) to $\mathbb{R}^{\mathtt{w}}$ such that $\int_{-\infty}^{0} \|w\|^2 dt < \infty$ ($\int_0^{\infty} \|w\|^2 dt < \infty$). When the dimension of the codomain is clear from the context, we denote these spaces by $\mathfrak{L}_2(\mathbb{R})$, $\mathfrak{L}_2(\mathbb{R}_-)$ and $\mathfrak{L}_2(\mathbb{R}_+)$. For a given function w on $\mathbb{R}$, we denote by $w|_{\mathbb{R}_-}$ and $w|_{\mathbb{R}_+}$ the restrictions of w to $\mathbb{R}_-$ and $\mathbb{R}_+$, respectively. $\mathbb{C}^-$ ($\mathbb{C}^+$) is the subset of $\mathbb{C}$ of all λ such that $\mathrm{Re}(\lambda) < 0$ ($\mathrm{Re}(\lambda) > 0$). For a given nonsingular, symmetric matrix $\Sigma \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ we denote by $\sigma_+(\Sigma)$ (the *positive signature of* Σ) the number of positive eigenvalues of Σ.

II. Reduction of dissipative linear differential behaviors

This paper deals with model reduction of dissipative *linear differential systems*. A subspace $\mathfrak{B} \subset \mathfrak{L}_2^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{\mathtt{w}})$ is called a linear differential system (or a linear differential behavior) if it is equal to the space of (weak) solutions $w : \mathbb{R} \to \mathbb{R}^{\mathtt{w}}$ of a system of linear, constant coefficient, higher order differential equations, i.e., there exists a polynomial matrix $R \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi]$ such that $\mathfrak{B} = \{w \in \mathfrak{L}_2^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid R(\frac{\mathrm{d}}{\mathrm{d}t})w = 0\}$ (see [9]). The variable w is called the *manifest variable* of the system $\mathfrak{B}$. The set of all linear differential systems with $\mathtt{w}$ variables is denoted by $\mathfrak{L}^{\mathtt{w}}$.

It is well known, see e.g. [9], [15], that any $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ admits state space representations. In this paper we will use mainly one type of state space representation, namely *driving variable representations* (DV-representations). Consider the equations

$$\dot{x} = Ax + Bv, \quad w = Cx + Dv, \tag{1}$$

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with $A \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$, $B \in \mathbb{R}^{\mathtt{n}\times\mathtt{m}}$, $C \in \mathbb{R}^{\mathtt{w}\times\mathtt{n}}$ and $D \in \mathbb{R}^{\mathtt{w}\times\mathtt{m}}$. These equations represent the *full behavior*

$$\begin{aligned}\mathfrak{B}_{DV}(A,B,C,D) \quad := \quad & \{(w,x,v) \in \mathfrak{L}_2^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \times \mathfrak{L}_2^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{\mathtt{n}}) \\ & \times \mathfrak{L}_2^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{\mathtt{v}}) \mid (1) \text{ holds}\}.\end{aligned}$$

The variable x is a state variable, taking its values in $\mathbb{R}^{\mathtt{n}}$, the state space, and v is called the *driving variable*, taking its values in $\mathbb{R}^{\mathtt{m}}$. The *external behavior* corresponding to this full behavior is defined as

$$\begin{aligned}\mathfrak{B}_{DV}(A,B,C,D)_{\mathrm{ext}} = \{ & w \in \mathfrak{L}_2^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{\mathtt{w}}) \mid \exists\ x \text{ and } v \text{ such} \\ & \text{that } (w,x,v) \in \mathfrak{B}_{DV}(A,B,C,D)\}.\end{aligned}$$

If $\mathfrak{B} = \mathfrak{B}_{DV}(A,B,C,D)_{\mathrm{ext}}$ then we call $\mathfrak{B}_{DV}(A,B,C,D)$ a driving variable representation of $\mathfrak{B}$.

We now review the notions of input and output. For a given system $\mathfrak{B}$, a partition of the manifest variable w into $w = \mathrm{col}(w_1, w_2)$ is called an *input-output partition* if w_1 is maximally free, meaning that it is free (i.e. for any $w_1 \in \mathfrak{L}_2^{\mathrm{loc}}(\mathbb{R},\mathbb{R}^{\mathtt{w}_1})$ there exists w_2 such that $\mathrm{col}(w_1, w_2) \in \mathfrak{B}$), and one can not enlarge w_1 to a new variable w_1' by adding components of w_2 such that the new variable w_1' is free. If $w = \mathrm{col}(w_1, w_2)$ is an input-output partition then w_1 is called input and w_2 is called output of $\mathfrak{B}$. For details we refer to [9]. The number of input components in any input-output partition of $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is an integer invariant of $\mathfrak{B}$, and is called the *input cardinality* of $\mathfrak{B}$, denoted by $\mathtt{m}(\mathfrak{B})$.

Now assume $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$, and partition $w = \mathrm{col}(w_1, w_2)$. Let $\mathfrak{B}_{DV}(A,B,C,D)$ be a driving variable representation of $\mathfrak{B}$. Of course, the partition of w into $\mathrm{col}(w_1, w_2)$ induces a partition of the matrices C and D into

$$C = \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}, \quad D = \begin{pmatrix} D_1 \\ D_2 \end{pmatrix}.$$

A natural question is now: under what conditions on the driving variable representation is the partition $w = \mathrm{col}(w_1, w_2)$ an input-output partition? The answer is given in the next lemma:

Lemma 2.1: Under the above assumptions, $w = \mathrm{col}(w_1, w_2)$ is an input-output partition for $\mathfrak{B}$ if and only if the rational matrix $G_1(\xi) := C_1(\xi I - A)^{-1}B + D_1$ is square and nonsingular. In that case, the transfer matrix from w_1 to w_2 is equal to $G_2(\xi)G_1^{-1}(\xi)$, with $G_2(\xi) := C_2(\xi I - A)^{-1}B + D_2$.

Another important integer invariant of a given behavior $\mathfrak{B}$ is the minimal dimension of the state space over all its state space representations. This integer is called the *McMillan degree* of $\mathfrak{B}$, denoted with $\mathtt{n}(\mathfrak{B})$.

A driving variable representation $\mathfrak{B}_{DV}(A,B,C,D)$ of $\mathfrak{B}$, with state space dimension $\mathtt{n}$ and driving variable dimension $\mathtt{m}$ is called *minimal* if $\mathtt{n}$ and $\mathtt{m}$ are minimal over all such driving variable representations. The minimal $\mathtt{n}$ is equal to the McMillan degree $\mathtt{n}(\mathfrak{B})$ and the minimal $\mathtt{m}$ is equal to the input cardinality $\mathtt{m}(\mathfrak{B})$. A given DV-representation $\mathfrak{B}_{DV}(A,B,C,D)$ of $\mathfrak{B}$ is a minimal DV-representation if and only if $(A,B,C,D$ is strongly observable (meaning that the pair $(C+DF, A+BF)$ is observable for every F) and D has full column rank (see [15]).

We restrict ourselves to controllable behaviors in this paper. A behavior $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}$ is called controllable if for all $w_1, w_2 \in \mathfrak{B}$ there exists $T \geq 0$ and $w \in \mathfrak{B}$ such that $w(t) = w_1(t)$ for $t < 0$, and $w(t) = w_2(t-T)$ for $t \geq 0$. Properties of controllable behaviors are discussed in [9]. $\mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ (a subset of $\mathfrak{L}^{\mathtt{w}}$) will denote the set of controllable behaviors.

If $\mathfrak{B}_{DV}(A,B,C,D)$ is a minimal DV-representation of $\mathfrak{B}$, then $\mathfrak{B}$ is controllable if and only if the pair (A,B) is controllable, see [15].

We will now review the basic material on dissipative behaviors. For an extensive treatment we refer to [12], [13], [17], [18]. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ and let $\Sigma = \Sigma^{\top} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular. The quadratic form $w^{\top}\Sigma w$ is called a *supply rate*. $\mathfrak{B}$ is said to be Σ*-dissipative* if $\int_{-\infty}^{+\infty} w^{\top}\Sigma w dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$. $\mathfrak{B}$ it is said to be Σ-dissipative on $\mathbb{R}_-$ if $\int_{-\infty}^{0} Q_{\Sigma}(w)\, dt \geq 0$ for all $w \in \mathfrak{B} \cap \mathfrak{D}$. We will also call such behaviors *half line dissipative*. It is easily seen that if $\mathfrak{B}$ is Σ-dissipative on $\mathbb{R}_-$, then it is Σ-dissipative. A controllable behavior $\mathfrak{B}$ is said to be *strictly* Σ*-dissipative* if there exists an $\epsilon > 0$ such that $\mathfrak{B}$ is $(\Sigma - \epsilon I)$-dissipative. We have the obvious definition for strict dissipativity on $\mathbb{R}_-$. If $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$, then it is strictly Σ-dissipative.

In this paper we deal with linear differential behaviors $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ that are strictly Σ-dissipative on $\mathbb{R}_-$. In addition, we assume that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$, i.e. the input cardinality of $\mathfrak{B}$ is equal to the positive signature of Σ. Two important special cases of such systems are

1. strictly bounded real input-output systems, where the manifest variable is partitioned as $w = \mathrm{col}(u, y)$, with u input and y output, and where $\Sigma = \mathrm{diag}(I_{\mathtt{m}}, -I_{\mathtt{p}})$, and
2. strictly passive input-output systems, where $w = \mathrm{col}(u, y)$, with u input and y output (having the same dimension $\mathtt{m}$, and where $\Sigma = \frac{1}{2}\begin{pmatrix} 0 & I_{\mathtt{m}} \\ I_{\mathtt{m}} & 0 \end{pmatrix}$.

As announced in the introduction, this paper deals with approximation of strictly passive input-output systems by a strictly passive input-output systems with a given lower McMillan degree, and approximation of strictly bounded real input-output systems by a strictly bounded real input-output systems with a given lower McMillan degree. By the above remarks, these problems can be put into one single framework, namely the following: given a strictly half line Σ-dissipative behavior $\mathfrak{B}$ with input cardinality $\mathtt{m}(\mathfrak{B})$ equal to the number $\sigma_+(\Sigma)$ of positive eigenvalues of Σ, and a given input-output partition of its manifest variable, approximate it by a strictly half line Σ-dissipative behavior of a given lower McMillan degree, with the same input-output partition. More precise, the problem can be formulated as follows:

Main Problem. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$, with system variable $w = \mathrm{col}(w_1, w_2)$, where w_1 is input and w_2 is output. Let $\Sigma = \Sigma^{\top} \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular and assume that $\mathtt{m}(\mathfrak{B})(= \dim(w_1)) = \sigma_+(\Sigma)$. Assume that $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}^-$ and let $\mathtt{k}$ be an integer such that $0 < \mathtt{k} < \mathtt{n}(\mathfrak{B})$. Find $\hat{\mathfrak{B}} \in \mathfrak{L}^{\mathtt{w}}_{\mathrm{cont}}$ such that

1) $\hat{\mathfrak{B}}$ is strictly Σ-dissipative on $\mathbb{R}^-$,
2) in $\hat{\mathfrak{B}}$ w_1 is input and w_2 is output,
3) $\mathtt{n}(\hat{\mathfrak{B}}) \leq \mathtt{k}$,
4) $\hat{\mathfrak{B}}$ is an approximation of $\mathfrak{B}$.

We will show that for the two special cases of strictly passive and strictly bounded real systems, i.e. $\Sigma = \frac{1}{2}\begin{pmatrix} 0 & I_{\mathtt{w}_1} \\ I_{\mathtt{w}_1} & 0 \end{pmatrix}$ and $\Sigma = \mathrm{diag}(I_{\mathtt{w}_1}, -I_{\mathtt{w}_2})$, respectively, reduction by balancing using normalized driving variable representations leads to a behavior $\hat{\mathfrak{B}}$ satisfying properties 1,2 and 3. Finally, the question whether our balanced truncation method leads to a reasonable approximation is studied afterwards, and amounts to finding reasonable error bounds.

III. Σ-CHARACTERISTIC VALUES OF SYSTEM BEHAVIORS

In this section we introduce the notion of Σ*-characteristic values* of behaviors that are strictly Σ-dissipative on $\mathbb{R}_-$ and that have the property $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$.

We will use the property that $\mathfrak{B}$ is strictly dissipative on $\mathbb{R}_-$ to endow the past behavior with an inner product, with the inner product given by the integral of the supply rate. In the same way, the supply rate will only yield an *indefinite* inner product on the future behavior. We will formulate a theorem that states that certain operators between past and future behavior allow singular value decompositions. This terminology should however be interpreted carefully, since the future behavior is not an inner product space. The "singular values" will form a set of invariants of the strictly Σ-dissipative behavior, and will be called the Σ*-characteristic values of* $\mathfrak{B}$.

Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\rm cont}$ and let a supply rate be given by the nonsingular symmetric matrix $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$. Assume $\mathfrak{B}$ is strictly Σ-dissipative. Introduce the following notation:

$$\mathfrak{B}_- := \{w|_{\mathbb{R}^-} \mid w \in \mathfrak{B}\}, \quad \mathfrak{B}_+ := \{w|_{\mathbb{R}^+} \mid w \in \mathfrak{B}\}.$$

Furthermore, for a given past trajectory $w_- \in \mathfrak{B}_-$ define the set of all future trajectories w_+ whose concatenation at time zero with past trajectory w_- is in $\mathfrak{B}$ by

$$\mathfrak{B}_+(w_-) := \{w_+ \in \mathfrak{B}_+ \mid \text{ there exists } w \in \mathfrak{B} \text{ such that } w|_{\mathbb{R}_-} = w_- \text{ and } w|_{\mathbb{R}_+} = w_+\},$$

and for a given future trajectory $w_+ \in \mathfrak{B}_+$ define the set of all past trajectories w_- whose concatenation at time zero with future trajectory w_+ is in $\mathfrak{B}$ by

$$\mathfrak{B}_-(w_+) := \{w_- \in \mathfrak{B}_- \mid \text{ there exists } w \in \mathfrak{B} \text{ such that } w|_{\mathbb{R}_-} = w_- \text{ and } w|_{\mathbb{R}_+} = w_+\}.$$

For a given past trajectory $w_- \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$ we define the associated *available storage* by

$$V_{av}(w_-) := \sup\{-\int_0^\infty w_+^\top \Sigma w_+ dt \mid w_+ \in \mathfrak{B}_+(w_-) \cap \mathfrak{L}_2(\mathbb{R}_+)\}, \tag{2}$$

and for a given future trajectory $w_+ \in \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ we define the associated *required supply* by

$$V_{req}(w_+) := \inf\{\int_{-\infty}^0 w_-^\top \Sigma w_- dt \mid w^- \in \mathfrak{B}_-(w_+) \cap \mathfrak{L}_2(\mathbb{R}_-)\}. \tag{3}$$

The available storage associated with past trajectory w_- is the maximal amount of supply that can be extracted from the system over all future trajectories $w_+ \in \mathfrak{B}_+(w_-) \cap \mathfrak{L}_2(\mathbb{R}_+)$. The required supply associated with future trajectory w_+ is the minimal amount of supply that has to be delivered to the system over all past trajectories $w_- \in \mathfrak{B}_-(w_+) \cap \mathfrak{L}_2(\mathbb{R}_-)$.

Due to Σ-dissipativity of $\mathfrak{B}$, the supremum and infimum above are finite for all w_- and w_+, respectively (see [12], [17], [18]). Also, by *strict* Σ-dissipativity, both the supremum and infimum are attained for all w_- and w_+. In particular, for given $w_- \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$ there is a *unique* $w_+^* \in \mathfrak{B}_+(w_-) \cap \mathfrak{L}_2(\mathbb{R}_+)$ such that

$$V_{av}(w_-) = -\int_0^\infty {w_+^*}^\top \Sigma w_+^* dt$$

and for given $w_+ \in \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ there is a unique $w_-^* \in \mathfrak{B}_-(w_+) \cap \mathfrak{L}_2(\mathbb{R}_-)$ such that

$$V_{req}(w_+) = \int_{-\infty}^0 {w_-^*}^\top \Sigma w_-^* dt.$$

By associating with any past trajectory $w_- \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$ the unique optimal future trajectory $w_+^* \in \mathfrak{B}_+(w_-) \cap \mathfrak{L}_2(\mathbb{R}_+)$ we obtain a map

$$\Gamma_- : \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-) \to \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+), \ \ \Gamma_-(w_-) = w_+^*,$$

and by associating with any future trajectory $w_+ \in \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ the unique optimal past trajectory $w_-^* \in \mathfrak{B}_-(w_+) \cap \mathfrak{L}_2(\mathbb{R}_-)$ we obtain a map

$$\Gamma_+ : \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+) \to \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-), \ \ \Gamma_+(w_+) = w_-^*.$$

In the remainder of this section, assume that $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$. This implies that there exists $\epsilon > 0$ such that

$$\int_{-\infty}^0 w^\top \Sigma w dt \geq \epsilon \int_{-\infty}^0 w^\top w dt$$

for all $\mathtt{w} \in \mathfrak{L}_2(\mathbb{R}_-)$. Consequently, the bilinear form

$$< w_1, w_2 >_{-,\Sigma} \ := \ \int_{-\infty}^0 w_1^\top \Sigma w_2 dt$$

defines an inner product on $\mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$. On $\mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ consider the bilinear form

$$< w_1, w_2 >_{+,\Sigma} \ := \ -\int_0^\infty w_1^\top \Sigma w_2 dt.$$

Since Σ is a nonsingular symmetric matrix, this defines an *indefinite* inner product on $\mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$. (Note that we have assumed strict Σ-dissipativity on $\mathbb{R}_-$, implying only positive definiteness of the bilinear form over the *past*, and not necessarily over the future!). Now, in the sequel it will be shown that the maps Γ_- and Γ_+ are linear. We will denote by $\Gamma_-^* : \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-) \to \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ the *adjoint* of Γ_-, i.e. the (unique) linear map $\Gamma_-^* : \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+) \to \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$ that satisfies

$$< w_1, \Gamma_-(w_2) >_{+,\Sigma} \ = \ < \Gamma_-^*(w_1), w_2 >_{-,\Sigma}$$

for all $w_1 \in \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ and $w_2 \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$. The existence and uniqueness of this adjoint can be easily proven, see e.g. [4], chapter 4. Likewise, $\Gamma_+^* : \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-) \to \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ will denote the adjoint of Γ_+, i.e. the unique linear map that satisfies

$$< w_1, \Gamma_+(w_2) >_{-,\Sigma} \ = \ < \Gamma_+^*(w_1), w_2 >_{+,\Sigma}$$

for all $w_1 \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$ and $w_2 \in \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$.

We now formulate a theorem stating that if $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$ and $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$, then the maps Γ_- and Γ_+ allow *singular value decompositions* that, in a certain sense, are compatible. It should however be understood that, strictly speaking, the terminology singular value decomposition is not appropriate in the present context, since our maps do not act between genuine inner product spaces: only the past behavior is an inner product space, on the future behavior we have an indefinite inner product. The notion singular value should therefore be interpreted in a generalized sense:

Theorem 3.1: Assume that $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$ and $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. The maps Γ_- and Γ_+ are linear. The map $\Gamma_-^*\Gamma_- : \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-) \to \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$ has a finite-dimensional image, and it is Hermitian and nonnegative. There exist positive real numbers $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_{\mathtt{n}} > 0$, where $\mathtt{n} = \mathtt{n}(\mathfrak{B})$, the McMillan degree of $\mathfrak{B}$, such that $\sigma_1^2 \geq \sigma_2^2 \geq \ldots \geq \sigma_{\mathtt{n}}^2 > 0$ are the nonzero eigenvalues of $\Gamma_-^*\Gamma_-$. There exists an orthonormal set $\{w_1^-, w_2^-, \ldots, w_{\mathtt{n}}^-\} \subset \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$, and an orthonormal set $\{w_1^+, w_2^+, \ldots, w_{\mathtt{n}}^+\} \subset \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ such that

$$\Gamma_- \ = \ \sum_{i=1}^{\mathtt{n}} \sigma_i < . \,, w_i^- >_{-,\Sigma} w_i^+, \tag{4}$$

$$\Gamma_+ \ = \ \sum_{i=1}^{\mathtt{n}} \frac{1}{\sigma_i} < . \,, w_i^+ >_{+,\Sigma} w_i^-. \tag{5}$$

An important result in the above theorem is the nonnegativity of the map $\Gamma_-^*\Gamma_-$. Of course, in genuine inner product spaces this nonnegativity is trivially satisfied. In the present context it is a statement that needs to be proven explicitely, and which follows from the fact that $\mathrm{im}(\Gamma_-)$ is a positive subspace for the indefinite future inner product. This follows from the nonnegativity of the available storage (which in turn follows from dissipativity on the negative half line and the assumption that $\mathtt{m}(\mathfrak{B}) = \sigma(\Sigma)$).

The positive real numbers $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$ will be called the Σ-*characteristic values of* $\mathfrak{B}$. As noted before, in a generalized sense these numbers are the singular values of the map Γ_-. In that sense, the pairs of functions (w_i^-, w_i^+) can be considered as Schmidt pairs of Γ_-. We note that for the special cases of strict passivity and strict bounded realness, the Σ-characteristic values coincide with the *positive real characteristic values* and *bounded real characteristic values*, respectively. This will follow immediately from the characterization of the Σ-characteristic values in terms of solutions of the algebraic Riccati equation in the next section.

IV. State space characterizations and representations

In this section we review the characterizations of (strict) Σ-dissipativity in terms of the algebraic Riccati equation associated with a minimal DV-representation of the given behavior $\mathfrak{B}$. We expliciteely compute representations of the linear maps (and their adjoints) that assign to each past (future) trajectory the unique state at time zero, and we characterize the extremal solutions of the Riccati equation in terms of these maps. We also compute the maps Γ_- and Γ_+ in terms of compositions of these maps. It will turn out that the Σ-characteristic values are the eigenvalues of the product of the inverse of the maximal solution and the minimal solution of the algebraic Riccati equation, and that $\mathfrak{B}$ admits a Σ-balanced minimal DV-representation. Much of the material in this section is an extension of results in [15] to the case that the future behavior is an indefinite inner product space.

Proposition 4.1: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ with minimal DV-representation $\mathfrak{B}_{DV}(A, B, C, D)$ and let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular. Assume $D^\top \Sigma D > 0$. Then

1) $\mathfrak{B}$ is Σ-dissipative if and only if there exists a real symmetric solution $P = P^\top \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ of the algebraic Riccati equation (ARE)
$$A^\top P + PA - C^\top \Sigma C + (PB - C^\top \Sigma D)(D^\top \Sigma D)^{-1}(B^\top P - D^\top \Sigma C) = 0. \tag{6}$$
If this is the case, then there exist real symmetric solutions P_- and P_+ such that every real symmetric solution P satisfies $P_- \leq P \leq P_+$.
2) $\mathfrak{B}$ is Σ-dissipative on $\mathbb{R}_-$ if and only if there exists a positive semidefinite solution $P = P^\top \in \mathbb{R}^{\mathtt{n}\times\mathtt{n}}$ of the ARE (6).
3) If $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$ then $\mathfrak{B}$ is Σ-dissipative on $\mathbb{R}_-$ if and only if all solutions of ARE (6) are positive definite, equivalently $P_- > 0$.

Proposition 4.2: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ with minimal DV-representation $\mathfrak{B}_{DV}(A, B, C, D)$ and let $\Sigma = \Sigma^\top \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}$ be nonsingular. If $\mathfrak{B}$ is strictly Σ-dissipative then $D^\top \Sigma D > 0$, and the minimal and maximal real symmetric solution P_- and P_+ of the ARE (6) satisfy $P_+ > P_-$. Furthermore, P_- and P_+ are stabilizing and anti-stabilizing, respectively, i.e., $\sigma(A_-) \subset \mathbb{C}^-$ and $\sigma(A_+) \subset \mathbb{C}^+$, where we denote
$$A_+ := A + B(D^\top \Sigma D)^{-1}(B^\top P_+ - D^\top \Sigma C), \tag{7}$$
$$A_- := A + B(D^\top \Sigma D)^{-1}(B^\top P_- - D^\top \Sigma C). \tag{8}$$
Finally, the following statements are *equivalent*:

1) $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}^-$,
2) $D^\top \Sigma D > 0$ and the maximal solution P_+ of the ARE (6) is positive definite and anti-stabilizing, i.e., $\sigma(A_+) \subset \mathbb{C}^+$.

We will now study the maps Γ_- and Γ_+ in terms of DV-representations of the given behavior $\mathfrak{B}$. Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ with minimal DV-representation $\mathfrak{B}_{DV}(A, B, C, D)$. Let $\mathtt{n} = \mathtt{n}(\mathfrak{B})$ be the McMillan degree of $\mathfrak{B}$. By minimality, for every $w \in \mathfrak{B}$ there is a unique state trajectory x. For any given $x_0 \in \mathbb{R}^{\mathtt{n}}$, let $\mathfrak{B}(x_0)$ denote the set of all $w \in \mathfrak{B}$ such that the corresponding state trajectory x satisfies $x(0) = x_0$. Thus, for every $w \in \mathfrak{B}$ there is a unique $x_0 \in \mathbb{R}^{\mathtt{n}}$ such that $w \in \mathfrak{B}(x_0)$. Moreover (see [15]), there exists linear surjective maps $R_- : \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-) \to \mathbb{R}^{\mathtt{n}}$ and $R_+ : \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+) \to \mathbb{R}^{\mathtt{n}}$ such that for all $x_0 \in \mathbb{R}^{\mathtt{n}}$ we have
$$w \in \mathfrak{B}(x_0) \quad \Leftrightarrow \quad \{R_-(w_-) = x_0 \text{ and } R_+(w_+) = x_0\},$$
where $w_- := w|_{\mathbb{R}_-}$ and $w_+ := w|_{\mathbb{R}_+}$. In the sequel we will explicitely compute representations of the maps R_- and R_+, and their adjoints R_-^* and R_+^* in terms of the systems matrices A, B, C and D. On $\mathbb{R}^{\mathtt{n}}$ we take the standard Euclidean inner product. Note that R_+^* denotes the generalized adjoint with respect to the indefinite inner product on $\mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$.

It is well known (see [13]) that the extremal solutions of the Riccati equation (6) are associated with the available storage and required supply as reviewed in the previous section:

Proposition 4.3: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ with minimal DV-representation $\mathfrak{B}_{DV}(A, B, C, D)$. Assume that $D^\top \Sigma D > 0$. Assume $\mathfrak{B}$ is Σ-dissipative and let P_- and P_+ be the minimal and maximal real symmetric solution of the ARE (6). Then for any $w_- \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$ we have $V_{av}(w_-) = x_0^\top P_- x_0$, where $x_0 := R_-(w_-)$. Also, for any $w_+ \in \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ we have $V_{req}(w_+) = x_0^\top P_+ x_0$, where $x_0 := R_+(w_+)$.
If $\mathfrak{B}$ is strictly Σ-dissipative then P_- and P_+ satisfy $\sigma(A_+) \subset \mathbb{C}^+$ and $\sigma(A_-) \subset \mathbb{C}^-$ (see Theorem 4.1). Introduce the following notation:
$$C_+ = C + D(D^\top \Sigma D)^{-1}(B^\top P_+ - D^\top \Sigma C), \tag{9}$$
$$C_- = C + D(D^\top \Sigma D)^{-1}(B^\top P_- - D^\top \Sigma C). \tag{10}$$
The following is also well-known (see also [16]):

Proposition 4.4: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ with minimal DV-representation $\mathfrak{B}_{DV}(A, B, C, D)$. Assume $\mathfrak{B}$ is strictly Σ-dissipative. Then for $w_- \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$ the unique optimal future trajectory w_+^* is given by $w_+^*(t) = C_- e^{A_- t} x_0$, where $x_0 := R_-(w_-)$. Also, for $w_+ \in \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ the unique optimal past trajectory w_-^* is given by $w_-^*(t) = C_+ e^{A_+ t} x_0$, where $x_0 := R_+(w_+)$.

In the remainder of this section we will assume that $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$ and that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. In that case, in addition we have $0 < P_- < P_+$ (see Prop. 4.2). The next theorem is the main result of this section. It computes representations of R_- and R_+ and their adjoints R_-^* and R_+^*, and shows that P_- and P_+ can be expressed in terms of compositions of these maps.

Theorem 4.5: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ with minimal DV-representation $\mathfrak{B}_{DV}(A, B, C, D)$. Assume that $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$ and that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Then for any $w_- \in \mathfrak{B}_- \cap \mathfrak{L}_2(\mathbb{R}_-)$ and $w_+ \in \mathfrak{B}_+ \cap \mathfrak{L}_2(\mathbb{R}_+)$ we have
$$R_-(w_-) = \int_{-\infty}^0 e^{-(A_+ - P_+^{-1} C_+^\top \Sigma C_+)s} P_+^{-1} C_+^\top \Sigma w_-(s) ds, \tag{11}$$
$$R_+(w_+) = -\int_0^\infty e^{-(A_- - P_-^{-1} C_-^\top \Sigma C_-)s} P_-^{-1} C_-^\top \Sigma w_+(s) ds. \tag{12}$$
Furthermore, for any $x_0 \in \mathbb{R}^{\mathtt{n}}$ we have
$$R_-^*(x_0) = C_+ P_+^{-1} e^{-(A_+ - P_+^{-1} C_+^\top \Sigma C_+)^\top t} x_0 \tag{13}$$
and
$$R_+^*(x_0) = C_- P_-^{-1} e^{-(A_- - P_-^{-1} C_-^\top \Sigma C_-)^\top t} x_0. \tag{14}$$
Finally, $P_+ = (R_- R_-^*)^{-1}$ and $P_- = (R_+ R_+^*)^{-1}$.

Remark 4.6: In the case that both the past and the future behavior are inner product spaces a result analogous to $P_+ = (R_- R_-^*)^{-1}$ and $P_- = (R_+ R_+^*)^{-1}$ was proven in [15] using a general least squares argument, without computing explicit representations of R_-, R_-^*, R_+ and R_+^*.

Corollary 4.7: Let $\mathfrak{B} \in \mathfrak{L}^{\mathtt{w}}_{\text{cont}}$ with minimal DV-representation $\mathfrak{B}_{DV}(A, B, C, D)$. Assume that $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$ and that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Then we have $\Gamma_- = R_+^*(R_+ R_+^*)^{-1} R_-$ and $\Gamma_+ = R_-^*(R_- R_-^*)^{-1} R_+$.

The eigenvalues $\sigma_1^2 \geq \sigma_2^2 \geq \ldots \geq \sigma_{\mathtt{n}}^2 > 0$ of $\Gamma_-^* \Gamma_-$ are in fact the eigenvalues of $P_+^{-1} P_-$, with $0 < P_- < P_+$ the extremal solutions of the ARE (6), for any minimal DV-representation of $\mathfrak{B}$.

Theorem 4.8: Assume that $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$ and $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Let $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n > 0$ be the Σ-characteristic values of $\mathfrak{B}$. Let $\mathfrak{B}_{DV}(A,B,C,D)$ be a minimal DV-representation of $\mathfrak{B}$ with $0 < P_- < P_+$ the extremal solutions of the ARE (6). Then $\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_\mathtt{n}^2\} = \sigma(P_+^{-1}P_-)$. Furthermore $0 < \sigma_i < 1$ for all i.

After a suitable coordinate transformation $\hat{x} = Tx$ in the state space of the DV-representation $\mathfrak{B}_{DV}(A,B,C,D)$, the maps R_- and R_+ transform to TR_- and TR_+. Thus $R_-R_-^*$ transforms to $TR_-R_-^*T^\top$ and $R_+R_+^*$ to $TR_+R_+^*T^\top$. This implies that P_+^{-1} transforms to $T^{-\top}P_+T^{-1}$ and P_- to $T^{-\top}P_-T^{-1}$. It is well known, see e.g. [20], that there exists a coordinate transformation T such that $T^{-\top}P_-T^{-1}$ and $T^{-\top}P_+T^{-1}$ are equal and diagonal. Since the set of eigenvalues of $P_+^{-1}P_-$ is $\{\sigma_1^2, \sigma_2^2, \ldots, \sigma_\mathtt{n}^2\}$ this diagonal matrix must be equal to the diagonal matrix $\Pi := \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_\mathtt{n})$. Thus we obtain:

Corollary 4.9: Assume that $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$ and $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Let $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_\mathtt{n} > 0$ be the Σ-characteristic values of $\mathfrak{B}$. There exists a minimal DV-representation $\mathfrak{B}_{DV}(A,B,C,D)$ of $\mathfrak{B}$ such that the corresponding extremal solutions P_- and P_+ of the ARE (6) satisfy $P_- = P_+^{-1} = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_\mathtt{n})$.

A minimal DV-representation of $\mathfrak{B}$ such that $P_- = P_+^{-1} = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_\mathtt{n})$ is called a Σ-*balanced DV-representation of* $\mathfrak{B}$.

V. Σ-NORMALIZED Σ-BALANCED DV-REPRESENTATIONS

In this section we show that if $\mathfrak{B}$ is strictly Σ-dissipative on $\mathbb{R}_-$ and $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$, then it has a Σ-balanced minimal DV-representation that is, in addition, Σ-normalized.

The idea of Σ-normalization originates from the concept *normalized coprime factorization*, see e.g. [20]. In the following lemma we state that strictly Σ-dissipative behaviors allow Σ-normalized DV-representations.

Lemma 5.1: Assume that $\mathfrak{B} \in \mathfrak{L}^\mathtt{w}_{\mathrm{cont}}$ is strictly Σ-dissipative. Then there exists a minimal driving variable representation $\mathfrak{B}_{DV}(A,B,C,D)$ of $\mathfrak{B}$ such that A is asymptotically stable and $G^\top(-\xi)\Sigma G(\xi) = I$, where $G(\xi) := C(\xi I - A)^{-1}B + D$.

A driving variable representation satisfying the two conditions of Lemma 5.1 is called a Σ-*normalized driving variable representation* of $\mathfrak{B}$. Σ-normalized representations have some nice properties. This is elaborated in the following lemma.

Lemma 5.2: Let $G(\xi)$ be proper rational and let $G(\xi) = C(\xi I - A)^{-1}B + D$ be a realization. Assume $\sigma(A) \subset \mathbb{C}^-$. Let M be the unique symmetric solution of $A^\top M + MA - C^\top \Sigma C = 0$. If $B^\top M - D^\top \Sigma C = 0$ and $D^\top \Sigma D = I$, then $G^\top(-\xi)\Sigma G(\xi) = I$. If, in addition, (A,B) is controllable and (C,A) is observable, then $B^\top M - D^\top \Sigma C = 0$ and $D^\top \Sigma D = I$ *if and only if* $G^\top(-\xi)\Sigma G(\xi) = I$

The above lemma suggests that for Σ-normalized representations the controllability and generalized observability (or Σ-observability) gramians are related to the maximal and minimal solutions of the ARE. Indeed, we have:

Lemma 5.3: Assume that $\mathfrak{B} \in \mathfrak{L}^\mathtt{w}_{\mathrm{cont}}$ is strictly Σ-dissipative. Let $\mathfrak{B}_{DV}(A,B,C,D)$ be a minimal, Σ-normalized driving variable representation of $\mathfrak{B}$. Let P_- and P_+ be the minimal and maximal real symmetric solutions of the ARE (6). Let M be the unique solution of $A^\top M + MA - C^\top \Sigma C = 0$ (called the Σ-*observability gramian*), and let W be the unique solution of $AW + WA^\top + BB^\top = 0$ (the controllability gramian). Then we have

1) $M = P_-$,
2) $W = (P_+ - P_-)^{-1}$.

If we start with a Σ-normalized driving variable representation, then transforming into Σ-balanced coordinates results in a Σ-normalized driving variable representation as well. This follows immediately from the fact that the transfer matrix associated with the DV-representation does not change under coordinate transformation. Thus we obtain:

Corollary 5.4: Let $\mathfrak{B} \in \mathfrak{L}^\mathtt{w}_{\mathrm{cont}}$ be strictly Σ-dissipative on $\mathbb{R}^-$ and assume that $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$. Then there exists a Σ-normalized, Σ-balanced minimal DV-representation $\mathfrak{B}_{DV}(A,B,C,D)$ of $\mathfrak{B}$.

For a Σ-normalized, Σ-balanced minimal DV-representations of $\mathfrak{B}$, the Σ-observability gramian and controllability gramian are diagonal matrices, and their diagonal elements can be expressed in terms of the Σ-characteristic values $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_\mathtt{n}$ of $\mathfrak{B}$:

Lemma 5.5: Let $\mathfrak{B}_{DV}(A,B,C,D)$ be a Σ-normalized, Σ-balanced minimal DV-representation of $\mathfrak{B}$. Then the Σ-observability gramian M and controllability gramian W are given by

1) $M = \Pi = \mathrm{diag}(\sigma_1, \ldots, \sigma_\mathtt{n})$,
2) $W = (\Pi^{-1} - \Pi)^{-1} = \mathrm{diag}(\frac{\sigma_1}{1-\sigma_1^2}, \ldots, \frac{\sigma_\mathtt{n}}{1-\sigma_\mathtt{n}^2})$.

VI. REDUCTION BY BALANCED TRUNCATION

Let $\mathfrak{B} \in \mathfrak{L}^\mathtt{w}_{\mathrm{cont}}$ be strictly Σ-dissipative on $\mathbb{R}^-$, and partition $w = \mathrm{col}(w_1, w_2)$ with w_1 input and w_2 output. Assume that $\mathtt{m}(\mathfrak{B}) = \dim(w_1) = \sigma_+(\Sigma)$. Let $\mathfrak{B}_{DV}(A,B,C,D)$ be a minimal Σ-normalized and Σ-balanced DV-representation of $\mathfrak{B}$. Define $G(\xi) = C(\xi I - A)^{-1}B + D$. We have $P_+^{-1} = P_- = \mathrm{diag}(\sigma_1, \sigma_2, \ldots, \sigma_\mathtt{n}) = \Pi$.

Pick $\mathtt{k} < \mathtt{n}$ such that $\sigma_\mathtt{k} > \sigma_{\mathtt{k}+1}$. We will to compute a reduced order approximation $\hat{\mathfrak{B}} \in \mathfrak{L}^\mathtt{w}_{\mathrm{cont}}$ of $\mathfrak{B}$ that inherits the input-output partition of $\mathfrak{B}$, i.e., also in $\hat{\mathfrak{B}}$ w_1 is input and w_2 is output, such that its McMillan degree $\mathtt{n}(\hat{\mathfrak{B}}) \leq \mathtt{k}$, and $\hat{\mathfrak{B}}$ is strictly Σ-dissipative on $\mathbb{R}_-$. This computation consists of the following steps:

step 1. Partition A, B and C:

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \tag{15}$$

with A_{11} $\mathtt{k} \times \mathtt{k}$, B_1 $\mathtt{k} \times \mathtt{m}$ and C_1 $\mathtt{p} \times \mathtt{k}$, and define the truncated system $\mathfrak{B}_{\mathrm{trunc}}$ by $\mathfrak{B}_{\mathrm{trunc}} := B_{DV}(A_{11}, B_1, C_1, D)$.

step 2. Define the reduced order approximation as the controllable part of $\mathfrak{B}_{\mathrm{trunc}}$:

$$\hat{\mathfrak{B}} := (\mathfrak{B}_{\mathrm{trunc}})_{\mathrm{cont}}. \tag{16}$$

A DV-representation of $\hat{\mathfrak{B}}$ can obtained by performing a Kalman controllability decomposition (see [7], proposition 22):

$$T^{-1}A_{11}T = \begin{bmatrix} \hat{A} & * \\ 0 & * \end{bmatrix}, T^{-1}B_1 = \begin{bmatrix} \hat{B} \\ 0 \end{bmatrix},$$

$$C_1 T = \begin{bmatrix} \hat{C} & * \end{bmatrix}, D = \hat{D}. \tag{17}$$

We then have $\hat{\mathfrak{B}} = \mathfrak{B}_{DV}(\hat{A}, \hat{B}, \hat{C}, \hat{D})_{\mathrm{ext}}$.

Our main result now states that the for the special cases of strictly passive and strictly bounded real systems, the reduced order behavior $\hat{\mathfrak{B}}$ obtained in this way satisfies the required properties.

Theorem 6.1: Assume

$$\Sigma = \frac{1}{2}\begin{pmatrix} 0 & I_{\mathtt{w}_1} \\ I_{\mathtt{w}_1} & 0 \end{pmatrix} \text{ or } \Sigma = \mathrm{diag}(I_{\mathtt{w}_1}, -I_{\mathtt{w}_2}).$$

Let $\hat{\mathfrak{B}}$ be defined by (16). Then

1. $\hat{\mathfrak{B}}$ is controllable,
2. $\sigma(\hat{A}) \subset \mathbb{C}^-$,
3. $\mathfrak{B}_{DV}(\hat{A}, \hat{B}, \hat{C}, \hat{D})$ is a Σ-normalized DV-representation of $\hat{\mathfrak{B}}$,
4. $\hat{\mathfrak{B}}$ is strictly Σ-dissipative on $\mathbb{R}^-$, $\mathtt{n}(\hat{\mathfrak{B}}) \leq \mathtt{k}$, and in $\hat{\mathfrak{B}}$ w_1 is input and w_2 is output.

VII. ERROR BOUNDS

A. One-step frequency domain inequality

Starting with $\mathfrak{B}$, strictly Σ-dissipative on $\mathbb{R}_-$, with $\mathtt{m}(\mathfrak{B}) = \sigma_+(\Sigma)$, let $\mathfrak{B}_{DV}(A,B,C,D)$ be a Σ-balanced, Σ-normalized minimal DV-representation. Let $G(\xi) = D + C(\xi I - A)^{-1}B$. Assume that the *distinct* Σ-characteristic values of $\mathfrak{B}$ are $\sigma_1 > \sigma_2 > \ldots > \sigma_N$, where σ_i appears $\mathtt{n}_i$ times. Then $\Pi = \mathrm{diag}(\sigma_1 I_1, \sigma_2 I_2, \ldots, \sigma_N I_N)$, with I_i the $\mathtt{n}_i \times \mathtt{n}_i$ identity matrix.

Suppose now that we do a one-step reduction by eliminating the state components corresponding to the last singular value σ_N: partition $\Pi = \mathrm{blockdiag}(\Pi_1, \Pi_2)$, with $\Pi_2 = \sigma_N I_N$. Let $\mathfrak{B}_{\mathrm{trunc}} = B_{DV}(A_{11}, B_1, C_1, D)_{\mathrm{ext}}$ be the truncated behavior as defined in step 2. of our algorithm. Let $G_{\mathrm{trunc}}(\xi) = D + C_1(\xi I - A_{11})^{-1}B_1$. Let $\hat{\mathfrak{B}} = \mathfrak{B}_{DV}(\hat{A}, \hat{B}, \hat{C}, \hat{D})_{\mathrm{ext}}$ be the reduced order behavior, and $\hat{G}(\xi) = \hat{D} + \hat{C}(\xi I - \hat{A})^{-1}\hat{B}$. Obviously, $G_{\mathrm{trunc}} = \hat{G}$. We will now derive some properties of the *error transfer matrix* $E := G - \hat{G}$.

Theorem 7.1: The rational matrix E is stable. For all $\omega \in \mathbb{R}$ we have

$$0 \leq -E^\top(-i\omega)\Sigma E(i\omega) \leq \frac{4\sigma_N^2}{1-\sigma_N^2} I_N. \qquad (18)$$

Of course, the question arises in what sense the inequality (18) can be interpreted as an error bound. Note that the supply rate Σ is still an arbitrary nonsingular symmetric matrix with the property that $\sigma_+(\Sigma) = \mathtt{m}(\mathfrak{B})$. In the following, denote $G^\sim(\xi) := G^\top(-\xi)$. Since $G^\sim \Sigma G = I$ and $\hat{G}^\sim \Sigma \hat{G} = I$, we have

$$I = (G-E)^\sim \Sigma (G-E) = I - E^\sim \Sigma G - G^\sim \Sigma E + E^\sim \Sigma E,$$

which implies that $E^\sim \Sigma E = E^\sim \Sigma G + G^\sim \Sigma E$. Thus (18) is equivalent with: for all $\omega \in \mathbb{R}$

$$0 \leq -[G^\top(-i\omega)\Sigma E(i\omega) + E^\top(-i\omega)\Sigma G(i\omega)] \leq \frac{4\sigma_N^2}{1-\sigma_N^2} I_N.$$

B. Error bounds for the strictly bounded real case

We will now study the case that our system $\mathfrak{B}$ comes with an input-output partition $w = \mathrm{col}(w_1, w_2)$, and that it is strictly bounded real, i.e. is is Σ-dissipative on $\mathbb{R}_-$ with Σ given by $\mathrm{diag}(I_{\mathtt{w}_1}, -I_{\mathtt{w}_2})$. We already know that our balanced truncation method retains the given input-output partition. In this section we will denote the input variable w_1 simply by u and the output variable w_2 by y.

Let the original system $\mathfrak{B}$ be represented by the minimal normalized DV-representation

$$\begin{aligned} \dot{x} &= Ax + Bv, \\ \begin{pmatrix} u \\ y \end{pmatrix} &= \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} x + \begin{pmatrix} D_1 \\ D_2 \end{pmatrix} v. \end{aligned}$$

Apply then the algorithm outlined in section VI to obtain a reduced order behavior $\hat{\mathfrak{B}}$ given in normalized DV representation by

$$\begin{aligned} \dot{z} &= \hat{A}z + \hat{B}v, \\ \begin{pmatrix} \hat{u} \\ \hat{y} \end{pmatrix} &= \begin{pmatrix} \hat{C}_1 \\ \hat{C}_2 \end{pmatrix} z + \begin{pmatrix} \hat{D}_1 \\ \hat{D}_2 \end{pmatrix} v. \end{aligned}$$

Next, we investigate the frequency domain inequality (18) for this special case. Let

$$G(\xi) = \begin{pmatrix} G_1(\xi) \\ G_2(\xi) \end{pmatrix} = \begin{pmatrix} C_1(\xi I - A)^{-1}B + D_1 \\ C_2(\xi I - A)^{-1}B + D_2 \end{pmatrix}$$

compatibly with the partition $w = (u, y)$ (i.e., $u = G_1 v$ and $y = G_2 v$). Likewise, define $\hat{G}_1$ and $\hat{G}_2$. Denote $E_1 := G_1 - \hat{G}_1$, $E_2 := G_2 - \hat{G}_2$. Assume now that we truncate one step, as explained in subsection VII-A. According to Theorem 7.1, for all $\omega \in \mathbb{R}$ we have

$$\begin{aligned} E_1(-i\omega)^\top E_1(-i\omega) &\leq E_2(-i\omega)^\top E_2(-i\omega) \\ &\leq \frac{4\sigma_N^2}{1-\sigma_N^2} I + E_1(-i\omega)^\top E_1(-i\omega) \end{aligned}$$

This implies that if we 'drive' both $\mathfrak{B}$ and $\hat{\mathfrak{B}}$ with the same driving variable trajectory $v \in \mathfrak{L}_2(\mathbb{R})$ with $\|v\|_2 = 1$, and denote the corresponding (unique) $\mathfrak{L}_2(\mathbb{R})$ input trajectories and output trajectories for $\mathfrak{B}$ and $\hat{\mathfrak{B}}$ by $u, \hat{u}$ and $y, \hat{y}$, respectively, then we have

$$\|u - \hat{u}\|_2^2 \leq \|y - \hat{y}\|_2^2 \leq \frac{4\sigma_N^2}{1-\sigma_N^2} + \|u - \hat{u}\|_2^2,$$

so

$$\|u - \hat{u}\|_2 \leq \|y - \hat{y}\|_2 \leq \frac{2\sigma_N}{\sqrt{1-\sigma_N^2}} + \|u - \hat{u}\|_2.$$

Thus, the norm of the difference $\|u - \hat{u}\|_2$ is a lower bound for the norm of the difference of the corresponding outputs $\|y - \hat{y}\|_2$, while for σ_N small the difference $\|y - \hat{y}\|_2$ is close to this lower bound. In plain words: for small σ_N, $\|y - \hat{y}\|_2$ is approximately $\|u - \hat{u}\|_2$. Also, of course, if the $\mathfrak{L}_2(\mathbb{R})$ inputs u and $\hat{u}$ of $\mathfrak{B}$ and $\hat{\mathfrak{B}}$ are equal, then the $\mathfrak{L}_2(\mathbb{R})$ outputs y and $\hat{y}$ are close in the sense that $\|y - \hat{y}\|_2 \leq \frac{2\sigma_N}{\sqrt{1-\sigma_N^2}}$. This statement applies to one-step truncation.

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