8. COUPLED PROBLEMS: (a) moving boundary problems

Efficient Numerical Modelling of Field Diffusion in High-Temperature Superconducting Wires

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***Abstract* — Multidimensional field diffusion problems with front-type behaviour, moving boundaries and non-linear material properties are analysed by a finite volume front fixing method. Advantages and implementation challenges of the method are discussed with special attention given to conservation properties of the algorithm and achieving accurate solutions close to the moving boundaries. The technique is validated using analytical solutions of diffusion problems with cylindrical symmetry.**

I. INTRODUCTION

Design and development of modern devices based on High Temperature Superconductors (HTS) requires numerical modelling since electromagnetic and thermal parts of the problem are coupled together via high sensitivity of HTS material properties to temperature [1, 2]. Both field variation and heat flow can be formulated in terms of diffusion. This allows utilising a standard modelling approach on fixed grids and thus simplifying the equipment design. But such approach often fails to deliver appropriate balance between accuracy and efficiency, especially when modelling pulse events or shallow field penetration. Special methods, such as adaptive meshes, front fixing and level sets methods [3], offer advantages in such applications but they have to be assessed and probably adapted for each particular problem. The paper focuses on the analysis of the front fixing technique [3] since it requires only a small modification of the computational algorithm in comparison with models based on fixed grids [4, 5]. The major challenges are an implementation of conservation laws and achieving accurate solutions close to the moving curved boundaries. The paper uses analytical solutions of common front type problems to evaluate the performance of the numerical method. Two types of the problem are considered, namely a current pulse and an imposed external magnetic flux.

II. PROBLEM FORMULATION

A. Governing equation and material properties

It is possible to describe the problem in terms of either magnetic or electric field diffusion [6, 7]. The electric field formulation is preferred for HTS materials with non-linear properties as it provides much more stable solutions [7]. The governing equation takes the diffusion-like form

$$\text{curl}\,\left(\text{curl}\,\mathbf{E}\right) = -\mu_0 \frac{\partial \mathbf{J}}{\partial t} \qquad (1)$$

expressed in terms of the electric field **E** and current density **J**. HTS materials exhibit strong flux creep *E-J* behaviour often described by Rhyner's power law [6], $E_c^{-1}E = (J_c^{-1}J)^{\alpha}$, where the critical current density $J_c \approx 10^9$ A m^{-2} corresponds to a critical electric field $E_c \approx 10^{-4}$ V m^{-1}. For practical HTS materials the power exponent α could be as high as 20. Substitution of the material properties into (1) results in a formulation of the problem in terms of the electric field only.

B. Boundary and initial conditions

A HTS wire with a round cross section of radius R is considered. For the first test, a pulse $I_z(r,t) = I_0\delta(r)\delta(t-t_0)$ of current is applied along the z axis at an instant t=t_0. The second test case assumes an external magnetic flux B_z to be switched suddenly in the centre of the wire and maintained at a constant value afterwards.

C. Analytical solutions

The existence of an axi-symmetric analytical solution provides an opportunity to evaluate the performance of the algorithm on curved boundaries using the Cartesian coordinate system. The dimensionless solution for (1) in the case of cylindrical symmetry under the conditions of the current pulse can be derived as shown in [8]

$$\frac{E(\rho,\tau)}{E_c} = \frac{1}{(\alpha\tau)}\left[\left(\frac{i_0}{4}\right)^{\frac{\alpha-1}{\alpha}} - \frac{\rho^2(\alpha-1)}{4\alpha\left(\alpha\tau\right)^{1/\alpha}}\right]^{\alpha/(\alpha-1)}, \qquad (2)$$

$$\rho = \frac{r}{R},\ \tau = \frac{\left(t-t_0\right)E_c}{\mu_0 J_c R^2},\ i_0 = \frac{I_0}{J_c \pi R^2}. \qquad (3)$$

The electric field and the current gradually spread from the centre of the wire towards the edges and there is a sharp interface between the region with a non-zero field and the outside part of the wire. A similar solution exists for the second test case of the applied external magnetic flux [9].

III. THE FRONT FIXING METHOD

The spatial transformation uses new positional variables [3] adjusted to the front position and, generally, introduces a co-ordinate system in which all of the spatial boundaries are fixed to 0 or 1. As a result, the new computational domains remain the same with an additional advection term in diffusion equation plus an implicit non-linear equation for the boundary motion. This allows treating the nodes close to the interface as being independent of the motion, which gives higher accuracy for the same number of nodes used [4, 5]. In practical applications it is often sufficient to apply the transformation in only one direction, resulting in additional simplification [3]. Equation (4) is an example of the transformed (1) in notations (3) for the case of cylindrical symmetry:

$$\frac{\partial\left(us^2e^{1/\alpha}\right)}{\partial\tau} = \frac{ds}{d\tau}\frac{\partial(su^2e^{1/\alpha})}{\partial u} + \frac{\partial}{\partial u}\left(\frac{\partial e}{\partial u}\right),\; e=\left(E_c^{-1}E\right). \quad (4)$$

with a boundary at $s(t)$ and a new coordinate $u=\rho/s(t)$. A divergent form of (4) ensures that there are no artificial energy sources [5].

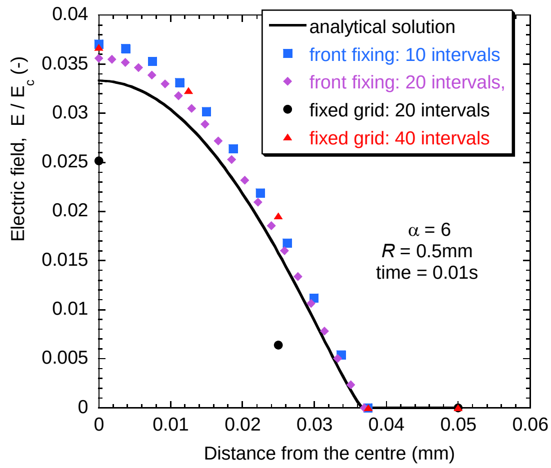

Fig. 1. Analytical and numerical predictions for a wire with I_0 = 2A, R=0.5mm and α=6: mesh size effects.
Dimensionless electric field $E_z(r)$ at t = 0.01s (time step 0.1ms).

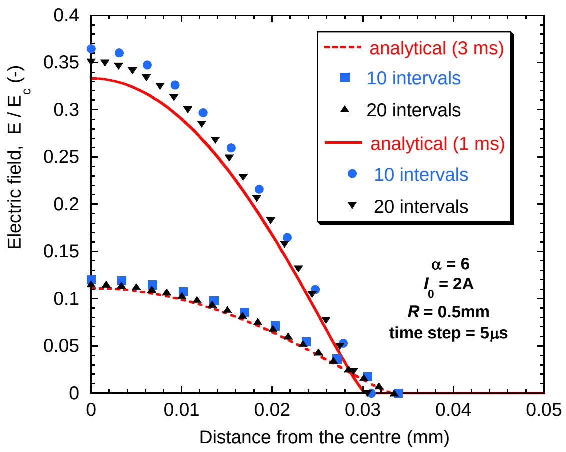

Fig. 2. Front-fixing method predictions of electric field inside the HTS wire after different times. Modelling conditions are similar to those in Fig. 1.

IV. COMPARISON OF COMPUTATIONAL TECHNIQUES

Predictions from fixed grid calculations and the front-fixing method are summarized in Figs. 1-3. Fixed nodes cannot adequately describe the field profile in the case of a shallow penetration (Fig. 1). At least 4 nodes per penetration depth are required, which could be computationally expensive for large devices. Placing nodes close to the boundary does not always solve the problem because the front propagates further into the material at later stages of the process. In contrast, the front fixing automatically adjusts the nodes towards the front boundary, Fig.2, and good accuracy is achieved even by only 10 nodes in total. The particular advantage of using a front-fixing method for modelling of superconductivity phenomena is that the high accuracy can be obtained with a small number of grid points. The interface motion can also be accurately predicted on a coarse moving mesh, Fig. 3.

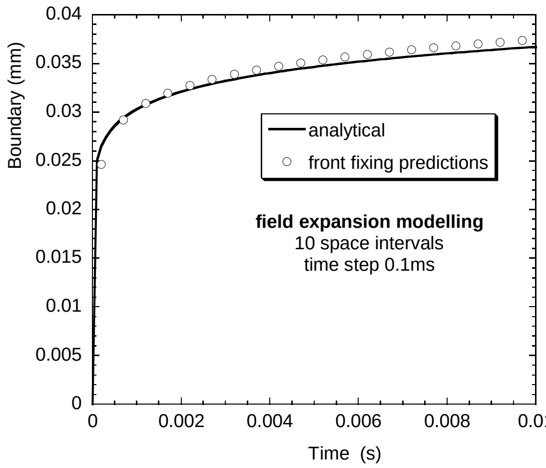

Fig. 3. Interface position as a function of time. Modelling conditions are similar to those on Fig. 1.

V. CONCLUSIONS

The application of a front-fixing method for modelling of shallow field penetration into HTS materials is demonstrated for 2D geometry cases. Efficient techniques for incorporating conservation laws are suggested and potential problems with complex boundary conditions are considered. It is shown that high accuracy can be achieved on a coarse mesh since the interface is fixed in new coordinates. The analysis of errors and further implementation details will be given in the extended version of the paper. The finite volume method has been utilised in the paper as an example; the finite element scheme can also be used for successful discretisation of space and time in the transformed equations.

VI. REFERENCES

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