On the equivalence of Finite Element and Finite Integration formulations

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Abstract—The paper offers a comparative study of numerical methods of analysis of electromagnetic fields. The focus is on the Finite Element Method (FEM) and Finite Integration Technique (FIT), but with the cell and equivalent network approaches also considered. It is shown how the approximate integrals describing coefficients of the FEM need to be derived for a mesh with parallelepiped elements to achieve consistency with FIT equations. The equivalence of FEM and FIT formulations for a triangular mesh in 2D is highlighted. The TEAM Workshops Problem No. 7 is used as an example for numerical comparisons. Edge values of magnetic vector potential $A$ and nodal values of electric scalar potential $V$ are used throughout.

1. Introduction

The finite element method (FEM) has established itself as the prime numerical technique for electromagnetic field computations, but some researchers prefer and promote the use of the finite integration technique (FIT) [1], the cell method (CM) [2] or the equivalent electric and magnetic networks (ENM) [3]. The similarities between CM, FIT and FEM were observed in [4, 5] and explored thoroughly in [6]. The main differences between the different approaches are related to the way in which space is discretised and equation coefficients set up, in particular the so-called ‘mass matrices’ of the FEM theory. [4]. The CM, FIT and ENM formulations rely on a discretisation which is equivalent to hexahedral FEM elements of 8 nodes and 12 edges (or curved rectangular parallelepipeds under cylindrical symmetry). The FEM mass matrices are non-diagonal, unlike the ones arising in CM, FIT and ENM. The purpose of this paper is to extend and enhance the previous comparative analysis of the methods. It is demonstrated that the CM, FIT and ENM equations may be considered a special case of the FEM formulation. The derived approximate integration formulae yield the equations equivalent (identical).

II. Equations of FEM and FIT

Both nodal elements using scalar potentials $A$, $V$ and edge elements in terms of vector potentials $A$, $T$ are considered. The FEM equations for scalar potentials correspond to the nodal equations of the edge network with branches coinciding with element edges (Fig. 1a) [6]. The permeances, conductances and capacitances forming the mass matrix may be found from

$$\Lambda_{ij}^{(\ell)} = \int_{V_0} [w_{ij} \mu w_{ij} \mu - \omega \sigma] dV, C_{ij}^{(\ell)} = \int_{V_0} [w_{ij} \omega + \omega p] w_{ij} \omega dV, (1a,b)$$

where $w_{ij}$, $w_{ij\ell}$ are interpolation functions of an edge element for the edges $P_j P_i$ and $P_i P_{i\ell}$ respectively, $p=d/dt$, and $V_0$ is the volume of the element. The FEM equations for vector potentials, on the other hand, represent loop equations of the facet network, the branches of which cross the element facets. A portion of a network of a parallelepiped element is shown in Fig. 1b. The reluctances and impedances of the element model relate to the mass matrix elements and are described by

$$R_{ij,q} = \int_{V_0} [w_{ij} \mu^{-1} w_{ij} \mu^{-1}] dV, Z_{ij,q} = \int_{V_0} (\sigma + p e^{-1}) w_{ij} dV, (2a,b)$$

where $w_{ij}$ are interpolation functions of a facet element for the facets $S_j, S_i$ [6].

The FEM mass matrices are non-diagonal; consequently so are the matrices of the equivalent network models. In the models of Fig. 1, the branches which are not perpendicular to each other will have a mutual coupling. Such couplings will also occur within the triangular 2D elements of Fig. 2. A model with mutual reluctances may be established using a facet model of a five sided prism [7].

Equations arising from the CM, FIT and ENM formulations may appear to be similar to those obtained from the FEM, but there is an important difference that they do not contain mutual couplings and thus the mass matrices are diagonal, for example $R_{5,5} = R_{6,6} = h_5/(2 \mu h_6 h_7), \Lambda_{ij}^{(3,7)} = \mu h_5 h_7/(4 h_8) (i=1,2,3,4), (3a,b)$ where $h_5, h_6, h_7, h_8$ are dimensions as in Fig. 1b. In the reluctance model of a triangle $R_{ij} = h/(\mu s)$, with $h$ and $s$ shown in Fig. 2.

Fig. 1. Edge (a) and facet (b) model of element
may be achieved by exploiting the condition with mutual couplings may be converted into an equivalent densities in a tetrahedra to be negligibly small. Assumptions regarding fluxes or currents; for example by may be made diagonal only if complemented by additional equations equivalent to the FIT method with a diagonal mass matrix – is perfectly acceptable without noticeable loss of accuracy. Moreover, the diagonal matrix is easy to invert, thus creating mutual conductances and reluctances. A block parameters for the FEM model were derived using (1) and (2), thousand nodal equations of the electric network. The relevant million loop equations related to the magnetic network and 20 reluctance-conductance network consisting of about half a region. The resultant system of equations corresponds to a some 16 thousand of which were placed in the conducting space has been subdivided into about 150 thousand elements, and nodal elements for the scalar potential

III. DERIVING FEM EQUIVALENT TO FIT

From circuit theory it is well known that a three branch star with mutual couplings may be converted into an equivalent one without any couplings, as demonstrated by Fig. 2. This may be achieved by exploiting the condition \( \Sigma_b = 0 \). It is therefore possible to start with different mass matrices for FEM and FIT and yet achieve identical matrix coefficients for both formulations. The above transposition, regrettably, does not apply to 3D systems. Notwithstanding, it is still possible to derive a FEM formulation which is equivalent to FIT by calculating the integrals (1) and (2) – required for setting up the mass matrix - using a simplified formula

\[
\int f \, dv = V_x / B_x \left| P_i \right|
\]

which results in models free of mutual couplings, thus with coefficients the same as if obtained from FIT. Unfortunately, the procedure described by (4) is only successful – in terms of making the matrix diagonal - in the case of parallelepped elements (it also works for curved rectangular paralleleppeds). The mass matrices of tetrahedral and five sided prism elements may be made diagonal only if complemented by additional assumptions regarding fluxes or currents; for example by imposing (or assuming) one of the facet flux or current densities in a tetrahedra to be negligibly small.

IV. EXAMPLE

The TEAM Workshops Problem No. 7 (Fig. 3) has been selected to illustrate the theoretical investigations [8]. The magnetic field and eddy current distributions have been calculated for a conducting plate with a hole, with the excitation provided by a multi-turn coil. An  \( \mathbf{A} \cdot \mathbf{V} \) formulation has been adopted with edge elements for the vector potential \( \mathbf{A} \) and nodal elements for the scalar potential \( \mathbf{V} \). The bounded space has been subdivided into about 150 thousand elements, some 16 thousand of which were placed in the conducting region. The resultant system of equations corresponds to a reluctance-conductance network consisting of about half a million loop equations related to the magnetic network and 20 thousand nodal equations of the electric network. The relevant parameters for the FEM model were derived using (1) and (2), thus creating mutual conductances and reluctances. A block relaxation method, combined with incomplete Cholesky decomposition, has been used to solve the final system of equations. Table I shows example values of the flux and current densities at selected points \( P_1 \) and \( P_2 \) as marked in Fig. 3.

V. REFERENCES