On the equivalence of Finite Element and Finite Integration formulations

A. Demenko*, J.K. Sykulski**, R. Wojciechowski*
*Poznań University of Technology, ** University of Southampton
*ul Piotrowo 3A, 60-965 Poznań, Poland, ** Southampton SO17 1BJ, UK
*andrzej.demenko@put.poznan.pl, ** J.K.Sykulski@soton.ac.uk

***Abstract* —The paper offers a comparative study of numerical methods of analysis of electromagnetic fields. The focus is on the Finite Element Method (FEM) and Finite Integration Technique (FIT), but with the cell and equivalent network approaches also considered. It is shown how the approximate integrals describing coefficients of the FEM need to be derived for a mesh with parallelepiped elements to achieve consistency with FIT equations. The equivalence of FEM and FIT formulations for a triangular mesh in 2D is highlighted. The TEAM Workshops Problem No. 7 is used as an example for numerical comparisons. Edge values of magnetic vector potential *A* and nodal values of electric scalar potential *V* are used throughout.**

I. INTRODUCTION

The finite element method (FEM) has established itself as the prime numerical technique for electromagnetic field computations, but some researchers prefer and promote the use of the finite integration technique (FIT) [1], the cell method (CM) [2] or the equivalent electric and magnetic networks (ENM) [3]. The similarities between CM, FIT and FEM were observed in [4, 5] and explored thoroughly in [6]. The main differences between the different approaches are related to the way in which space is discretised and equation coefficients set up, in particular the so-called 'mass matrices' of the FEM theory. [4]. The CM, FIT and ENM formulations rely on a discretisation which is equivalent to hexahedral FEM elements of 8 nodes and 12 edges (or curved rectangular parallelepipeds under cylindrical symmetry). The FEM mass matrices are non-diagonal, unlike the ones arising in CM, FIT and ENM. The purpose of this paper is to extend and enhance the previous comparative analysis of the methods. It is demonstrated that the CM, FIT and ENM equations may be considered a special case of the FEM formulation. The derived approximate integration formulae yield the equations equivalent (identical).

II. EQUATIONS OF FEM AND FIT

Both nodal elements using scalar potentials Ω, V and edge elements in terms of vector potentials $\boldsymbol{A}$, $\boldsymbol{T}$ are considered. The FEM equations for scalar potentials correspond to the nodal equations of the edge network with branches coinciding with element edges (Fig. 1a) [6]. The permeances, conductances and capacitances forming the mass matrix may be found from

$$\Lambda_{i,j}^{(p,q)}=\int_{V_e}\boldsymbol{w}_{ei,j}\mu\boldsymbol{w}_{ep,q}\mathrm{d}v,\ G_{i,j}^{(p,q)}+\mathrm{p}C_{i,j}^{(p,q)}=\int_{V_e}\boldsymbol{w}_{ei,j}(\sigma+\mathrm{p}\varepsilon)\boldsymbol{w}_{ep,q}\mathrm{d}v\,, \quad (1a,b)$$

where $\boldsymbol{w}_{ei,j}$, $\boldsymbol{w}_{ep,q}$ are interpolation functions of an edge element for the edges P_iP_j and P_pP_q respectively, p=d/dt, and V_e is the volume of the element. The FEM equations for vector potentials, on the other hand, represent loop equations of the facet network, the branches of which cross the element facets. A portion of a network of a parallelepiped element is shown in Fig. 1b. The reluctances and impedances of the element model relate to the mass matrix elements and are described by

$$R_{\mu i,q}=\int_{V_e}\boldsymbol{w}_{fi}\mu^{-1}\boldsymbol{w}_{fq}\mathrm{d}v\,,\ Z_{i,q}=\int_{V_e}\boldsymbol{w}_{fi}(\sigma+\mathrm{p}\varepsilon)^{-1}\boldsymbol{w}_{fq}\mathrm{d}v\,, \quad (2a,b)$$

where $\boldsymbol{w}_{fi}$, $\boldsymbol{w}_{fq}$ are interpolation functions of a facet element for the facets S_i, S_q.[6].

The FEM mass matrices are non-diagonal; consequently so are the matrices of the equivalent network models. In the models of Fig. 1, the branches which are not perpendicular to each other will have a mutual coupling. Such couplings will also occur within the triangular 2D elements of Fig. 2. A model with mutual reluctances may be established using a facet model of a five sided prism [7].

Equations arising from the CM, FIT and ENM formulations may appear to be similar to those obtained from the FEM, but there is an important difference that they do not contain mutual couplings and thus the mass matrices are diagonal, for example

$$R_{\mu5,5}=R_{\mu6,6}=h_z/(2\mu h_x h_y),\ \Lambda_{i,i+4}^{(i,i+4)}=\mu h_x h_y/(4h_z)\ (i=1,2,3,4), \quad (3a,b)$$

where h_y, h_x, h_z are dimensions as in Fig. 1.b. In the reluctance model of a triangle $R_{\mu i,i}=h_i/(\mu s_i)$, with h_i and s_i shown in Fig. 2.

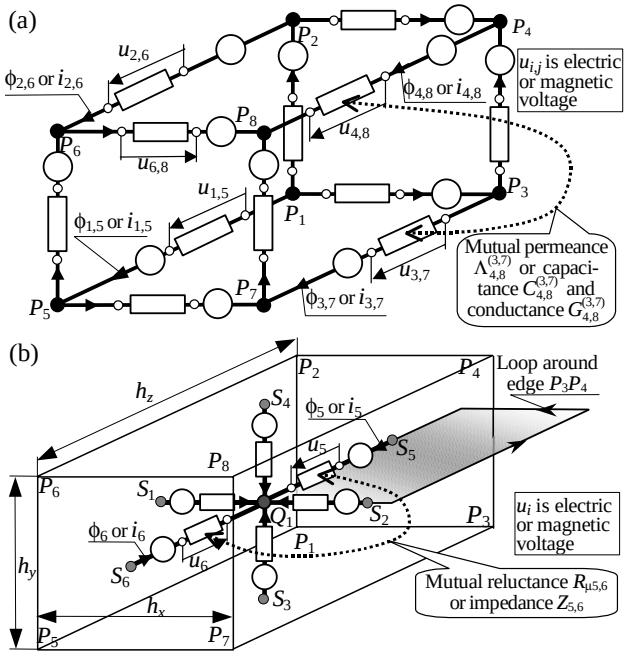

Fig. 1. Edge (a) and facet (b) model of element

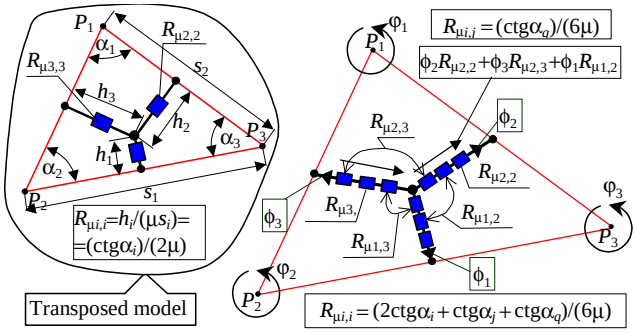

Fig. 2 Reluctance (facet) model of triangle

III. DERIVING FEM EQUIVALENT TO FIT

From circuit theory it is well known that a three branch star with mutual couplings may be converted into an equivalent one without any couplings, as demonstrated by Fig. 2. This may be achieved by exploiting the condition $\Sigma\phi_i=0$. It is therefore possible to start with different mass matrices for FEM and FIT and yet achieve identical matrix coefficients for both formulations. The above transposition, regrettably, does not apply to 3D systems. Notwithstanding, it is still possible to derive a FEM formulation which is equivalent to FIT by calculating the integrals (1) and (2) – required for setting up the mass matrix – using a simplified formula

$$\int_{V_e} f\,\mathrm{d}v = V_e/8 \sum_{i=1}^{8} f(P_i)\,, \qquad (4)$$

which results in models free of mutual couplings, thus with coefficients the same as if obtained from FIT. Unfortunately, the procedure described by (4) is only successful – in terms of making the matrix diagonal – in the case of parallelepiped elements (it also works for curved rectangular parallelepipeds). The mass matrices of tetrahedral and five sided prism elements may be made diagonal only if complemented by additional assumptions regarding fluxes or currents; for example by imposing (or assuming) one of the facet flux or current densities in a tetrahedra to be negligibly small.

IV. EXAMPLE

The TEAM Workshops Problem No. 7 (Fig. 3) has been selected to illustrate the theoretical investigations [8]. The magnetic field and eddy current distributions have been calculated for a conducting plate with a hole, with the excitation provided by a multi-turn coil. An ***A*-*V*** formulation has been adopted with edge elements for the vector potential ***A*** and nodal elements for the scalar potential *V*. The bounded space has been subdivided into about 150 thousand elements, some 16 thousand of which were placed in the conducting region. The resultant system of equations corresponds to a reluctance-conductance network consisting of about half a million loop equations related to the magnetic network and 20 thousand nodal equations of the electric network. The relevant parameters for the FEM model were derived using (1) and (2), thus creating mutual conductances and reluctances. A block relaxation method, combined with incomplete Cholesky decomposition, has been used to solve the final system of equations. Table I shows example values of the flux and current densities at selected points P_1 and P_2 as marked in Fig. 3.

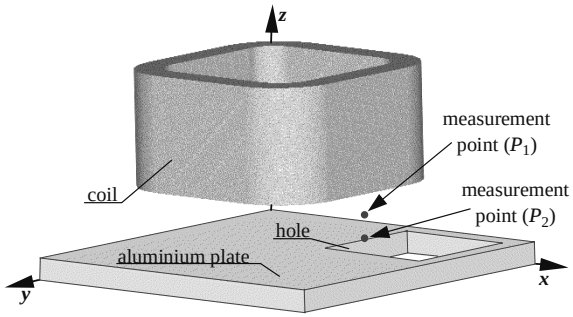

Fig. 3 TEAM Workshops Problem No.7

TABLE I
COMPARISON BETWEEN FIT AND FEM RESULTS

Quantity \ Method	FIT	FEM
Flux density in point P_1		
B_x [T]	-0.010689	-0.010747
B_y [T]	0.003581	0.003583
B_z [T]	0.008145	0.008165
Current density in point P_2		
J_x [A/m^2]	83275.85	83196.15
J_y [A/m^2]	1713894.52	1710454.18
J_z [A/m^2]	-39703.18	-39469.81

The values are given for an instant of time when the coil current was at its maximum (a 50Hz supply has been assumed).

For all points considered, the differences between the FIT and FEM results do not exceed 0.6% for flux density and 0.7% for current density, respectively. It appears therefore reasonable to conclude that the proposed approximation (4) – which leads to equations equivalent to the FIT method with a diagonal mass matrix – is perfectly acceptable without noticeable loss of accuracy. Moreover, the diagonal matrix is easy to invert, thus seeking edge values of ***A***, representing loop fluxes in the model of Fig. 1b, may be conveniently replaced by an easier task of finding nodal potentials associated with element centres (nodes Q_i). In the case of diffusion problems the additional advantage of making the mass matrix diagonal is a possibility of applying explicit numerical schemes [4].

V. REFERENCES

[1] T. Weiland, "A discretization method for the solution of Maxwell's equations for six-component fields," *Electron. Commun. AEÜ*, 31(3), pp. 116-120, 1977.
[2] E. Tonti, "A direct formulation of field laws: The cell method," *CMES*, 2(2), pp. 237–258, 2001.
[3] J. Sykulski et al, *Computational Magnetics*, Chapman & Hall, 1995.
[4] A. Bossavit and L. Kettunen, "Yee-like schemes on staggered cellular grids: A synthesis between FIT and FEM approaches," *IEEE Trans. on Magnetics*, 36(4), pp. 861-867, 2000.
[5] M Clemens, Stefan Feigh, T. Weiland "Construction principles of multigrid smoothers for curl–curl equations," *IEEE Trans. on Magnetics*, 41(5), pp.1680-83, 2005.
[6] A. Demenko and J. K. Sykulski, "Network equivalents of nodal and edge elements in electromagnetics," *IEEE Trans. on Magnetics*, 38 (2), pp. 1305-08, 2002.
[7] A. Demenko, L. Nowak, W. Szeląg, "Reluctance network formed by means of edge element method", *IEEE Trans. on Magnetics*, 34(5), pp. 2485-88, 1998.
[8] O. Biro, K. Preis, W. Renhart , K.R. Richter , G. Vrisk , "Performance of different vector potential formulations in solving multiply connected 3D eddy current problems", *IEEE Trans. Magnetics*, 26 (2), pp. 438-41, 1990.