Mixed $\mu$ Robust Finite Word Length Controller Design

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1 Motivations
   • Existing Approaches
   • Our Novelty

2 Robust FWL Controller Design
   • Problem Formulation
   • FWL Robust Measure
   • FWL Robust Control Design

3 Design Examples
   • Numerical Example One
   • Numerical Example Two

4 Conclusions
Outline

1 Motivations
   • Existing Approaches
   • Our Novelty

2 Robust FWL Controller Design
   • Problem Formulation
   • FWL Robust Measure
   • FWL Robust Control Design

3 Design Examples
   • Numerical Example One
   • Numerical Example Two

4 Conclusions
Fragility Problem

- Control system designed by maximising its robustness to plant uncertainty alone may exhibit poor stability margin with respect to controller coefficient perturbation.

- Two types of finite word length errors in controller implementation are:
  - Rounding errors that occur in arithmetic operations, and
  - Controller parameter representation errors

These two types of errors are typically investigated separately for mathematical tractability.

- We consider second type of FWL errors, which has critically influence on close-loop stability.
Existing Approaches

- Two strategies for considering FWL controller parameter representation errors
  - Indirect approach: search for an “optimal” realisation of the given controller that is most robust to FWL errors
  - Direct approach: design controller realisation by considering both robust control criteria and FWL errors
- In literature, direct approach is also referred to as non-fragile, defragile or resilient control
  - Some works assume controller parameter perturbation block is 2-norm bounded
  - More realistic ones assume parameter perturbation is independent and magnitude bounded
- Yang et al. [20] design robust FWL $H_2$ controller by considering all vertices of FWL perturbation hypercube
1 Motivations
   - Existing Approaches
   - Our Novelty

2 Robust FWL Controller Design
   - Problem Formulation
   - FWL Robust Measure
   - FWL Robust Control Design

3 Design Examples
   - Numerical Example One
   - Numerical Example Two

4 Conclusions
Our Contributions

With similar hypothesis to Yang et al. [20], we study robust FWL $H_\infty$ output feedback controller, and our contributions are

- FWL robust control performance measure is proposed, which takes into account robust control requirements and FWL effects on controller implementation.
- Robust FWL controller design problem is naturally formulated as a mixed $\mu$ problem which can be solved effectively with the aid of mixed $\mu$ theory.
- Our proposed method is computationally more attractive than Yang et al. [20].
Outline

1 Motivations
   - Existing Approaches
   - Our Novelty

2 Robust FWL Controller Design
   - Problem Formulation
   - FWL Robust Measure
   - FWL Robust Control Design

3 Design Examples
   - Numerical Example One
   - Numerical Example Two

4 Conclusions
Plant is described by known nominal model $\hat{P}_g(w)$ and unknown but bounded structured uncertainty $\hat{U}(w)$, where $w \in \mathbb{C}$.

$\hat{P}_g(w)$ is given as

$$
x_P(k+1) = A_P x_P(k) + B_v v(k) + B_w w(k) + B_P u_P(k) \\
h(k) = C_h x_P(k) D_{1,1} v(k) + D_{1,2} w(k) \\
z(k) = C_z x_P(k) + D_{2,1} v(k) + D_{2,2} w(k) + D_{2,3} u_P(k) \\
y_P(k) = C_P x_P(k) + D_{3,2} w(k)
$$

$x_P(k) \in \mathbb{R}^n$: state, $v(k) \in \mathbb{R}^{n_1}$: uncertainty-linked input, $w(k) \in \mathbb{R}^{n_2}$: external disturbance input, $u_P(k) \in \mathbb{R}^s$: control input, $h(k) \in \mathbb{R}^{n_1}$: uncertainty-linked output, $z(k) \in \mathbb{R}^{n_2}$: controlled output, $y_P(k) \in \mathbb{R}^t$: measured output.

$\hat{P}_g(w)$ connects with $\hat{U}(w)$ through $h$ and $v$

$$
v = \hat{U}(w) h
$$
Structured Uncertainty

- Unknown structured uncertainty $\hat{U}(w)$ takes the form

$$\hat{U}(w) = \text{diag}\left(\hat{U}_1(w), \cdots, \hat{U}_{b+d}(w)\right)$$

where $\hat{U}_i(w) = \varphi_i(w)I_{p_i}$ with $\varphi_i(w) \in \mathbb{C}$, $\forall w \in \mathbb{C}$, $\forall i \in \{1, \cdots, b\}$; and $\hat{U}_i(w) \in \mathbb{C}^{p_i \times p_i}$, $\forall w \in \mathbb{C}$, $\forall i \in \{b+1, \cdots, b+d\}$, while

$$\sum_{i=1}^{b+d} p_i = n_1, p_i \geq 1$$

- Given a constant $\tau > 0$, $\hat{U}(w)$ is included in the set

$$\mathcal{H}_\tau \triangleq \left\{ \hat{U}(w) \right\vert \hat{U}(w) = \text{diag}\left(\hat{U}_1(w), \cdots, \hat{U}_{b+d}(w)\right), \hat{U}(w) \in \mathcal{F}, \hat{U}(w) \text{ is stable, } \|\hat{U}(w)\|_\infty < \tau \right\}$$

with $\mathcal{F}$: the set of all causal finite linear time-invariant systems
Controller

Controller $\hat{C}(w)$ of $m$th-order is described by

$$\begin{align*}
x_C(k+1) &= A_C x_C(k) + B_C y_P(k) \\
u_P(k) &= C_C x_C(k) + D_C y_P(k)
\end{align*}$$

and the controller is also denoted by its parameters as

$$X \triangleq \begin{bmatrix} D_C & C_C \\ B_C & A_C \end{bmatrix} \in \mathbb{R}^{(s+m) \times (t+m)}$$

$X$ is perturbed to $X + \Delta$ due to FWL fixed-point implementation, with $\Delta$ belonging to the hypercube

$$\mathcal{D}_\beta \triangleq \{ \Delta \mid \Delta \in \mathbb{R}^{(s+m) \times (t+m)}, \| \Delta \|_m \leq \beta \}$$

where $0 \leq \beta \in \mathbb{R}$ is the maximum representation error, $\Delta = [\delta_{i,j}]$ and $\| \Delta \|_m = \max_{i,j} |\delta_{i,j}|$
Closed-Loop System

- Closed-loop system, which consists of $\hat{P}_g(w)$, $\hat{U}(w)$, $X$ and $\Lambda$, is denoted as $\hat{\Phi}(w, \hat{U}(w), X, \Lambda)$, where $\Lambda$ is equivalent to $\Delta$ as
  $$\Lambda \triangleq \text{diag}(\delta_{1,1}, \delta_{2,1}, \cdots, \delta_{s+m,1}, \delta_{1,2}, \cdots, \delta_{1,t+m}, \cdots, \delta_{s+m,t+m})$$
  $$\Lambda \in \mathcal{O}_\beta \triangleq \{ Q \mid Q \in \mathbb{R}^{N \times N}, Q \text{ is diagonal}, \sigma(Q) \leq \beta \}$$
  with $\sigma(Q)$ denoting the maximum singular value of $Q$.

- Further denote the closed-loop transfer function from $w(k)$ to $z(k)$ by $\hat{\Phi}_{wz}(w, \hat{U}(w), X, \Lambda)$.

- For $0 < \xi \in \mathbb{R}$, the set of all $m$th-order robust $H_\infty$ controllers, which do not consider FWL effect, is defined by
  $$\mathcal{X}_m \triangleq \{ X \mid X \in \mathbb{R}^{(s+m) \times (t+m)}, \hat{\Phi}(w, \hat{U}(w), X, 0) \text{ is stable, } \forall \hat{U}(w) \in \mathcal{H}_\tau, \| \hat{\Phi}_{wz}(w, \hat{U}(w), X, 0) \|_\infty \leq \xi \}$$
Outline

1 Motivations
   • Existing Approaches
   • Our Novelty

2 Robust FWL Controller Design
   • Problem Formulation
   • FWL Robust Measure
   • FWL Robust Control Design

3 Design Examples
   • Numerical Example One
   • Numerical Example Two

4 Conclusions
Theoretical Measure

- For a controller $X \in \mathcal{X}_m$, the FWL robust measure

$$
\nu(X) \triangleq \sup_{0 \leq \beta \in \mathbb{R}} \left\{ \beta \left| \forall \hat{U}(w) \in \mathcal{H}_\tau, \hat{\Phi}(w, \hat{U}(w), X, \Lambda) \text{ is stable,} \right. \right. \\
\left. \left. \forall \Lambda \in \mathcal{O}_\beta, \| \hat{\Phi}_{wz}(w, \hat{U}(w), X, \Lambda) \|_{\infty} \leq \xi \right\} \right.
$$

characterises “robustness” of $X$ to controller perturbation $\Lambda$

- $\mathcal{H}_\tau$ is the set of structured uncertainty
- $\mathcal{O}_\beta$ defines FWL perturbation hypercube
- $\hat{\Phi}(w, \hat{U}(w), X, \Lambda)$ is the whole closed-loop system
- $\hat{\Phi}_{wz}(w, \hat{U}(w), X, \Lambda)$ is the closed-loop transfer function from external perturbation input $w(k)$ to controlled output $z(k)$

- However, how to compute the value of $\nu(X)$ is unknown
- With aid of mixed $\mu$ theorem, we derive a tractable lower bound for $\nu(X)$
“Substitute out” $\hat{U}(\omega)$ from $\hat{\Phi}(\omega, \hat{U}(\omega), X, \Lambda)$ ⇒ composite system of $\hat{P}_g(\omega)$, $X$ and $\Lambda$, described by:

$$
\begin{align*}
    x_{PC}(k + 1) &= (\overline{A}(X) + B_u \Lambda C_u) x_{PC}(k) + B_v v(k) + \overline{B}(X)w(k) \\
    h(k) &= C_h x_{PC}(k) + D_{1,1} v(k) + D_{1,2} w(k) \\
    z(k) &= C(X)x_{PC}(k) + D_{2,1} v(k) + \overline{D}(X)w(k)
\end{align*}
$$

Define the matrix

$$
\Theta(X, \beta) \triangleq \begin{bmatrix}
    \overline{A}(X) & B_u & B_v & \overline{B}(X) \\
    \beta C_u & 0 & 0 & 0 \\
    \tau C_{\overline{h}} & 0 & \tau D_{1,1} & \tau D_{1,2} \\
    \frac{1}{\xi} C(X) & 0 & \frac{1}{\xi} D_{2,1} & \frac{1}{\xi} \overline{D}(X)
\end{bmatrix}
$$

and the related set of allowable perturbations $\mathcal{K}_\theta$

We can obtain a computable mixed $\mu$: $\alpha_{\mathcal{K}_\theta}(\Theta(X, \beta))$
**Tractable Measure**

- **Result**: \( \exists \ 0 \leq \beta \in \mathbb{R} \) such that \( \alpha_{K_\theta}(\Theta(X, \beta)) < 1 \), then \( X \in \mathcal{X}_m \) and \( \forall \hat{U}(w) \in \mathcal{H}_\tau, \ \forall \Lambda \in \mathcal{O}_\beta \)

\[
\hat{\Phi}(w, \hat{U}(w), X, \Lambda) \text{ is stable, } \| \hat{\Phi}_{wz}(w, \hat{U}(w), X, \Lambda) \|_\infty \leq \xi
\]

- Define a subset of \( \mathcal{X}_m \) as

\[
\tilde{\mathcal{X}}_m \triangleq \{ X | X \in \mathbb{R}^{(s+m) \times (t+m)}, \alpha_{K_\theta}(\Theta(X, 0)) < 1 \}
\]

- For \( X \in \tilde{\mathcal{X}}_m \), the FWL robust measure

\[
\tilde{\nu}(X) \triangleq \sup_{0 \leq \beta \in \mathbb{R}} \{ \beta | \alpha_{K_\theta}(\Theta(X, \beta)) < 1 \}
\]

is a lower bound of \( \nu(X) \)

- \( \tilde{\nu}(X) \) can be computed using combined linear matrix inequality technique and bisection search
Motivations
- Existing Approaches
- Our Novelty

Robust FWL Controller Design
- Problem Formulation
- FWL Robust Measure
- FWL Robust Control Design

Design Examples
- Numerical Example One
- Numerical Example Two

Conclusions
Design Problem

- Robust FWL controller design: given $\hat{P}_g(w)$, $\tau$, $\xi$, $m$ and nonempty $\tilde{X}_m$, find a controller $X_{\text{opt}} \in \tilde{X}_m$ that achieves
  \[ \gamma = \sup_{X \in eX_m} \tilde{\nu}(X) \]

- This design makes the FWL tolerance as large as possible, while satisfying a suboptimal robust control requirement

- This robust FWL controller design can be solved with aid of bilinear matrix inequality

- Complexity comparison with Yang et al. [20]
  - Our FWL robust $H_\infty$ controller design solves one BMI of size $2(n + m + N + n_1 + n_2)$
  - FWL robust $H_2$ controller design [20] requires to solve at least $2^N$ BMIs of size no less than $4n$
Outline

1. Motivations
   - Existing Approaches
   - Our Novelty

2. Robust FWL Controller Design
   - Problem Formulation
   - FWL Robust Measure
   - FWL Robust Control Design

3. Design Examples
   - Numerical Example One
   - Numerical Example Two

4. Conclusions
Design Problem

- Nominal plant model $\hat{P}_g(w)$ is given by

$$\hat{P}_0(w) = \frac{3.3750 \times 10^{-3} w + 1.3669 \times 10^{-2} w^2 + 3.4605 \times 10^{-3} w^3}{1 - 3.0488 w + 3.1001 w^2 - 1.0513 w^3},$$

$$\hat{W}_1(w) = \frac{4.9875 \times 10^{-3} w}{1 - 9.9501 \times 10^{-1} w}, \quad \hat{W}_2(w) = \frac{5.8512 \times 10^{-1} w - 5.5933 \times 10^{-1} w^2}{1 - 1.3390 w + 3.7908 \times 10^{-1} w^2}.$$

- Plant model uncertainty $\hat{U}(w) \in \mathcal{H}_\tau$ with $\tau = 0.4$
Design Solution

- Constant that bounds closed-loop $H_\infty$ norm from $w$ to $z$ was set to $\xi = 0.3$, and controller order was chosen to be $m = 2$.

- Solving optimal FWL robust design problem yields the controller

$$X_{opt1} = \begin{bmatrix}
-103.44 & -15.600 & -1.4984 \\
-16.070 & -1.4261 & 0.25055 \\
-19.469 & -3.0400 & 0.37517
\end{bmatrix}$$

with $\tilde{\nu}(X_{opt1}) = 8.2842 \times 10^{-3}$

- For any FWL perturbation to $X_{opt1}$ smaller than $8.2842 \times 10^{-3}$ and for any $\hat{U}(w) \in H_\tau$ with $\tau = 0.4$,
  
  - the closed-loop system maintains stability, and
  
  - closed-loop $H_\infty$ norm from $w$ to $z$ is always less than $0.3$.
Bit Length Estimate

- Using fixed point processor of $c$-bit length to implement $X$, $c$ bits are assigned as:
  - 1 sign bit, $c_{int}$ bits for integer part, $c_{fra}$ bits for fraction part
- To guarantee dynamic range of $X$, $c_{int} = \lceil \log_2 \|X\|_m \rceil$,
- Fraction bit length bounds the absolute values of FWL errors by $2^{-(c_{fra}+1)}$, and to maintain closed-loop performance, at least
  \[ c_{fra} = \lceil -\log_2 \tilde{\nu}(X) \rceil - 1 \]
- Minimal bit length guaranteeing closed-loop performance, estimated based on $\tilde{\nu}(X)$, is
  \[ \tilde{c}(X) \triangleq \lceil \log_2 \|X\|_m \rceil + \lceil -\log_2 \tilde{\nu}(X) \rceil \]

In this example, \( \tilde{c}(X_{opt1}) = 14 \)
Outline

1. Motivations
   - Existing Approaches
   - Our Novelty

2. Robust FWL Controller Design
   - Problem Formulation
   - FWL Robust Measure
   - FWL Robust Control Design

3. Design Examples
   - Numerical Example One
   - Numerical Example Two

4. Conclusions
Design Problem

- Example from Yang et al. [20] was for FWL H₂ control under plant parameter uncertainty.

- Noting $\|\hat{\Phi}_{wz}\|_\infty \geq \|\hat{\Phi}_{wz}\|_2$ and structured uncertainty includes parameter uncertainty, we substituted $\|\hat{\Phi}_{wz}\|_\infty$ for $\|\hat{\Phi}_{wz}\|_2$ and plant structured uncertainty for plant parameter uncertainty.

- Nominal plant model $\hat{P}_g(w)$ is defined by:

  $$A_P = \begin{bmatrix} 0.5 & 0.1 \\ 0.2 & 0 \end{bmatrix}, \quad B_v = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad B_w = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \quad B_P = \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

  $$C_h = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad C_z = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad D_{2,2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad D_{2,3} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

  $$C_P = [0, -1], \quad D_{3,2} = \begin{bmatrix} 1 & 1 \end{bmatrix}, \quad D_{1,1} = D_{1,2} = D_{2,1} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

- Plant structure uncertainty is defined by:

  $$\hat{U}(w) = \varphi(w) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \mathcal{H}_\tau \text{ with } \varphi(w) \in \mathbb{C} \text{ and } \tau = 0.13.$$
Set constant $\xi = 4.9676$. We designed 1st-order controller by solving the FWL robust design problem, leading to

$$X_{opt2} = \begin{bmatrix} 1.0853 & -0.36600 \\ 1.1031 & -0.34734 \end{bmatrix}$$

with $\tilde{v}(X_{opt2}) = 0.0275$, which can be implemented with a fixed point processor of $\tilde{c}(X_{opt2}) = 7$ bits.

As $\|\hat{\Phi}_{wz}\|_\infty \geq \|\hat{\Phi}_{wz}\|_2$, system was guaranteed to be closed-loop stable and $\|\hat{\Phi}_{wz}\|_2 < 4.9676$ when $\tau = 0.13$ and the FWL bound was 0.0275.

Yang et al. [20] obtained a controller achieving $\|\hat{\Phi}_{wz}\|_2 < 3.0822$ when $\tau = 0.13$ and the FWL bound 0.0275.

Our method required to solve one BMI of size 22, while Yang et al. [20] required to solve 32 BMIs of size 8.
We have used mixed $\mu$ theory to directly design optimal robust FWL controllers, and our novel contributions include:

- A robust FWL control performance measure taking into account both robust control requirements and FWL implementation considerations
- This robust FWL control performance measure can be computed conveniently using LMI
- Optimal robust FWL controller design is formulated as a mixed $\mu$ problem, which can be solved by means of BMI
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