Lyapunov Stability Analysis of Higher-Order 2-D Systems

Chiaki Kojima, Paolo Rapisarda, and Kiyotsugu Takaba

Abstract. We prove a necessary and sufficient condition for the asymptotic stability of a 2-D system described by a system of higher-order linear partial difference equations. We use the definition of asymptotic stability given by Valcher in "Characteristic Cones and Stability Properties of Two-Dimensional Autonomous Behaviors", *IEEE Trans. Circ. Syst. — Part I: Fundamental Theory and Applications*, vol. 47, no. 3, pp. 290–302, 2000. This property is shown to be equivalent to the existence of a vector Lyapunov functional satisfying certain positivity conditions together with its divergence along the system trajectories. We use the behavioral framework and the calculus of quadratic difference forms based on four variable polynomial algebra.

1 Introduction

Discrete- and continuous-time two-dimensional (in the following abbreviated as 2-D) systems have application in all those situations when the evolution of the to-be-modeled system depends on two independent variables. In this paper we adopt the behavioral framework pioneered by J. C. Willems in the 1-D case (see [13]),

Chiaki Kojima
Department of Information Physics and Computing, Graduate School of Information Science and Technology, The University of Tokyo, Hongo, Bunkyo-Ku, Tokyo 113-0033, Japan
e-mail: chiaki_kojima@ipc.i.u-tokyo.ac.jp

Paolo Rapisarda
Information: Signals, Images, Systems group, School of Electronics and Computer Science, University of Southampton, SO171BJ Southampton, United Kingdom
e-mail: pr3@ecs.soton.ac.uk

Kiyotsugu Takaba
Dept. of Applied Mathematics and Physics, Graduate School of Informatics, Kyoto University, Kyoto 606–8501, Japan
e-mail: takaba@amp.i.kyoto-u.ac.jp

S. Hara et al. (Eds.): Perspec. in Math. Sys. Theory, Ctrl., & Signal Process., LNCIS 398, pp. 197–206.
springerlink.com © Springer-Verlag Berlin Heidelberg 2010

and extended to the 2-D case by P. Rocha (see [14]) and other authors. In this setting the main object of study is the behavior, the set consisting of all the trajectories admissible by the physical laws describing the system trajectories.

The notion of stability, because of its important consequences in the analysis and design of control systems and of filters, has attracted considerable interest also in the case of 2-D systems. The issue of what the correct definition of stability is for this situation presents first and foremost the difficulty of extending the notion of "past" and "future", self-evident in the 1-D framework, to the case of two independent variables, where there is no obvious such splitting of the independent variables domain. An eminently reasonable position is to let the laws describing the physical phenomenon themselves dictate what the direction is of the evolution of the system. This is the approach pioneered by M. E. Valcher in [15] and followed in this paper.

In this paper we present a necessary and sufficient condition for the asymptotic stability of 2-D systems based on Lyapunov functions. This idea is by no means original, having been applied already in [10, 1]; however, those approaches relied entirely on a specific ("state-space") type of representation of the system, while we deal with systems described in a general form, namely as the solutions of a system of partial difference equations. Moreover, the "generalized Bézoutian" introduced in [4] is shown in this paper to be the scalar version of a generalized Bézoutian arising naturally as a Lyapunov function for 2-D systems.

Sections 2 and 3 of this paper contain background material on 2-D systems and quadratic difference forms, respectively. Section 4 contains the main result of this paper, namely a stability criterion for higher-order systems of difference equations based on Lyapunov analysis.

In this paper, the concepts and tools of the behavioral approach, and of quadratic difference forms will be put to strenuous use. The reader not familiar with them is referred to [12, 13, 14, 16] for a thorough exposition.

Notation: We denote with $\mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1,\xi_2]$ (respectively, $\mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$) the set of all $\mathtt{r}\times\mathtt{w}$ matrices with entries in the ring $\mathbb{R}[\xi_1,\xi_2]$ of polynomials in 2 indeterminates, with real coefficients (respectively in the ring $\mathbb{R}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$ of Laurent polynomials in 2 indeterminates with real coefficients). Given a nonzero Laurent polynomial $p(\xi_1,\xi_2)=\sum_{\ell,m}p_{\ell,m}\xi_1^{\ell}\xi_2^{m}\in\mathbb{R}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$, the *Laurent variety* of p is defined as

$$\mathscr{V}_L(p):=\{(\alpha,\beta)\in\mathbb{C}\times\mathbb{C}\mid\alpha\beta\neq 0,p(\alpha,\beta)=0\}$$

This definition extends to sets $\mathscr{I}$ of Laurent polynomials, with $\mathscr{V}(\mathscr{I})$ being the intersection of the Laurent varieties of all polynomials in the set. For $R\in\mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$, the *characteristic ideal* is the ideal of $\mathbb{R}[\xi_1,\xi_2]$ generated by the determinants of all $\mathtt{w}\times\mathtt{w}$ minors of R, and the *characteristic variety* is the set of roots common to all polynomials in the ideal. Further properties and definitions can be found in [3].

A set $\mathscr{K}\subset\mathbb{R}\times\mathbb{R}$ is called a *cone* if $\alpha\mathscr{K}\subset\mathscr{K}$ for all $\alpha\geq 0$. A cone is *solid* if it contains an open ball in $\mathbb{R}\times\mathbb{R}$, and *pointed* if $\mathscr{K}\cap-\mathscr{K}=\{(0,0)\}$. A cone is *proper* if it is closed, pointed, solid, and convex. It is easy to see that a proper cone is uniquely identified as the set of nonnegative linear combinations of two linearly independent vectors $v_1, v_2 \in \mathbb{R}^2$. In the following we will often consider the intersection of a cone $\mathscr{K}$ with $\mathbb{Z} \times \mathbb{Z}$; whenever it will be clear from the context, we will be denoting this set with $\mathscr{K}$ instead of with $\mathscr{K} \cap \mathbb{Z} \times \mathbb{Z}$.

We denote with $\overline{\mathscr{P}_1}$ the closed unit polydisk:

$$\overline{\mathscr{P}_1} := \{(\alpha, \beta) \in \mathbb{C} \times \mathbb{C} \mid |\alpha| \leq 1, |\beta| \leq 1\}$$

Given a set $\mathscr{S} \subset \mathbb{Z} \times \mathbb{Z}$, its *(discrete) convex hull* is the intersection of the convex hull of $\mathscr{S}$ (seen as a subset of $\mathbb{R} \times \mathbb{R}$) and of $\mathbb{Z} \times \mathbb{Z}$. In the following we will also refer to the (discrete) convex hull associated with an element $p \in \mathbb{R}[\xi_1, \xi_2, \xi_1^{-1}, \xi_2^{-1}]$, meaning the (discrete) convex hull of the *support of* p, i.e. the set

$$\begin{aligned} \text{supp}(p) := \{(x_1, x_2) \in \mathbb{Z} \times \mathbb{Z} \mid & \text{ the coefficient of } \xi_1^h \xi_2^k \\ & \text{in } p(\xi_1, \xi_2, \xi_1^{-1}, \xi_2^{-1}) \text{ is } \neq 0\} \end{aligned}$$

We denote with $\mathbb{W}^{\mathbb{T}}$ the set consisting of all trajectories from $\mathbb{T}$ to $\mathbb{W}$. We denote with σ_1, σ_2 the *shift operators* defined as

$$\begin{aligned} \sigma_i : (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} &\to (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \quad i = 1, 2 \\ (\sigma_1 w)(x_1, x_2) &:= w(x_1 - 1, x_2) \\ (\sigma_2 w)(x_1, x_2) &:= w(x_1, x_2 - 1) \end{aligned}$$

2 2-D Behaviors

We call $\mathfrak{B}$ a *linear discrete-time complete 2-D behavior* if it is the subset of $(\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$ consisting of all solutions to

$$R(\sigma_1, \sigma_2) w = 0 \tag{1}$$

where $R \in \mathbb{R}^{\mathtt{r} \times \mathtt{w}}[\xi_1, \xi_2, \xi_1^{-1}, \xi_2^{-1}]$. We call (1) a *kernel representation* of $\mathfrak{B}$. The set of all such behaviors is denoted with $\mathscr{L}_2^{\mathtt{w}}$.

$\mathfrak{B} \in \mathscr{L}_2^{\mathtt{w}}$ is *autonomous* if there exists a proper cone $\mathscr{K} \subset \mathbb{R} \times \mathbb{R}$ such that

$$\left[w_1, w_2 \in \mathfrak{B} \text{ and } w_{1|\mathscr{K} \cap \mathbb{Z} \times \mathbb{Z}} = w_{2|\mathscr{K} \cap \mathbb{Z} \times \mathbb{Z}}\right] \Longrightarrow [w_1 = w_2]$$

Such a cone $\mathscr{K} \cap \mathbb{Z} \times \mathbb{Z}$ will be called a proper *characteristic* cone for $\mathfrak{B}$. Observe that if $w \in \mathfrak{B}$ is such that $w_{|\mathscr{K} \cap \mathbb{Z} \times \mathbb{Z}} = 0$, then $w = 0$. The following result holds.

Theorem 1. *Let $\mathfrak{B} \in \mathscr{L}_2^{\mathtt{w}}$ be autonomous, and let $\mathfrak{B} = \ker\ R(\sigma_1, \sigma_2)$ for some $R \in \mathbb{R}^{\mathtt{r} \times \mathtt{w}}[\xi_1, \xi_2, \xi_1^{-1}, \xi_2^{-1}]$. Then there exist $H \in \mathbb{R}^{\bullet \times \bullet}[\xi_1, \xi_2, \xi_1^{-1}, \xi_2^{-1}]$ right factor prime, and $\Delta \in \mathbb{R}^{\mathtt{w} \times \mathtt{w}}[\xi_1, \xi_2, \xi_1^{-1}, \xi_2^{-1}]$ nonsingular, such that $R = H \cdot \Delta$.*

Moreover, denote $\delta := \det(\Delta) \in \mathbb{R}[\xi_1, \xi_2, \xi_1^{-1}, \xi_2^{-1}]$. The following statements are equivalent:

1. *The proper cone $\mathscr{K}$ is characteristic for $\mathfrak{B}$;*
2. *The proper cone $\mathscr{K}$ is characteristic for* $\ker\ \Delta(\sigma_1, \sigma_2)$*;*

3. *The proper cone* $\mathscr{K}$ *is characteristic for* ker $\delta(\sigma_1,\sigma_2)$*;*
4. *The discrete convex hull* $\mathscr{H}_\delta$ *of* δ *satisfies the following two conditions:*

   4*a*. $-\mathscr{H}_\delta \subset \mathscr{K}$*;*
   4.*b*. $-\mathscr{H}_\delta \subset \mathscr{K}$ *intersects the generating lines of* $\mathscr{K}$ *only in* $(0,0)$.

It can be shown (see [2]) that if $\mathfrak{B} \in \mathscr{L}_2^{\mathtt{w}}$ is such that $\mathfrak{B} = \ker R(\sigma_1,\sigma_2)$ for some right factor prime matrix $R \in \mathbb{R}^{\mathtt{r}\times\mathtt{w}}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$, then $\mathfrak{B}$ is autonomous and finite-dimensional; then (see Lemma 2.4 of [15]) every proper cone is characteristic.

If $\mathfrak{B}$ is autonomous, and $\mathfrak{B} = \ker R(\sigma_1,\sigma_2)$ for some nonsingular Laurent matrix R, then $\mathfrak{B}$ is called a *square autonomous behavior*. Observe that Theorem 1 shows that for any autonomous behavior $\mathfrak{B}$ whose kernel representation can be factored as $H\Delta$ with H right factor prime and Δ nonsingular, the characteristic cone is determined by its "square autonomous part" ker $\Delta(\sigma_1,\sigma_2)$.

We now introduce the concept of stability introduced by Valcher in [15]. We examine the finite-dimensional case first.

Definition 1. Let $\mathfrak{B} \in \mathscr{L}_2^{\mathtt{w}}$ be autonomous and finite-dimensional, and let $\mathscr{K}$ be any proper cone of $\mathbb{Z}\times\mathbb{Z}$. $\mathfrak{B}$ is $\mathscr{K}$*-stable* if

$$[w \in \mathfrak{B}] \Longrightarrow \left[\lim_{\substack{(i,j)\in\mathscr{K} \\ |i|+|j|\to+\infty}} \|w(i,j)\| = 0\right]$$

The following algebraic characterization of finite-dimensional stable behaviors (see [15, Theorem 3.3, p. 297]) holds. In order to avoid cumbersome details, we follow [15], and only consider proper cones generated by unimodular integer matrices, which are then isomorphic to the first orthant of $\mathbb{Z}\times\mathbb{Z}$, in the sense that there exists a (linear, bijective) transformation $T : \mathbb{Z}^2 \to Z^2$ such that $T(\mathscr{K})$ is the first orthant.

Theorem 2. *Let* $\mathfrak{B} = \ker H(\sigma_1,\sigma_2)$*, with*

$$H(\xi_1,\xi_2) = \sum_{\ell,m} H_{\ell,m}\xi_1^\ell \xi_2^m \in \mathbb{R}^{\bullet\times\mathtt{w}}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$$

right factor prime (see [3] *for the definition), and let* $\mathscr{K}$ *be a proper cone isomorphic to the first orthant. Denote with* T *the transformation mapping* $\mathscr{K}$ *to the first orthant, and denote with* $(t_1(\ell,m),t_2(\ell,m))$ *the image of* $(\ell,m)\in\mathbb{Z}\times\mathbb{Z}$ *under* T*. Define*

$$H_T(\xi_1,\xi_2) := \sum_{\ell,m} H_{\ell,m}\xi_1^{t_2(\ell,m)}\xi_2^{t_1(\ell,m)}$$

Then the following two statements are equivalent:

1. $\mathfrak{B}$ *is* $\mathscr{K}$*-stable;*
2. *Every* (α,β) *in the Laurent variety of the maximal order minors of* H_T *satisfies* $|\alpha| > 1$ *and* $|\beta| > 1$.

In order to state the definition of stability for the square case, we need to introduce the following notation: given a proper cone $\mathscr{C}$, we denote with $\delta(\mathscr{C})$ its *boundary*, i.e. the generating lines of $\mathscr{C}$. We denote with $(\delta(\mathscr{C}))^n$ the set consisting of the points of $\mathbb{Z}\times\mathbb{Z}$ whose distance from $\delta(\mathscr{C})$ is less than n:

$$(\delta(\mathscr{C}))^n := \{(i,j)\in\mathbb{Z}\times\mathbb{Z} \mid \min\{|i-h|+|j-k| \text{ with } (x_1,x_2)\in\delta(\mathscr{C})\}\leq n\}$$

Definition 2. Let $\mathscr{K}$ be a proper cone such that $-\mathscr{K}$ is characteristic for a square autonomous behavior $\mathfrak{B}\in\mathscr{L}_2^{\mathtt{w}}$. $\mathfrak{B}$ is $\mathscr{K}$-stable if there exists $n\in\mathbb{N}$, $n>0$ such that

$$[w\in\mathfrak{B}, w \text{ bounded in } (\delta(-\mathscr{K}))^n] \Longrightarrow \left[\lim_{\substack{(i,j)\in\mathscr{K}\\ |i|+|j|\to+\infty}} \|w(i,j)\|=0\right]$$

The following is Theorem 3.6 of [15].

Theorem 3. *Let* $\mathfrak{B}=\ker\Delta(\sigma_1,\sigma_2)$ *be a square autonomous behavior, and let* $\mathscr{K}$ *be a proper cone for* $\mathfrak{B}$ *which is* T*-isomorphic to the first orthant. Denote* $\delta := \det(\Delta)$, *and assume without loss of generality that* $\mathscr{H}_\delta\subset\mathscr{K}$ *and that* $\mathscr{H}_\delta\cap\delta\mathscr{K}=\{(0,0)\}$. *Denote with* $(t_1(\ell,m),t_2(\ell,m))$ *the image of* $(\ell,m)\in\mathbb{Z}\times\mathbb{Z}$ *under* T. *Define*

$$\Delta_T(\xi_1,\xi_2) := \sum_{\ell,m}\Delta_{\ell,m}\xi_1^{t_1(\ell,m)}\xi_2^{t_2(\ell,m)}$$

Then the following two statements are equivalent:

1. $\mathfrak{B}$ *is* $\mathscr{K}$*-stable;*
2. *The Laurent variety of* $\det\Delta_T$ *does not intersect the closed unit polydisk* $\overline{\mathscr{P}_1}$.

3 Bilinear and Quadratic Difference Forms for 2-D Systems

In order to represent bilinear and quadratic functionals of the variables of continuous-time 2-D systems, 4-variable polynomial matrices are used (see [12]). We now illustrate quadratic difference forms for 2-D discrete systems; some preliminary results are in [9].

In order to simplify the notation, define the multi-indices $\mathbf{k} := (k_1,k_2)$, $\mathbf{l} := (l_1,l_2)$, and the notation $\zeta := (\zeta_1,\zeta_2)$ and $\eta := (\eta_1,\eta_2)$, and $\zeta^{\mathbf{k}}\eta^{\mathbf{l}} := \zeta_1^{k_1}\zeta_2^{k_2}\eta_1^{l_1}\eta_2^{l_2}$. Let $\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta,\eta]$ denote the set of real polynomial $\mathtt{w}_1\times\mathtt{w}_2$ matrices in the 4 indeterminates ζ_i and η_i, $i=1,2$; that is, an element of $\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta,\eta]$ is of the form

$$\Phi(\zeta,\eta) = \sum_{\mathbf{k},\mathbf{l}}\Phi_{\mathbf{k},\mathbf{l}}\zeta^{\mathbf{k}}\eta^{\mathbf{l}}$$

where $\Phi_{\mathbf{k},\mathbf{l}}\in\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}$; the sum ranges over the nonnegative multi-indices $\mathbf{k}$ and $\mathbf{l}$, and is assumed to be finite. This matrix induces a *bilinear difference form* (*BDF* in the following) L_Φ

$$L_\Phi : (\mathbb{R}^{\mathtt{w}_1})^{\mathbb{Z}^2} \times (\mathbb{R}^{\mathtt{w}_2})^{\mathbb{Z}^2} \longrightarrow (\mathbb{R})^{\mathbb{Z}^2}$$
$$L_\Phi(v,w) := \sum_{\mathbf{k},\mathbf{l}} (\sigma^{\mathbf{k}} v)^\top \Phi_{\mathbf{k},\mathbf{l}} (\sigma^{\mathbf{l}} w)$$

where the **k**-th shift operator $\sigma^{\mathbf{k}}$ is defined as $\sigma^{\mathbf{k}} := \sigma_1^{k_1}\sigma_2^{k_2}$, and analogously for $\sigma^{\mathbf{l}}$.

The 4-variable polynomial matrix $\Phi(\zeta_1,\zeta_2,\eta_1,\eta_2)$ is called *symmetric* if $\mathtt{w}_1 = \mathtt{w}_2 =: \mathtt{w}$ and $\Phi(\zeta_1,\zeta_2,\eta_1,\eta_2) = \Phi(\eta_1,\eta_2,\zeta_1,\zeta_2)^\top$, concisely written as $\Phi(\zeta,\eta) = \Phi(\eta,\zeta)^\top$. In this case, Φ induces also a quadratic functional

$$Q_\Phi : (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \longrightarrow (\mathbb{R})^{\mathbb{Z}^2}$$
$$Q_\Phi(w) := L_\Phi(w,w)$$

We will call Q_Φ the *quadratic difference form* (in the following abbreviated with QDF) associated with the four-variable polynomial matrix Φ.

In this paper we also consider vectors $\Psi \in (\mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta,\eta])^2$, i.e.

$$\Psi(\zeta,\eta) = \begin{bmatrix} \Psi_1(\zeta,\eta) \\ \Psi_2(\zeta,\eta) \end{bmatrix} =: \mathrm{col}(\Psi_i(\zeta,\eta))_{i=1,2}$$

with $\Psi_i \in \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta,\eta]$ and with $\mathrm{col}(A_i)_{i=1,2}$ the matrix obtained by stacking the two matrices A_i, both with the same number of columns, on top of each other. Such Ψ induces a vector bilinear difference form (abbreviated *VBDF*), defined as

$$L_\Psi : (\mathbb{R}^{\mathtt{w}_1})^{\mathbb{Z}^2} \times (\mathbb{R}^{\mathtt{w}_2})^{\mathbb{Z}^2} \longrightarrow (\mathbb{R}^2)^{\mathbb{Z}^2}$$
$$L_\Psi(v,w) := \begin{bmatrix} L_{\Psi_1}(v,w) \\ L_{\Psi_2}(v,w) \end{bmatrix}.$$

Finally, we introduce the notion of (discrete) divergence of a VBDF. Given a VBDF $L_\Psi = \mathrm{col}(L_{\Psi_1}, L_{\Psi_2})^\top$, we define its *divergence* as the BDF defined by

$$\begin{aligned}(\mathrm{div}\, L_\Psi)(w_1,w_2) :=\ & \big(L_{\Psi_1}(w_1,w_2) - \sigma_1(L_{\Psi_1}(w_1,w_2))\big) \\ +\ & \big(L_{\Psi_2}(w_1,w_2) - \sigma_2(L_{\Psi_2}(w_1,w_2))\big)\end{aligned} \tag{2}$$

for all w_1, w_2. It is straightforward to verify that in terms of the 4-variable polynomial matrices associated with the BDF's, the relationship between a VBDF L_Ψ and its divergence $L_\Phi = \mathrm{div}\, L_\Psi$ is expressed as

$$\Phi(\zeta_1,\zeta_2,\eta_1,\eta_2) = (1-\zeta_1\eta_1)\Psi_1(\zeta_1,\zeta_2,\eta_1,\eta_2) + (1-\zeta_2\eta_2)\Psi_2(\zeta_1,\zeta_2,\eta_1,\eta_2)$$

In order to characterize those BDFs which are the divergence of some VBDF, we need to introduce the "del" operator, defined as

$$\partial : \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\zeta_1,\zeta_2,\eta_1,\eta_2] \longrightarrow \mathbb{R}^{\mathtt{w}_1\times\mathtt{w}_2}[\xi_1,\xi_2,\xi_1^{-1},\xi_2^{-1}]$$
$$\partial\Phi(\xi_1,\xi_2) := \Phi(\xi_1^{-1},\xi_2^{-1},\xi_1,\xi_2)$$

The following result holds true.

Proposition 1. *A BDF L_Φ is the divergence of some VBDF L_Ψ if and only if $\partial\Phi(\xi_1,\xi_2) = 0$.*

Proof. Necessity is straightforward. Sufficiency can be proved using a Gröbner basis argument, which can be extended entrywise to polynomial matrices. □

The definition and properties described above can be adapted to a vector quadratic difference form (VQDF) in a obvious manner.

A QDF Q_Δ induced by $\Delta \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1,\zeta_2,\eta_1,\eta_2]$ is *nonnegative* if $Q_\Delta(w(x_1,x_2)) \geq 0\ \forall(x_1,x_2) \in \mathbb{Z}^2$ and $\forall\, w \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$. This will be denoted with $Q_\Delta \geq 0$ or $\Delta(\zeta,\eta) \geq 0$. We call Q_Δ positive, denoted $Q_\Delta > 0$ or $\Delta(\zeta,\eta) > 0$, if $Q_\Delta \geq 0$ and $Q_\Delta(w(x_1,x_2)) = 0\ \forall(x_1,x_2) \in \mathbb{Z}^2$ implies $w = 0$. Often in the following we will also consider QDFs induced by matrices of the form $\Delta(e^{-i\omega},\zeta_2,e^{i\omega},\eta_2)$, i.e. matrices in the indeterminates ζ_2,η_2 with coefficients being polynomials in $e^{i\omega}$ for some $\omega \in \mathbb{R}$. The definition of nonnegativity and positivity in this case is readily adapted from above.

4 Necessary and Sufficient Lyapunov Conditions for Stability of 2-D Systems

Using Theorem 3, we now concentrate on stability with respect to the proper cone consisting of the first orthant of $\mathbb{Z}\times\mathbb{Z}$; we denote this set with $\mathscr{K}_0$ in the following. Moreover, we only consider the case of square autonomous systems. We begin this section with a straightforward but important refinement of Proposition 3.5 of [15].

Proposition 2. *Let $\mathfrak{B} \in \mathscr{L}_2^{\mathtt{w}}$ be square and autonomous, and let $\Delta \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi_1,\xi_2]$ nonsingular be such that $\mathfrak{B} = \ker\ \Delta(\sigma_1,\sigma_2)$. Assume that $\delta := \det\ \Delta$ is such that $\mathscr{H}_\delta$ is a subset of $\mathscr{K}_0$, the first orthant of $\mathbb{Z}\times\mathbb{Z}$, that intersects the coordinate axes only in the origin. Then the following statements are equivalent:*

1. *$\mathfrak{B}$ is $\mathscr{K}_0$-stable;*
2. *For all $\omega \in \mathbb{R}$, the polynomial $\delta(e^{j\omega},\xi_2)$ has all its roots outside of the closed unit disk $\{z_2 \in \mathbb{C} \mid |z_2| \geq 1\}$, and the polynomial $\delta(\xi_1,e^{j\omega})$ has all its roots outside of the closed unit disk $\{z_1 \in \mathbb{C} \mid |z_1| \geq 1\}$.*

Proof. The proof follows from Theorem 3 and from the equivalence of statements *i*) and *iv*) in Proposition 3.1 of [5]. □

In order to state the main result of this paper we need some notation; we denote with $\mathrm{Per}_2 \subset (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$ the set consisting of all trajectories $v \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2}$ such that the restriction of v to the lines $\{(x_1,x_2) \mid x_2 \in \mathbb{Z}\}$ is periodic for all fixed $x_1 \in \mathbb{Z}$, i.e.

$$\mathrm{Per}_2 := \left\{ v \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \mid v(x_1,\cdot) \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{R}} \text{ is periodic for all fixed } x_1 \in \mathbb{Z} \right\} ;$$

analogously

$$\mathrm{Per}_1 := \left\{ v \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{Z}^2} \mid v(\cdot,x_2) \in (\mathbb{R}^{\mathtt{w}})^{\mathbb{R}} \text{ is periodic for all fixed } x_2 \in \mathbb{Z} \right\} .$$

Theorem 4. *Let $\mathfrak{B} \in \mathscr{L}_2^{\mathtt{w}}$ be square and autonomous, and $R \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\xi_1, \xi_2]$ nonsingular be such that $\mathfrak{B} = \ker\ R(\sigma_1, \sigma_2)$. The following statements are equivalent:*

(1) $\mathfrak{B}$ is $\mathscr{K}_0$-stable;
(2) There exists a VQDF $Q_\Phi = \mathrm{col}(Q_{\Phi_1}, Q_{\Phi_2})$ and a QDF Q_Δ such that

(2a) *div* $Q_\Phi \stackrel{\mathfrak{B}}{=} -Q_\Delta$;
(2b) $Q_{\Phi_1}(w), Q_\Delta(w) > 0$ *for all* $w \in \mathfrak{B} \cap \mathrm{Per}_2$, *and* $Q_{\Phi_2}(w), Q_\Delta(w) > 0$ *for all* $w \in \mathfrak{B} \cap \mathrm{Per}_1$.

(3) There exist $\Phi = \mathrm{col}(\Phi_1, \Phi_2)$ and Δ, with Φ_1, Φ_2, $Y \in \mathbb{R}^{\mathtt{w}\times\mathtt{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2]$, $\Delta \in \mathbb{R}_s^{\mathtt{w}\times\mathtt{w}}[\zeta_1, \zeta_2, \eta_1, \eta_2]$ such that

(3a) $$\begin{aligned}(1-\zeta_1\eta_1)\Phi_1(\zeta_1,\zeta_2,\eta_1,\eta_2) + (1-\zeta_2\eta_2)\Phi_2(\zeta_1,\zeta_2,\eta_1,\eta_2) \\ = -\Delta(\zeta_1,\zeta_2,\eta_1,\eta_2) \\ + R(\zeta_1,\zeta_2)^\top Y(\zeta_1,\zeta_2,\eta_1,\eta_2) + Y(\eta_1,\eta_2,\zeta_1,\zeta_2)^\top R(\eta_1,\eta_2);\end{aligned}$$

(3b) $\Phi_1(\zeta_1,\zeta_2,\eta_1,\eta_2) \stackrel{\mathfrak{B}\cap\mathrm{Per}_2}{>} 0$, $\Phi_2(\zeta_1,\zeta_2,\eta_1,\eta_2) \stackrel{\mathfrak{B}\cap\mathrm{Per}_1}{>} 0$, *and* $\Delta(\zeta_1,\zeta_2,\eta_1,\eta_2) \stackrel{\mathfrak{B}\cap\mathrm{Per}_i}{>} 0$, $i = 1, 2$.

We refer to a VQDF Q_Φ satisfying (2a) *and* (2b) *as a Lyapunov function for* $\mathfrak{B}$.

Proof. The equivalence of statements (2) and (3) is straightforward.

We now prove the implication (3) ⇒ (1). Consider any trajectory in $\mathfrak{B}$ of the form $w(t_1,t_2) = v\,\lambda^{t_1}\,\mu^{t_2}$ for some $v \in \mathbb{C}^{\mathtt{w}}$ and $\lambda, \mu \in \mathbb{C}$. We now prove that if μ lies on the unit circle, i.e. $\mu = e^{i\omega}$ for some $\omega \in \mathbb{R}$, then $|\lambda| > 1$. Once this will have been established, statement (1) follows from Proposition 2.

Let $\zeta_1 = \overline{\lambda}$, $\eta_1 = \lambda$, $\zeta_2 = \overline{\mu} = e^{-i\omega}$, $\eta_2 = \mu = e^{i\omega}$ in (3a):

$$(1-\overline{\lambda}\lambda)\, v^\top \Phi_1(\overline{\lambda}, e^{-i\omega}, \lambda, e^{i\omega}) v = -v^\top \Delta(\overline{\lambda}, e^{-i\omega}, \lambda, e^{i\omega}) v$$

The right-hand side of this equation is strictly negative; on the left-hand side $v^\top \Phi_1(\overline{\lambda}, e^{-i\omega}, \lambda, e^{i\omega}) v > 0$, and consequently it follows that $1 - \overline{\lambda}\lambda < 0$. An analogous argument is used when $w(t_1,t_2) = v\, e^{i\omega t_1}\mu^{t_2}$. This proves the claim.

The proof of implication (1) ⇒ (3) is established by producing matrices $\Phi_i \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta_1,\zeta_2,\eta_1,\eta_2]$, $i = 1, 2$, and $\Delta \in \mathbb{R}_S^{\mathtt{w}\times\mathtt{w}}[\zeta_1,\zeta_2,\eta_1,\eta_2]$ such that (3a)–(3b) hold.

Write $R(\xi_1,\xi_2) = \sum_{i=0}^{L_1} R_i(\xi_2)\xi_1^{L_1} = \sum_{i=0}^{L_2} R'_i(\xi_1)\xi_2^{L_2}$, where L_i is the highest power of ξ_i in R, $i = 1, 2$. Define the four-variable polynomial matrix

$$H(\zeta_1,\zeta_2,\eta_1,\eta_2) := R(\zeta_1,\zeta_2)^\top R(\eta_1,\eta_2) - \zeta_1^{L_1}\zeta_2^{L_2}\eta_1^{L_1}\eta_2^{L_2} R(\eta_1^{-1},\eta_2^{-1})^\top R(\zeta_1^{-1},\zeta_2^{-1})\,. \quad (3)$$

Observe that $\partial H = 0$; conclude from Proposition 1 that there exists $\Phi = \mathrm{col}(\Phi_1, \Phi_2) \in \mathbb{R}^{2\mathtt{w}\times\mathtt{w}}[\zeta_1,\zeta_2,\eta_1,\eta_2]$ such that div $\Phi(\zeta_1,\zeta_2,\eta_1,\eta_2) = H(\zeta_1,\zeta_2,\eta_1,\eta_2)$. Moreover, it is easy to see using Proposition 3.2 of [6] that

$$\Phi_1(\zeta_1, e^{-i\omega}, \eta_1, e^{i\omega}) = \frac{R(\zeta_1, e^{-i\omega})^\top R(\eta_1, e^{i\omega}) - \zeta_1^{L_1}\eta_1^{L_1} R(\eta_1^{-1}, e^{-i\omega})^\top R(\zeta_1^{-1}, e^{i\omega})}{1-\zeta_1\eta_1}\,. \quad (4)$$

From Proposition 2 it follows that since $\mathfrak{B}$ is $\mathscr{K}_0$-stable the polynomial det $\left(R(\xi_1, e^{i\omega})\right)$ is anti-Schur (meaning all its roots have modulus greater than one) for all $\omega \in \mathbb{R}$. It follows from Corollary 1 of [8] that for all $\omega \in \mathbb{R}$ $\Phi_1(\zeta_1, e^{-i\omega}, \eta_1, e^{i\omega}) > 0$, since (4) is equivalent with Φ_1 being the R-canonical solution of an ω-dependent polynomial Lyapunov equation in two variables (see equation (4) of [8]) for the behavior described in kernel form by $R(\xi_1, e^{i\omega})$. From this it follows that $\Phi_1 \overset{\mathfrak{B}\cap\mathrm{Per}_2}{>} 0$.

An analogous argument based on the same considerations and on the fact that $R(e^{i\omega}, \xi_2)$ is anti-Schur for all $\omega \in \mathbb{R}$, shows that $\Phi_2(e^{-i\omega}, \zeta_2, e^{i\omega}, \eta_2) > 0$ for all $\omega \in \mathbb{R}$.

In order to conclude the proof, define

$$Y(\xi_1, \xi_2) := \frac{1}{2} R(\xi_1, \xi_2)$$
$$\Delta(\zeta_1, \zeta_2, \eta_1, \eta_2) := \zeta_1^{L_1} \eta_1^{L_1} \zeta_2^{L_2} \eta_2^{L_2} R(\eta_1^{-1}, \eta_2^{-1})^\top R(\zeta_1^{-1}, \zeta_2^{-1})$$

The fact that $\Delta(\zeta_1, e^{-i\omega}, \eta_1, e^{i\omega}) > 0$ and $\Delta(e^{-i\omega}, \zeta_2, e^{i\omega}, \eta_2) > 0$ for all $\omega \in \mathbb{R}$ follows from the $\mathscr{K}_0$-stability of $\mathfrak{B}$, which implies for all $\omega \in \mathbb{R}$ that $R(\xi_1, e^{i\omega})$ and $R(e^{i\omega}, \xi_2)$ are anti-Schur. □

Remark 1. The 4-variable polynomial matrices $\Phi = \mathrm{col}(\Phi_1, \Phi_2)$ and Δ given in the proof of Theorem 4 are germane to the multivariable Bézoutian

$$\frac{R(\zeta)^\top R(\eta) - R(-\eta)^\top R(-\zeta)}{\zeta + \eta}$$

used in analyzing stability of 1-D continuous-time systems (see section 3 of [16]). In the 2-D single-variable ($\mathtt{w} = 1$) case, stability conditions based on the positivity of the coefficient matrix of an ω-dependent Bézoutian have been obtained in [4, 5].

Of course, there are more Lyapunov functions than the Bézoutian. The computation of Lyapunov functions via a (4-variable) polynomial Lyapunov equation as in [11, 16] is the subject of an ongoing investigation.

Acknowledgements. The second author gratefully acknowledges the financial support of *the Great Britain Sasakawa Foundation* for financially supporting the visit to Kyoto University during which the results presented in this paper were obtained.

References

1. Fornasini, E., Marchesini, G.: Stability Analysis of 2-D Systems. IEEE Trans. Circ. Syst. 27(12), 1210–1217 (1980)
2. Fornasini, E., Rocha, P., Zampieri, S.: State-space realization of 2-D finite-dimensional behaviors. SIAM J. Control and Optim. 31, 1502–1517 (1993)
3. Fornasini, E., Valcher, M.E.: *n*D polynomial matrices with applications to multidimensional signal analysis. Multidimensional Syst. Sign. Proc. 8, 387–407 (1997)
4. Geronimo, J.S., Woerdeman, H.J.: Positive extensions, Fejér-Riesz factorization and autoregressive filters in two-variables. Ann. Math. 160, 839–906 (2004)

5. Geronimo, J.S., Woerdeman, H.J.: Two-Variable Polynomials: Intersecting Zeros and Stability. IEEE Trans. Circ. Syst., Part I: Regular Papers 53(5), 1130–1139 (2006)
6. Kaneko, O., Fujii, T.: Discrete-time average positivity and spectral factorization in a behavioral framework. Systems & Control Letters 39, 31–44 (2000)
7. Kojima, C., Rapisarda, P., Takaba, K.: Canonical forms for polynomial and quadratic differential operators. Systems & Control Letters 56, 678–684 (2007)
8. Kojima, C., Takaba, K.: A generalized Lyapunov stability theorem for discrete-time systems based on quadratic difference forms. In: Proc. 44th IEEE Conference on Decision and Control and the European Control Conference, Seville, Spain, pp. 2911–2916 (2005)
9. Kojima, C., Takaba, K.: A Lyapunov stability analysis of 2-D discrete-time behaviors. In: Proc. 17th MTNS, Kyoto, Japan, pp. 2504–2512 (2006)
10. Lu, W.-S., Lee, E.B.: Stability analysis of two-dimensional systems via a Lyapunov approach. IEEE Trans. Circ. Syst. 32(1), 61–68 (1985)
11. Peeters, R., Rapisarda, P.: A two-variable approach to solve the polynomial Lyapunov equation. System & Control Letters 42, 117–126 (2001)
12. Pillai, H.K., Willems, J.C.: Lossless and dissipative distributed systems. SIAM J. Control Optim. 40, 1406–1430 (2002)
13. Polderman, J.W., Willems, J.C.: Introduction to Mathematical System Theory: A Behavioral Approach. Springer, Berlin (1997)
14. Rocha, P.: Structure and representation of 2-D systems. Ph.D. thesis, Univ. of Groningen, The Netherlands (1990)
15. Valcher, M.E.: Characteristic cones and stability properties of two-dimensional autonomous behaviors. IEEE Trans. Circ. Syst., Part I: Fundamental Theory and Applications 47(3), 290–302 (2000)
16. Willems, J.C., Trentelman, H.L.: On quadratic differential forms. SIAM J. Control Optim. 36(5), 1703–1749 (1998)